ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\psi = \int_0^1 \frac{x+1}{x}\log(2x+1)\,dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pratham Prasad-India

$$\psi = \int_0^1 \frac{x+1}{x}\log(2x+1)\,dx = \int_0^1 \log(2x+1)\,dx + \int_0^1 \frac{\log(2x+1)}{x}\,dx$$

$$= [x\log(2x+1)]_0^1 - \int_0^1 \frac{2x}{2x+1}dx + \int_0^1 \frac{\log(2x+1)}{x}\,dx$$

$$= \log(3) - \int_0^1 1\,dx + \int_0^1 \frac{1}{2x+1}\,dx + \int_0^1 \frac{\log(2x+1)}{x}\,dx$$

$$= \log(3) - 1 + \left[\frac{1}{2}\log(2x+1)\right]_0^1 + \int_0^1 \frac{\log(2x+1)}{x}\,dx$$

$$= \log(3) - 1 + \frac{1}{2}\log(3) + \int_0^1 \frac{\log(2x+1)}{x}\,dx = \frac{3}{2}\log(3) - 1 + \int_0^2 \frac{\log(x+1)}{x}\,dx =$$

$$= \frac{3}{2}\log(3) - 1 + \int_0^1 \frac{\log(x+1)}{x}\,dx + \int_1^2 \frac{\log(x+1)}{x}\,dx$$

$$= \frac{3}{2}\log(3) - 1 + \frac{1}{2}\zeta(2) + \int_{\frac{1}{2}}^1 \frac{\log\left(\frac{1}{x}+1\right)}{x}\,dx$$

$$= \frac{3}{2}\log(3) - 1 + \frac{1}{2}\zeta(2) + \int_{\frac{1}{2}}^1 \frac{\log(1+x)}{x}\,dx - \int_{\frac{1}{2}}^1 \frac{\log(x)}{x}\,dx$$

$$= \frac{3}{2}\log(3) - 1 + \frac{1}{2}\zeta(2) + \int_0^1 \frac{\log(1+x)}{x}\,dx - \int_0^{\frac{1}{2}} \frac{\log(1+x)}{x}\,dx - \left[\frac{1}{2}\log^2(x)\right]_{\frac{1}{2}}^1$$

$$= \frac{3}{2}\log(3) - 1 + \frac{1}{2}\zeta(2) + \frac{1}{2}\zeta(2) + Li_2\left(-\frac{1}{2}\right) + \frac{1}{2}\log^2(2)$$

$$= \frac{3}{2}\log(3) - 1 + \zeta(2) + Li_2\left(-\frac{1}{2}\right) + \frac{1}{2}\log^2(2)$$

$$\psi = \frac{3}{2}\log(3) - 1 + \frac{\pi^2}{6} + Li_2\left(-\frac{1}{2}\right) + \frac{1}{2}\log^2(2)$$

$$\int_0^1 \frac{x+1}{x}\log(2x+1)\,dx = \frac{3}{2}\log(3) - 1 + \frac{\pi^2}{6} + Li_2\left(-\frac{1}{2}\right) + \frac{1}{2}\log^2(2)$$