ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^2\ln\left(\sin^2(x)\right)+x^2(1+\sin(x^2))\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\therefore \ln(\sin(x))=-\ln(2)-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k},\qquad for\ \ 0<x<\pi$$

$$\therefore \int x^{n-1}\cos(ax)\,dx=\sum_{j=0}^{n-1}j!\binom{n-1}{j}\frac{x^{n-1-j}}{a^{j+1}}\sin\left(ax+\frac{j\pi}{2}\right)$$

$$For\ \ n,\ \ m,\qquad k\in Z^+;\quad m\in R;\ a,b\ \in R$$

$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^n\ln\left(\sin^k(x)\right)+x^m(a+b\sin(x^m))\right)dx=\underbrace{\int_0^{\frac{\pi}{2}}x^n\ln\left(\sin^k(x)\right)dx}_{A}+$$

$$\underbrace{a\int_0^{\frac{\pi}{2}}x^m dx}_{B}+\underbrace{b\int_0^{\frac{\pi}{2}}x^m(\sin(x^m))dx}_{C}$$

$$A=\int_0^{\frac{\pi}{2}}x^n\ln\left(\sin^k(x)\right)dx=k\int_0^{\frac{\pi}{2}}x^n\ln(\sin(x))\,dx$$

$$=-k\int_0^{\frac{\pi}{2}}x^n\left(\ln(2)+\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}\right)dx=$$

$$-k(\ln(2)\int_0^{\frac{\pi}{2}}x^{n-1}dx+\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}}x^{n-1}\cos(2kx))=-k(\frac{\pi^n\ln(2)}{2^n.n}+$$

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=0}^{\infty}j!\binom{n-1}{j}\frac{x^{n-1-j}}{(2k)^{j+1}}\sin\left(2kx+\frac{j\pi}{2}\right)\Bigg|_0^{\frac{\pi}{2}})=-k(\frac{\pi^{n+1}\ln(2)}{2^{n+1}.(n+1)}-$$

$$\frac{n!}{2^{n+1}}\sin\left(\frac{\pi x}{2}\right)\sum_{k=1}^{\infty}\frac{1}{k^{n+2}}+\sum_{j=0}^{n}j!\binom{n}{j}\frac{\pi^{n-j}}{2^{n+1}}\sum_{k=1}^{\infty}\frac{\sin\left(k\pi+\frac{j\pi}{2}\right)}{k^{j+1}})=-k(\frac{\pi^{n+1}\ln(2)}{2^{n+1}.(n+1)}-$$

$$\sum_{j=0}^{n}j!\binom{n}{j}\frac{\pi^{n-j}}{2^{n+1}}\sin\left(\frac{j\pi}{2}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{j+1}}-\frac{n!}{2^{n+1}}\sin\left(\frac{n\pi}{2}\right)\zeta(n+2))=-(\frac{\pi^{n+1}\ln(2)}{2^{n+1}.(n+1)}-$$

$$\sum_{j=0}^{n}j!\binom{n}{j}\frac{\pi^{n-j}}{2^{n+1}}\sin\left(\frac{j\pi}{2}\right)(1-2^{-j})\zeta(n+1)-\frac{n!}{2^{n+1}}\sin\left(\frac{n\pi}{2}\right)\zeta(n+2))$$

$$B=\int_0^{\frac{\pi}{2}}x^m dx=\frac{x^{m+1}}{m+1}\Bigg|_0^{\frac{\pi}{2}}=\frac{1}{m+1}\left(\frac{\pi}{2}\right)^{m+1}$$

$$C=\int_0^{\frac{\pi}{2}}x^m(\sin(x^m))dx=-\frac{1}{m}\int_0^{\frac{\pi}{2}}xd(\cos(x^m))\overset{I.B.P}{\cong}-\frac{1}{m}\left(x\cos(x^m)|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\cos(x^m)\right)=$$

