ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\frac{\pi}{2}}\left(x\ln(1+\cos(x))+x\ln(\sin(x))\right)dx=\pi G-\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}\ln(2)$$

$$G\to\ Catalan's\ constant\ ,\qquad \zeta(3)\to Apery's\ constant$$

Proposed by Shirvan Tahirov, Elsen Kerimov-Azerbaijan

Solution 1 by Quadri Faruk Temitope-Nigeria

$$I=\underbrace{\int_0^{\frac{\pi}{2}}x\ln(1+\cos(x))dx}_{A}+\underbrace{\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx}_{B}=A+B$$

$$A=\int_0^{\frac{\pi}{2}}x\ln(1+\cos(x))dx=\int_0^{\frac{\pi}{2}}x\ln\left(2\cos^2\left(\frac{x}{2}\right)\right)dx=\int_0^{\frac{\pi}{2}}x\ln(2)dx+\int_0^{\frac{\pi}{2}}x\ln\left(\cos^2\left(\frac{x}{2}\right)\right)=$$

$$=\ln(2)\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2\right]+2\int_0^{\frac{\pi}{2}}x\ln\left(\cos\left(\frac{x}{2}\right)\right)dx=\frac{\pi^2}{8}\ln(2)+2\int_0^{\frac{\pi}{2}}x\left[-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n}\right]dx=$$

$$\frac{\pi^2}{8}\ln(2)-2\ln(2)\frac{x^2}{8}\Bigg|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x\left[\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n}\right]dx=\frac{\pi^2}{8}\ln(2)-\frac{\pi^2}{4}\ln(2)-$$

$$2\Re\int_0^{\frac{\pi}{2}}x\left[\sum_{n=1}^{\infty}\frac{(-1)^n e^{-inx}}{n}\right]dx=\frac{\pi^2}{8}\ln(2)-\frac{\pi^2}{4}\ln(2)-2\Re\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{2}}x\,e^{-inx}dx=$$

$$-\frac{\pi^2}{8}\ln(2)-2\Re\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[-\frac{1}{n^2}+\frac{e^{-\frac{i\pi n}{2}}}{n^2}+\frac{i\pi n e^{-\frac{i\pi n}{2}}}{2n^2}\right]=-\frac{\pi^2}{8}\ln(2)-2\left[-\frac{\pi G}{2}+\frac{21}{32}\zeta(3)\right]$$

$$A=-\frac{\pi^2}{8}\ln(2)+\pi G-\frac{21}{16}\zeta(3)$$

Working on B

$$B=\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx=\int_0^{\frac{\pi}{2}}x\left[-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right]dx=-\ln(2)\int_0^{\frac{\pi}{2}}xdx-$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}}x\cos(2nx)dx=-\ln(2)\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2\right]-\sum_{n=1}^{\infty}\frac{1}{n}\Re\int_0^{\frac{\pi}{2}}xe^{-2inx}dx=-\frac{\pi^2}{8}\ln(2)-$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\Re\left[\frac{-1+e^{-i\pi n}+i\pi n e^{-i\pi n}}{4n^2}\right]=-\frac{\pi^2}{8}\ln(2)-\frac{1}{4}\Re\left[\sum_{n=1}^{\infty}-\frac{1}{n^3}+\sum_{n=1}^{\infty}\frac{e^{-i\pi n}}{n^3}+i\pi\sum_{n=1}^{\infty}\frac{e^{-i\pi n}}{n^2}\right]=$$

$$-\frac{\pi^2}{8}\ln(2)-\frac{1}{4}\Re\left[-\zeta(3)-\frac{3}{4}\zeta(3)-i\pi\frac{\zeta(2)}{2}\right]=-\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)$$

$$B=-\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)$$

$$I=\underbrace{\int_0^{\frac{\pi}{2}}x\ln(1+\cos(x))dx}_{A}+\underbrace{\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx}_{B}=-\frac{\pi^2}{8}\ln(2)+\pi G-\frac{21}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)$$

$$I=\pi G-\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}ln(2)$$

Solution 2 by Pratham Prasad-India

$$\psi=\int_0^{\frac{\pi}{2}}xln(1+cosx)+xlnsinx\,dx=\int_0^{\frac{\pi}{2}}xln\left(2\cos^2\left(\frac{x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}}xln(sinx)\,dx$$

$$=4\int_0^{\frac{\pi}{4}}uln\left(2\cos^2(u)\right)du+\int_0^{\frac{\pi}{2}}xln(sinx)\,dx$$

$$=4\int_0^{\frac{\pi}{4}}uln(2)\,du+8\int_0^{\frac{\pi}{4}}uln(cosu)\,du+\int_0^{\frac{\pi}{2}}xln(sinx)\,dx$$

