ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \frac{(x-1)^2 Li_3(-x)}{x+1}dx, \qquad \zeta(3) \to Apery's\ constant$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$$\int_0^1 \frac{(x-1)^2 Li_3(-x)}{x+1}dx \to \begin{cases} u = Li_3(-x) \\ dv = \frac{(x-1)^2}{1+x}dx \end{cases} \to \begin{cases} du = \frac{Li_2(-u)}{x} \\ v = \left(\frac{x^2}{2} - 3x + 4\ln(1+x)\right) \end{cases}$$

$$\Omega = \left(\frac{x^2}{2} - 3x + 4\ln(1+x)\right)Li_3(-x)\Big|_0^1 - \int_0^1 \frac{Li_2(-u)}{x}\left(\frac{x^2}{2} - 3x + 4\ln(1+x)\right)dx =$$

$$\frac{15}{8}\zeta(3) - 3\ln(2)\zeta(3) - \frac{1}{2}\int_0^1 xLi_2(-x)dx + 3\int_0^1 Li_2(-x)dx - 4\int_0^1 \frac{\ln(1+x)Li_2(-x)}{x}dx$$

$$A = \int_0^1 xLi_2(-x)dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\int_0^1 x^{n+1}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2(n+2)} = -\frac{\pi^2}{24} + \frac{1}{8}$$

$$B = \int_0^1 Li_2(-x)dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\int_0^1 x^n dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2(n+1)} = -\frac{\pi^2}{12} + 2\ln(2) - 1$$

$$C = \int_0^1 \frac{\ln(1+x)Li_2(-x)}{x}dx = -\int_0^1 Li_2(-x)d\big(Li_2(-x)\big) = -\frac{1}{2}Li_2^2(-x)\Big|_0^1 = -\frac{\pi^4}{288}$$

$$\Omega = \frac{15}{8}\zeta(3) - 3\ln(2)\zeta(3) - \frac{1}{2}A + 3B - 4C = \frac{15}{8}\zeta(3) - 3\ln(2)\zeta(3) - \frac{1}{2}\left(-\frac{\pi^2}{24} + \frac{1}{8}\right) +$$

$$3\left(-\frac{\pi^2}{12} + 2\ln(2) - 1\right) - 4\left(-\frac{\pi^4}{288}\right)$$

$$\int_0^1 \frac{(x-1)^2 Li_3(-x)}{x+1}dx = \frac{15}{8}\zeta(3) - 3\ln(2)\zeta(3) + \frac{\pi^4}{72} - \frac{11}{48}\pi^2 + 6ln(2) - \frac{49}{16}$$

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Solution 2 by Pratham Prasad-India

$$\int_0^1 \frac{(x-1)^2 Li_3(-x)}{x+1}dx = \int_0^1 \frac{(x+1)^2 Li_3(-x)}{x+1}d\,x - 4\int_0^1 Li_3(-x)dx + 4\int_0^1 \frac{Li_3(-x)}{1+x}dx =$$

$$\int_0^1 xLi_3(-x)dx - 3\int_0^1 Li_3(-x)dx + 4\int_0^1 \frac{Li_3(-x)}{1+x}dx = I_1 - 3I_2 + 4I_3$$

$$I_1 = \int_0^1 xLi_3(-x)dx = \int_0^1 x\sum_{n=1}^{\infty}\frac{(-x)^n}{n^3}dx = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^{n+1}\,dx =$$

$$\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{n^2(n+2)}\right\} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n+2} + \frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} =$$

$$-\frac{3}{8}\zeta(3) + \frac{1}{8}\zeta(2) - \frac{1}{8}\left(\frac{1}{2} - \ln(2)\right) + \frac{1}{8}(-\ln(2)) = -\frac{3}{8}\zeta(3) + \frac{1}{8}\zeta(2) - \frac{1}{16}$$

$$I_2 = \int_0^1 Li_3(-x)dx = \int_0^1 \sum_{n=1}^{\infty}\frac{(-x)^n}{n^3}dx = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^n\,dx = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n+1)} =$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\frac{3}{4}\zeta(3) + \frac{1}{2}\zeta(2) - \ln(2) - \ln(2) + 1 =$$

$$-\frac{3}{4}\zeta(3) + \frac{1}{2}\zeta(2) - 2\ln(2) + 1$$

$$I_3 = \int_0^1 \frac{Li_3(-x)}{1+x}dx = Li_3(-x)\ln(x+1)\Big|_0^1 - \int_0^1 \frac{Li_2(-x)}{x}\ln(1+x)\,dx = Li_3(-1)\ln(2) +$$

$$\frac{(Li_2(-x))^2}{2}\Bigg|_0^1 = \ln(2)\left(-\frac{3}{4}\zeta(3)\right) + \frac{1}{2}(Li_2(-2))^2 = \ln(2)\left(-\frac{3}{4}\zeta(3)\right) + \frac{1}{2}\left(-\frac{1}{2}\zeta(2)\right)^2 =$$

$$\frac{\pi^4}{288} - \frac{3}{4}\zeta(3)\ln(2)$$

$$\int_0^1 \frac{(x-1)^2 Li_3(-x)}{x+1}dx = I_1 - 3I_2 + 4I_3 = -\frac{3}{8}\zeta(3) + \frac{1}{8}\zeta(2) - \frac{1}{16} -$$

$$3\left(-\frac{3}{4}\zeta(3) + \frac{1}{2}\zeta(2) - 2\ln(2) + 1\right) + 4\left(\frac{\pi^4}{288} - \frac{3}{4}\zeta(3)\ln(2)\right)$$

$$\int_0^1 \frac{(x-1)^2 Li_3(-x)}{x+1}dx = \frac{15}{8}\zeta(3) - 3\ln(2)\,\zeta(3) + \frac{\pi^4}{72} - \frac{11}{48}\pi^2 + 6ln(2) - \frac{49}{16}$$