$$-\frac{1}{m}\left(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-\int_0^\infty cos(x^m)dx+\int_{\frac{\pi}{2}}^\infty cos(x^m)dx\right)=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)cos\frac{\pi}{2m}+\frac{1}{2m}\int_{\frac{\pi}{2}}^\infty\frac{e^{ix^m}+e^{-ix^m}}{(x^m)^{\frac{m-1}{m}}}d(x^m))=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)cos\frac{\pi}{2m}+\frac{1}{2m}\int_1^\infty\frac{e^{i\left(\frac{\pi}{2}\right)^m t}+e^{-i\left(\frac{\pi}{2}\right)^m t}}{\left(\left(\frac{\pi}{2}\right)^n t\right)^{\frac{m-1}{m}}}\left(\frac{\pi}{2}\right)dt)=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)cos\frac{\pi}{2m}+\frac{1}{4m}\left(\int_1^\infty\frac{e^{i\left(\frac{\pi}{2}\right)^m t}}{t^{\frac{m-1}{m}}}dt+\int_1^\infty\frac{e^{-i\left(\frac{\pi}{2}\right)^m t}}{t^{\frac{m-1}{m}}}dt\right))=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)cos\frac{\pi}{2m}+\frac{1}{4m}\left(E_{\frac{m-1}{m}}\left(-i\left(\frac{\pi}{2}\right)^m\right)+E_{\frac{m-1}{m}}\left(i\left(\frac{\pi}{2}\right)^m\right)\right)$$

Note :

$$\therefore\int_0^\infty\cos(x^m)\,dx=\Re\left(\int_0^\infty e^{ix^m}\,dx\right)=\Re\left(\Gamma\left(1+\frac{1}{m}\right)\left(cos\frac{\pi}{2m}+isin\frac{\pi}{2m}\right)\right)$$

$$=\Gamma\left(1+\frac{1}{m}\right)cos\frac{\pi}{2m}$$

$$\therefore E_n(x)=\int_1^\infty\frac{e^{-xt}}{t^n}dt:\ \textit{The exponential integral}$$

$$n=2,\quad k=2,\quad m=2,\ a=1,b=1$$

$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^2\ln\left(sin^2(x)\right)+x^2(1+\sin(x^2))\right)dx$$

$$\Omega=-\frac{\ln(2)\pi^3}{12}+\frac{3\pi}{8}\zeta(3)+\frac{\pi^3}{24}-\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right)+\frac{1}{2}\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{\pi}{2}}\right)$$

Solution 2 by Pratham Prasad-India

$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^2\ln(sin^2(x))+x^2(1+\sin(x^2))\right)$$

$$\int_0^{\frac{\pi}{2}}\left(x^2\ln(sin^2(x))+x^2(1+\sin(x^2))\right)$$

$$=\int_0^{\frac{\pi}{2}}x^2\ln(sin^2(x))\,dx+\int_0^{\frac{\pi}{2}}x^2\,dx+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$2\int_0^{\frac{\pi}{2}}x^2\ln(sinx)\,dx+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$2\int_0^{\frac{\pi}{2}}x^2\left(-ln(2)-\sum_{k=1}^\infty\frac{\cos(2kx)}{k}\right)dx+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$-2\ln(2)\int_0^{\frac{\pi}{2}} x^2\,dx - 2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}} x^2\cos(2x)\,dx + \frac{\pi^3}{24} + \int_0^{\frac{\pi}{2}} x^2\sin(x^2)\,dx =$$

$$-\frac{\ln(2)\,\pi^3}{12} - 2\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\pi(-1)^k}{4k^2}\right)dx + \frac{\pi^3}{24} + \int_0^{\frac{\pi}{2}} x^2\sin(x^2)\,dx =$$

$$-\frac{\ln(2)\,\pi^3}{12} - \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3} + \frac{\pi^3}{24} + \int_0^{\frac{\pi}{2}} x^2\sin(x^2)\,dx =$$

$$-\frac{\ln(2)\,\pi^3}{12} + \frac{3\pi}{8}\zeta(3) + \frac{\pi^3}{24} + \int_0^{\frac{\pi}{2}} x^2\sin(x^2)\,dx$$