By expanding and evaluating using Fourier series of the second and third integral

$$\int_0^{\frac{\pi}{4}}uln(cosu)\,du=\int_0^{\frac{\pi}{4}}u\left(-\ln(2)+\sum_{K=1}^{\infty}\frac{(-1)^{k-1}\cos(2ku)}{k}\right)du=$$

$$=-\frac{\pi^2}{32}\ln(2)+\int_0^{\frac{\pi}{4}}u\left(\sum_{K=1}^{\infty}\frac{(-1)^{k-1}\cos(2ku)}{k}\right)du$$

$$=-\frac{\pi^2}{32}\ln(2)+\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^{\frac{\pi}{4}}(u\cos(2ku))\,du$$

$$=-\frac{\pi^2}{32}\ln(2)+\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{\pi}{8k}\sin\left(\frac{\pi k}{2}\right)+\frac{1}{4k^2}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{4k^2}\right)$$

$$=-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{8}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\left(\sin\left(\frac{\pi k}{2}\right)\right)+\frac{1}{4}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}\left(\cos\left(\frac{\pi k}{2}\right)\right)-\frac{1}{4}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$$=-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{8}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{32}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}-\frac{1}{4}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$$=\frac{1}{128}\left(16\pi G-21\zeta(3)-4\pi^2\ln(2)\right)$$

$$\int_0^{\frac{\pi}{2}} x\ln(\sin x)\,dx=\int_0^{\frac{\pi}{2}} x\left(-\ln(2)-\sum_{K=1}^{\infty}\frac{\cos(2ku)}{k}\right)dx=$$

$$=-\frac{\pi^2}{8}\ln(2)-\int_0^{\frac{\pi}{2}} x\left(\sum_{K=1}^{\infty}\frac{\cos(2ku)}{k}\right)dx=-\frac{\pi^2}{8}\ln(2)-\sum_{K=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}} x(\cos(2ku))\,dx=$$

$$=-\frac{\pi^2}{8}\ln(2)-\sum_{K=1}^{\infty}\frac{1}{k}\left(\frac{(-1)^k}{4k^2}-\frac{1}{4k^2}\right)=\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}ln(2)$$

Putting everything back :

$$\psi=\frac{\pi^2}{8}\ln(2)+8\left(\frac{1}{128}\left(16\pi G-21\zeta(3)-4\pi^2\ln(2)\right)\right)+\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}ln(2)$$

$$=\frac{\pi^2}{8}\ln(2)+\pi G-\frac{21}{16}\zeta(3)-\frac{\pi^2}{4}\ln(2)+\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}ln(2)$$

$$\psi=\pi G-\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}ln(2)$$

Solution 3 by Exodo Halcalias-Angola

$$\Omega=\int_0^{\frac{\pi}{2}}(x\ln(1+\cos(x))+x\ln(\sin(x)))dx=\int_0^{\frac{\pi}{2}} x\ln\left(\sin(x)+\frac{1}{2}\sin(x)\cos(x)\right)dx=$$

$$=\int_0^{\frac{\pi}{2}} x\ln\left(4\sin\left(\frac{x}{2}\right)cos^3\left(\frac{x}{2}\right)\right)dx=4\int_0^{\frac{\pi}{4}} x\ln(\sin(2x))dx+4\ln(2)\int_0^{\frac{\pi}{4}} xdx+8\int_0^{\frac{\pi}{4}} x\ln(\cos(x))dx=$$

$$\int_0^{\frac{\pi}{2}} x\left(\ln\left(\frac{1}{2}\right)-\sum_{k\in N}\frac{\cos(2kx)}{k}\right)dx-\frac{\pi^2}{8}ln\left(\frac{1}{2}\right)+8\int_0^{\frac{\pi}{4}} x\left(\ln\left(\frac{1}{2}\right)+\sum_{k\in N}\frac{(-1)^{k-1}\cos(2kx)}{k}\right)dx=\frac{\pi^2}{4}ln\left(\frac{1}{2}\right)-$$

$$-\sum_{k\in N}\frac{1}{k}\left(\frac{\pi\sin(\pi k)}{4k}+\frac{cos(\pi k)}{4k^2}-\frac{1}{k^2}\right)+\sum_{k\in N}\frac{(-1)^{k-1}}{k}\left(\frac{\pi}{k}\sin\left(\frac{k\pi}{2}\right)+\frac{1}{2k^2}\cos\left(\frac{k\pi}{2}\right)-\frac{1}{2k^2}\right)=$$

$$\frac{\pi^2}{4}ln\left(\frac{1}{2}\right)-\frac{1}{4}\sum_{k\in N}\frac{(-1)^k}{k^3}+\frac{1}{4}\sum_{k\in N}\frac{1}{k^3}+\pi\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k)^3}-\frac{1}{2}\sum_{k\in N}\frac{1}{k^3}=$$

$$\frac{\pi^2}{4}ln\left(\frac{1}{2}\right)+\pi\beta(2)-\frac{7}{8}\sum_{k\in N}\frac{1}{k^3}=\pi G-\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}ln(2)$$

$$\Omega=\pi G-\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}ln(2)$$