ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \frac{\sqrt{x}\ln^2(x)}{1+x+x^2}dx$$

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Lemma: If

$$I(a)=\int_0^1 \frac{x^a}{1+x+x^2}dx$$

then:

$$I''(a)=\frac{1}{27}\left(\psi_2\left(\frac{a+1}{3}\right)-\psi_2\left(\frac{a+2}{3}\right)\right)$$

Proof:

$$I(a)=\int_0^1 \frac{x^a(1-x)}{1-x^3}dx$$

$$I(a)=\int_0^1 (x^a-x^{a+1})\sum_{r=0}^{\infty} x^{3r}\,dx$$

$$I(a)=\sum_{r=0}^{\infty}\int_0^1 (x^{a+3r}-x^{a+1+3r})\,dx,\quad I(a)=\sum_{r=0}^{\infty}\left(\frac{1}{3r+a+1}-\frac{1}{3r+a+2}\right)$$

$$I(a)=\frac{1}{3}\left(\psi\left(\frac{a+1}{3}\right)-\psi\left(\frac{a+2}{3}\right)\right),\quad I'(a)=\frac{1}{9}\left(\psi_1\left(\frac{a+1}{3}\right)-\psi_1\left(\frac{a+2}{3}\right)\right)$$

$$I''(a)=\frac{1}{27}\left(\psi_2\left(\frac{a+1}{3}\right)-\psi_2\left(\frac{a+2}{3}\right)\right),\qquad I''\left(\frac{1}{2}\right)=\frac{1}{27}\left(\psi_2\left(\frac{1}{2}\right)-\psi_2\left(\frac{5}{6}\right)\right)$$

$$I''\left(\frac{1}{2}\right)=\int_0^1 \frac{\sqrt{x}\ln^2(x)}{1+x+x^2}dx$$