ROMANIAN MATHEMATICAL MAGAZINE

If $1 \le a \le b$ then:

$$\int_a^b \int_a^b \int_a^b \left(\frac{x}{x+yz} + \frac{y}{y+xz}\right) dxdydz \ge 2(b-a)^2 \log\left(\frac{b+1}{a+1}\right)$$

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$$\frac{x}{x+yz} + \frac{y}{y+zx} = \frac{2xy + z(x^2+y^2)}{xy + z(x^2+y^2) + xyz^2}$$

$$We\ will\ prove:$$

$$\frac{2xy + z(x^2+y^2)}{xy + z(x^2+y^2) + xyz^2} \ge \frac{2}{z+1}$$

$$or, 2xy(z+1) + (z^2+z)(x^2+y^2) \ge 2xy + 2z(x^2+y^2) + 2xyz^2$$

$$or, 2xyz + 2xy + z^2(x^2+y^2) + z(x^2+y^2) \ge 2xy + 2z(x^2+y^2) + 2xyz^2$$

$$2xyz + z^2(x^2+y^2) \ge z(x^2+y^2) + 2xyz^2$$

$$2xy(1-z) - (x^2+y^2)(1-z) \ge 0$$

$$(z-1)(x-y)^2 \ge 0\ true\ (as\ 1 \le a \le z \le b)$$

$$Hence\ we\ can\ \ say:$$

$$\frac{x}{x+yz} + \frac{y}{y+zx} = \frac{2xy + z(x^2+y^2)}{xy + z(x^2+y^2) + xyz^2} \ge \frac{2}{z+1}\ (1)$$

$$\int_a^b \int_a^b \int_a^b \left(\frac{x}{x+yz} + \frac{y}{y+xz}\right) dxdydz \overset{(1)}{\ge} \int_a^b \int_a^b \int_a^b \left(\frac{2}{1+z}\right) dxdydz =$$

$$= 2[\log(z+1)]_a^b [x]_a^b [y]_a^b = 2\ (b-a)^2 \log\left(\frac{b+1}{a+1}\right)$$

Equality holds for $a = b$.