ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1 \frac{x\ln^2(x)}{(1-x)(2-x)(3-x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\Omega=\int_0^1 \frac{x\ln^2(x)}{(1-x)(2-x)(3-x)}dx=-2\int_0^1\frac{\ln^2(x)}{2-x}dx-\frac{1}{2}\int_0^1\frac{\ln^2(x)}{x-1}dx+\frac{3}{2}\int_0^1\frac{\ln^2(x)}{3-x}dx=$$

$$-2\int_0^1\frac{\frac{1}{2}\ln^2(x)}{1-\frac{1}{2}x}dx+\frac{1}{2}\int_0^1\frac{\ln^2(x)}{1-x}dx+\frac{3}{2}\int_0^1\frac{\frac{1}{3}\ln^2(x)}{1-\frac{1}{3}x}dx=-4Li_3\left(\frac{1}{2}\right)+Li_3(1)+3Li_3\left(\frac{1}{3}\right)=$$

$$-\frac{1}{6}\left(21\zeta(3)+4\ln^3(2)+12\zeta(2)\ln\left(\frac{1}{2}\right)\right)+\zeta(3)+3Li_3\left(\frac{1}{3}\right)$$

$$Note\ :$$

$$\therefore\int_0^1\frac{y\ln^n(z)}{1-yz}dz=(-1)^n n!\,Li_{n+1}(y)$$

$$Li_{s+1}(z)=\int_0^z\frac{Li_s(t)}{t}dt$$

Solution 2 by Pratham Prasad-India

$$\Omega=\int_0^1 \frac{x\ln^2(x)}{(1-x)(2-x)(3-x)}dx=$$

$$=\frac{3}{2}\int_0^1\frac{\ln^2(x)}{3-x}dx+\frac{1}{2}\int_0^1\frac{\ln^2(x)}{1-x}dx-2\int_0^1\frac{\ln^2(x)}{2-x}dx=\frac{3}{2}I(3)+\frac{1}{2}I(1)-2I(2)$$

$$Define:$$

$$I(a)=\int_0^1\frac{\ln^2(x)}{a-x}dx=\frac{1}{a}\int_0^1\frac{\ln^2(x)}{1-\frac{x}{a}}dx$$

$$=\frac{1}{a}\int_0^1\sum_{r=0}^{\infty}\left(\frac{x}{a}\right)^r\ln^2(x)\,dx=\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\int_0^1 x^r\ln^2(x)\,dx=$$

$$\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\frac{d^2}{dr^2}\left(\int_0^1 x^r\,dx\right)=\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\frac{d^2}{dr^2}\left(\frac{1}{r+1}\right)=\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\left(\frac{2}{(r+1)^3}\right)=2\sum_{r=1}^{\infty}\frac{\left(\frac{1}{a}\right)^r}{r^3}$$

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$$I(a)=2Li_3\left(\frac{1}{a}\right)$$

$$\psi=3Li_3\left(\frac{1}{3}\right)+Li_3(1)-4Li_2\left(\frac{1}{2}\right)$$

$$\int_0^1\frac{x\ln^2(x)}{(1-x)(2-x)(3-x)}dx=3Li_3\left(\frac{1}{3}\right)+\zeta(3)-4Li_2\left(\frac{1}{2}\right)$$

Solution 3 by Shobhit Jain-India

$$\Delta=\int_0^1\frac{x\ln^2x}{(1-x)(2-x)(3-x)}dx=\int_0^1\ln^2x\left(\frac{\frac{1}{2}}{1-x}-\frac{2}{2-x}+\frac{\frac{3}{2}}{3-x}\right)dx$$

$$=\int_0^1\ln^2x\left(\frac{1}{2}(1-x)^{-1}-\left(1-\frac{x}{2}\right)^{-1}+\frac{1}{2}\left(1-\frac{x}{3}\right)^{-1}\right)dx=\frac{1}{2}F(1)-F\left(\frac{1}{2}\right)+\frac{1}{2}F\left(\frac{1}{3}\right)$$

$$here,\ \ F(a)=\int_0^1(\ln^2x)(1-ax)^{-1}dx=\sum_{n=1}^{\infty}\int_0^1(\ln^2x)a^{n-1}x^{n-1}dx=\Gamma 3\sum_{n=1}^{\infty}\frac{a^{n-1}}{n^3}=\frac{2}{a}Li_3(a)$$

$$\Rightarrow\Delta=\frac{1}{2}F(1)-F\left(\frac{1}{2}\right)+\frac{1}{2}F\left(\frac{1}{3}\right)=Li_3(1)-4Li_3\left(\frac{1}{2}\right)+3Li_3\left(\frac{1}{3}\right)$$

By Landen's Trilogarithmic identity,

$$Li_3(z)+Li_3(1-z)+Li_3\left(1-\frac{1}{z}\right)=\zeta(3)+\frac{1}{6}\ln^3z+\zeta(2)\ln z-\frac{1}{2}(\ln^2z)\ln(1-z)$$

$$\underbrace{\Rightarrow}_{z=1}Li_3(1)=\zeta(3)\underbrace{\Rightarrow}_{z=\frac{1}{2}}2Li_3\left(\frac{1}{2}\right)+Li_3(-1)=\zeta(3)+\frac{1}{3}\ln^32-\zeta(2)\ln2$$

$$\Rightarrow 2Li_3\left(\frac{1}{2}\right)-\eta(3)=\zeta(3)+\frac{1}{3}\ln^32-\frac{\pi^2}{6}\ln2\Rightarrow 2Li_3\left(\frac{1}{2}\right)-\frac{3}{4}\zeta(3)=\zeta(3)+\frac{1}{3}\ln^32-\frac{\pi^2}{6}\ln2$$

$$\Rightarrow Li_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln2+\frac{1}{6}\ln^32\Rightarrow\Delta=\zeta(3)-4\left(\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln2+\frac{1}{6}\ln^32\right)+3Li_3\left(\frac{1}{3}\right)$$

$$\Rightarrow\Delta=3Li_3\left(\frac{1}{3}\right)+\frac{\pi^2}{3}\ln2-\frac{5}{2}\zeta(3)-\frac{2}{3}\ln^32$$