ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} \in \mathbb{R}, a \leq \mathrm{b}$ then :

$$(\mathrm{b}-a)^2 \int_a^{\mathrm{b}} \mathrm{e}^{5x^2} dx \geq \left(\int_a^{\mathrm{b}} \mathrm{e}^{x^2} dx\right)\left(\int_a^{\mathrm{b}} \mathrm{e}^{2x^2} dx\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ $x \leq 0$ and then, since $\left(\mathrm{e}^{2x^2}\right)' = 4x\mathrm{e}^{2x^2} \leq 0$ and $\left(\mathrm{e}^{x^2}\right)' = 2x\mathrm{e}^{x^2} \leq 0 \therefore$ both e^{2x^2} and e^{x^2} are $\downarrow$

$$\therefore \left(\int_a^{\mathrm{b}} \mathrm{e}^{5x^2} dx\right)\left(\int_a^{\mathrm{b}} \mathrm{e}^{x^2} dx\right) \geq \left(\int_a^{\mathrm{b}} \left(\mathrm{e}^{\frac{5x^2}{2}} . \mathrm{e}^{\frac{x^2}{2}}\right) dx\right)^2 =$$

$$(\text{via Integral form of CBS inequality}) = \left(\int_a^{\mathrm{b}} \mathrm{e}^{3x^2} dx\right)^2$$

$$\therefore (\mathrm{b}-a)^2 \left(\int_a^{\mathrm{b}} \mathrm{e}^{5x^2} dx\right)\left(\int_a^{\mathrm{b}} \mathrm{e}^{x^2} dx\right) \geq (\mathrm{b}-a)^2 . \left(\int_a^{\mathrm{b}} \left(\mathrm{e}^{2x^2} . \mathrm{e}^{x^2}\right) dx\right)^2 \geq$$

$$\geq \left(\left(\int_a^{\mathrm{b}} \mathrm{e}^{x^2} dx\right)\left(\int_a^{\mathrm{b}} \mathrm{e}^{2x^2} dx\right)\right)^2 \ (\text{via Integral form of Chebyshev inequality})$$

$$\therefore (\mathrm{b}-a)^2 \int_a^{\mathrm{b}} \mathrm{e}^{5x^2} dx \geq \left(\int_a^{\mathrm{b}} \mathrm{e}^{x^2} dx\right)\left(\int_a^{\mathrm{b}} \mathrm{e}^{2x^2} dx\right)^2$$

$\boxed{\text{Case 2}}$ $x \geq 0$ and then, since $\left(\mathrm{e}^{2x^2}\right)' = 4x\mathrm{e}^{2x^2} \geq 0$ and $\left(\mathrm{e}^{x^2}\right)' = 2x\mathrm{e}^{x^2} \geq 0$ $\therefore$ both e^{2x^2} and e^{x^2} are $\uparrow$ and then, proceeding in a similar manner as in Case 1,

$$\text{we get : } (\mathrm{b}-a)^2 \int_a^{\mathrm{b}} \mathrm{e}^{5x^2} dx \geq \left(\int_a^{\mathrm{b}} \mathrm{e}^{x^2} dx\right)\left(\int_a^{\mathrm{b}} \mathrm{e}^{2x^2} dx\right)^2 \ (\text{QED})$$

Equality holds for $a = b$.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\left(\int_a^b e^{2x^2}dx\right)^2=\left(\int_a^b e^{2x^2}dx\right)\left(\int_a^b e^{2x^2}dx\right)\overset{CEBYSHEV}{\underset{\smile}{\leq}}(b-a)\int_a^b\left(e^{2x^2}\cdot e^{2x^2}\right)dx$$

$$\left(\int_a^b e^{x^2}dx\right)\left(\int_a^b e^{2x^2}dx\right)^2\leq\left(\int_a^b e^{x^2}dx\right)\cdot(b-a)\int_a^b\left(e^{2x^2}\cdot e^{2x^2}\right)dx=$$

$$=(b-a)\left(\int_a^b e^{x^2}dx\right)\left(\int_a^b e^{4x^2}dx\right)\overset{CEBYSHEV}{\underset{\smile}{\leq}}$$

$$\leq(b-a)(b-a)\left(\int_a^b e^{x^2+4x^2}dx\right)=(b-a)^2\left(\int_a^b e^{5x^2}dx\right)$$

Equality holds for $a=b$.