Now,

$$\int_0^{\frac{\pi}{2}} x^2\sin(x^2)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} x(2x\sin(x^2))\,dx = \left[-\frac{1}{2}x\cos(x^2)\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(x^2)\,dx =$$

$$= -\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right) + \frac{1}{2}\left[\sqrt{\frac{\pi}{2}}\,C\left(\sqrt{\frac{2}{\pi}}\,x\right)\right]_0^{\frac{\pi}{2}} = -\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right) + \frac{1}{2}\sqrt{\frac{\pi}{2}}\,C\left(\sqrt{\frac{\pi}{2}}\right)$$

$$\psi = \int_0^{\frac{\pi}{2}}\left(x^2\ln(\sin^2(x)) + x^2(1+\sin(x^2))\right)$$

$$\Omega = -\frac{\ln(2)\,\pi^3}{12} + \frac{3\pi}{8}\zeta(3) + \frac{\pi^3}{24} - \frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right) + \frac{1}{2}\sqrt{\frac{\pi}{2}}\,C\left(\sqrt{\frac{\pi}{2}}\right)$$

Solution 3 by Shobhit Jain-India

$$\Omega = \int_0^{\frac{\pi}{2}} x^2\ln(\sin^2 x)\,dx + \int_0^{\frac{\pi}{2}} x^2 dx + \int_0^{\frac{\pi}{2}} x^2\sin(x^2)dx$$

$$= \int_0^{\frac{\pi}{2}} 2x^2\ln(2\sin x)\,dx + (1-2\ln 2)\int_0^{\frac{\pi}{2}} x^2 dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} x\sin(x^2)2x\,dx$$

$$\underbrace{=}_{\theta=2x}\frac{1}{4}\int_0^{\pi}\theta^2\ln\left(2\sin\left(\frac{\theta}{2}\right)\right)d\theta + (1-2\ln 2)\frac{\pi^3}{24} + \frac{1}{2}\int_0^{\frac{\pi}{2}} x\sin(x^2)d(x^2)$$

$$= -\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\int_0^{\pi}\theta^2\cos(n\theta)\,d\theta}_{I_1} + (1-2\ln 2)\frac{\pi^3}{24} - \frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}} x\,d[\cos(x^2)]}_{I_2}$$

$$I_1 = \int_0^{\pi}\theta^2\cos(n\theta)\,d\theta = \left[\frac{\theta^2\sin(n\theta)}{n} + 2\frac{\theta\cos(n\theta)}{n^2} - 2\frac{\sin(n\theta)}{n^3}\right]_0^{\pi} = \frac{2\pi\cos(n\pi)}{n^2} = \frac{2\pi(-1)^n}{n^2}$$

$$I_2=\int_0^{\frac{\pi}{2}} xd[\cos(x^2)]=\frac{\pi}{2}\cos\left(\frac{\pi^2}{4}\right)-\int_0^{\frac{\pi}{2}}\cos(x^2)dx=\frac{\pi}{2}\cos\left(\frac{\pi^2}{4}\right)-\sqrt{\frac{\pi}{2}}C\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \Omega=-\frac{1}{4}\sum_{n=1}^{\infty}\frac{2\pi(-1)^n}{n^3}+(1-2\ln 2)\frac{\pi^3}{24}-\frac{1}{2}\left(\frac{\pi}{2}\cos\left(\frac{\pi^2}{4}\right)-\sqrt{\frac{\pi}{2}}C\left(\frac{\pi}{2}\right)\right)$$

$$=\frac{\pi}{2}\eta(3)+(1-2\ln 2)\frac{\pi^3}{24}-\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right)+\sqrt{\frac{\pi}{8}}C\left(\frac{\pi}{2}\right)$$

$$\Omega=\frac{3\pi}{8}\zeta(3)+(1-2\ln 2)\frac{\pi^3}{24}-\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right)+\sqrt{\frac{\pi}{8}}C\left(\frac{\pi}{2}\right)$$

NOTES:

$$\eta(x)-\text{Dirichlet's Eta Function}$$

$$\zeta(x)-\text{Riemann's Zeta Function}$$

$$\eta(x)=(1-2^{1-x})\zeta(x)$$

$$C(x)-\text{Fresenel's cosine integral function}$$

$$C(x)=\sqrt{\frac{2}{\pi}}\int_0^x\cos(t^2)dt=\frac{1}{2}\int_0^{x^2}J_{-\frac{1}{2}}(t)dt$$

$$J_{-\frac{1}{2}}(t)=\sqrt{\frac{2}{\pi t}}\cos(t)-\text{Bessel's cosine function}$$