ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\psi=\int_0^1\int_0^1\frac{x^2\ln(x+1)\ln(y+2)}{(x+1)(y+1)}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pratham Prasad-India

$$\psi=\int_0^1\int_0^1\frac{x^2\ln(x+1)\ln(y+2)}{(x+1)(y+1)}dxdy$$

$$=\left(\int_0^1\frac{x^2\ln(x+1)}{(x+1)}dx\right)\left(\int_0^1\frac{\ln(y+2)}{(y+1)}dy\right)$$

$$=\left(\int_0^1\frac{(x+1)^2\ln(x+1)}{(x+1)}dx-2\int_0^1\frac{(x+1)\ln(x+1)}{(x+1)}dx+\int_0^1\frac{\ln(x+1)}{(x+1)}dx\right)\left(\int_0^1\frac{\ln(y+2)}{(y+1)}dy\right)$$

$$=\left(\int_0^1(x+1)\ln(x+1)dx-2\int_0^1\ln(x+1)\,dx+\int_0^1\frac{\ln(x+1)}{(x+1)}dx\right)\left(\int_0^1\frac{\ln(y+2)}{(y+1)}dy\right)$$

$$=(I_1-2I_2+I_3)(I_4)$$

$$I_1=\int_0^1(x+1)\ln(x+1)dx$$

$$I_1\overset{IBP}{\cong}\left[\frac{(x+1)^2}{2}\ln(x+1)\right]_0^1-\frac{1}{2}\int_0^1(x+1)\,dx$$

$$I_1=2\ln(2)-\frac{1}{2}\left(\frac{1}{2}+1\right)$$

$$I_1=2\ln(2)-\frac{3}{4}$$

$$I_2=\int_0^1\ln(x+1)\,dx$$

$$I_2\overset{IBP}{\cong}[(x+1)\ln(1+x)]_0^1-\int_0^1\frac{x+1}{x+1}dx$$

$$I_2=2\ln(2)-\int_0^1 1\,dx$$

$$I_2=2\ln(2)-1$$

$$I_3=\int_0^1\frac{\ln(x+1)}{(x+1)}dx$$

$$I_3\overset{\ln(1+x)=t}{\cong}\int_0^{\ln(2)}tdt$$

$$I_3=\frac{1}{2}\ln^2(2)$$

$$I_1-2I_2+I_3=2\ln(2)-\frac{3}{4}-2(2\ln(2)-1)+\frac{1}{2}\ln^2(2)$$

$$I_1-2I_2+I_3=-2\ln(2)+\frac{5}{4}+\frac{1}{2}\ln^2(2)$$

$$I_4=\int_0^1\frac{\ln(y+2)}{(y+1)}dy$$

$$I_4\overset{x=y+1}{\cong}\int_1^2\frac{\ln(y+1)}{y}dy$$

$$I_4=\int_0^2\frac{\ln(y+1)}{y}dy-\int_0^1\frac{\ln(y+1)}{y}dy$$

$$I_4=-Li_2(-2)-\frac{1}{2}\zeta(2)$$

$$\psi=(I_1-2I_2+I_3)(I_4)$$

$$\psi=\left(-2\ln(2)+\frac{5}{4}+\frac{1}{2}\ln^2(2)\right)\left(-Li_2(-2)-\frac{1}{2}\zeta(2)\right)$$

$$\psi=\left(2\ln(2)-\frac{5}{4}-\frac{1}{2}\ln^2(2)\right)\left(Li_2(-2)+\frac{1}{2}\zeta(2)\right)$$

$$\psi=2\ln(2)Li_2(-2)-\frac{5}{4}Li_2(-2)-\frac{1}{2}\ln^2(2)\,Li_2(-2)+\zeta(2)\ln(2)-\frac{5}{4}\zeta(2)-\frac{1}{2}\zeta(2)\ln^2(2)$$

$$\boxed{\int_0^1\int_0^1\frac{x^2\ln(x+1)\ln(y+2)}{(x+1)(y+1)}dxdy=2\ln(2)Li_2(-2)-\frac{5}{4}Li_2(-2)-\frac{1}{2}\ln^2(2)\,Li_2(-2)+\zeta(2)\ln(2)-\frac{5}{4}\zeta(2)-\frac{1}{2}\zeta(2)\ln^2(2)}$$