ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz+\sqrt{1-xyz}}dxdydz$$

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By the Property:

$$\int_0^1\int_0^1\int_0^1 f(xyz)dxdydz = \frac{1}{2}\int_0^1 f(x)\ln^2(x)dx$$

$$\psi = \int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz+\sqrt{1-xyz}}dxdydz = \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{1-x+\sqrt{1-x}}dx$$

$$= \frac{1}{2}\int_0^1 \frac{\ln^2(1-x)}{x+\sqrt{x}}dx = \frac{1}{2}\int_0^1 \frac{(1-\sqrt{x})\ln^2(1-x)}{\sqrt{x}(1-x)}dx =$$

$$= -\frac{1}{6}\int_0^1 \frac{\left(1-\sqrt{x}\right)}{\sqrt{x}}d(\ln^3(1-x)) \overset{IBP}{=} -\frac{1}{12}\int_0^1 \ln^3(1-x)x^{-\frac{3}{2}}dx =$$

$$= -\frac{1}{12}\lim_{\substack{x\to 1\\ y\to -\frac{1}{2}}} \frac{\partial^3}{\partial x^3}B(x,y) = -\frac{1}{12}\left(-24\zeta(3)-16\ln^3(2)+24\zeta(2)\ln(2)\right) =$$

$$= 2\zeta(3)+\frac{4}{3}\ln^3(2)-\frac{\pi^2}{3}\ln(2)\ \psi = 2\zeta(3)+\frac{1}{6}\ln^3(4)-\frac{\pi^2}{6}\ln(4)$$

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz+\sqrt{1-xyz}}dxdydz = 2\zeta(3)+\frac{1}{6}\ln^3(4)-\frac{\pi^2}{6}\ln(4)$$