ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\int_0^\infty \frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

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Solution by Pratham Prasad-India

$$I=\int_0^\infty \frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$Replace\ z\ by -z$$

$$I=\int_0^{-\infty} \frac{e^{-3z}-e^{-z}}{z(e^{-2z}+1)^2}dz, \qquad I=\int_{-\infty}^0 \frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$2I=I+I=\int_{-\infty}^0 \frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz+\int_0^\infty \frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$2I=\int_{-\infty}^\infty \frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$Let, e^z=x$$

$$2I=\int_0^\infty \frac{x^2-1}{ln(x)(1+x^2)^2}dx, \qquad 2I=\int_0^\infty \frac{1}{(1+x^2)^2}\int_0^2 x^a\,da\,dx$$

$$2I=\int_0^\infty\int_0^2 \frac{x^a}{(1+x^2)^2}da\,dx, \qquad 2I=\int_0^2\int_0^\infty \frac{x^a}{(1+x^2)^2}dx\,da$$

$$2I=\frac{1}{2}\int_0^2\int_0^\infty \frac{x^{\frac{a-1}{2}}}{(1+x)^2}dx\,da, \qquad 2I=\frac{1}{2}\int_0^2 B\left(\frac{a+1}{2},2-\frac{a+1}{2}\right)da$$

$$4I=\int_0^2 \frac{\left(\frac{1-a}{2}\right)\pi}{\sin\left(\frac{\pi(a+1)}{2}\right)}da,$$

$$Let\left(\frac{1+a}{2}\right)\pi=u$$

$$4I=-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\pi-u}{\sin(u)}du$$

$$Replace\ \pi-u\ by\ u$$

$$4I=-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{u}{\sin(u)}du$$

$$\boldsymbol{Replace\ u\ by\ u+\pi}$$

$$4I=-\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(u+\pi)}{\sin(u+\pi)}du,\qquad 4I=\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(u+\pi)}{\sin(u)}du$$

$$4I=\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{u}{\sin(u)}du+2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\sin(u)}du$$

$$4I=\frac{4}{\pi}\int_{0}^{\frac{\pi}{2}}\frac{u}{\sin(u)}du+2(0),\qquad 4I=\frac{4}{\pi}\int_{0}^{\frac{\pi}{2}}u\,d\left(\ln\left(tan\left(\frac{u}{2}\right)\right)\right)$$

$$\boldsymbol{Apply\ Integration\ By\ parts}$$

$$4I=\frac{4}{\pi}\int_{0}^{\frac{\pi}{2}}\ln\left(tan\left(\frac{u}{2}\right)\right)du$$

$$\boldsymbol{Replace\,\frac{u}{2}\,by\ u}$$

$$4I=\frac{4}{\pi}\left(2\int_{0}^{\frac{\pi}{4}}\ln(tan(u))\,du\right),\qquad 4I=\frac{4}{\pi}(2G),\qquad I=\frac{2G}{\pi}$$