ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$-\frac{4}{5i\sqrt{\pi}}\int_0^\infty\int_0^\infty\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{z(x+y)^2}ln\left(1-e^{-(x+y)^2}\right)ln\left(1+e^{-(x+y)^2}\right)}{(x+y)z\sqrt{z}}dzdxdy=\zeta(3)$$

$$here,\ \ Re(\sigma)>0$$

Proposed by Shobhit Jain-India

Solution by Pham Duc Nam-Vietnam

$$\Omega=-\frac{8\sqrt{\pi}}{5}\cdot\frac{1}{2i\pi}\int_0^\infty\int_0^\infty\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{z(x+y)^2}ln\left(1-e^{-(x+y)^2}\right)ln\left(1+e^{-(x+y)^2}\right)}{(x+y)z\sqrt{z}}dzdxdy$$

$$=-\frac{8\sqrt{\pi}}{5}\int_0^\infty\int_0^\infty\frac{ln\left(1-e^{-(x+y)^2}\right)ln\left(1+e^{-(x+y)^2}\right)}{(x+y)}\mathcal{L}^{-1}_{(z)}\left(\frac{1}{z\sqrt{z}}\right)_{t=(x+y)^2}dxdy$$

$$=-\frac{16}{5}\int_0^\infty\int_0^\infty ln\left(1-e^{-(x+y)^2}\right)ln\left(1+e^{-(x+y)^2}\right)dxdy$$

$$=-\frac{16}{5}\sum_{n=1}^\infty\frac{1}{n}\left(H_n-H_{2n}-\frac{1}{2n}\right)\int_0^\infty\int_0^\infty\left(e^{-(x+y)^2}\right)^{2n}dxdy$$

$$=-\frac{4}{5}\sum_{n=1}^\infty\frac{1}{n^2}\left(H_n-H_{2n}-\frac{1}{2n}\right)=\frac{2}{5}\zeta(3)-\frac{4}{5}\left(\sum_{n=1}^\infty\frac{H_n}{n^2}-\sum_{n=1}^\infty\frac{H_{2n}}{n^2}\right)$$

$$Using\ known\ Euler\ sums:\sum_{n=1}^\infty\frac{H_n}{n^2}=2\zeta(3),$$

$$and\sum_{n=1}^\infty\frac{H_{2n}}{n^2}=4\sum_{n=1}^\infty\frac{H_{2n}}{(2n)^2}=2\left(\sum_{n=1}^\infty\frac{H_n}{n^2}+\sum_{n=1}^\infty\frac{(-1)^nH_n}{n^2}\right)=$$

$$=2\left(2\zeta(3)-\frac{5}{8}\zeta(3)\right)=\frac{11}{4}\zeta(3)\Rightarrow\Omega=\zeta(3)\left(\frac{2}{5}-\frac{4}{5}\left(2-\frac{11}{4}\right)\right)=\zeta(3)$$