Find:

$$\Omega=\lim_{n\to\infty}\left(\log 2.\log n-\sum_{k=n}^{2n-1}\frac{\log(k+1)}{k+1}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Shobhit Jain-India

$$\Omega=\lim_{n\to\infty}\left(\log 2.\log n-\sum_{k=n}^{2n-1}\frac{\log(k+1)}{k+1}\right)=\lim_{n\to\infty}\left(\log 2.\log n-\sum_{k=n+1}^{2n}\frac{\log(k)}{k}\right)$$

$$=\lim_{n\to\infty}\left(\log 2.\log n+\sum_{k=1}^{n}\frac{\log(k)}{k}-\sum_{k=1}^{2n}\frac{\log(k)}{k}\right)$$

$$Now,\ \ By\ the\ Euler-Maclaurin\ \ Summation\ \ formula$$

$$\sum_{k=1}^{n}f(k)=\int_1^n f(x)dx+\frac{f(n)+f(1)}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(f^{2k-1}(n)-f^{2k-1}(1)\right)$$

$$here,\ \ B_n\ is\ the\ n\_th\ Bernoulli\ number$$

$$Now,\ \ use\ f(x)=\frac{\log(x)}{x}$$

$$\Rightarrow\sum_{k=1}^{n}\frac{\log(k)}{k}=\int_1^n\frac{\log(x)}{x}dx+\frac{1}{2}\left(\frac{\log n}{n}+\frac{\log(1)}{1}\right)+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(f^{2k-1}(n)-f^{2k-1}(1)\right)$$

$$\Rightarrow\sum_{k=1}^{n}\frac{\log(k)}{k}\sim C+\frac{(\log n)^2}{2}+O\left(\frac{\log n}{n}\right)\qquad for\ very\ large\ (n)$$

$$here,\ \ C=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\log(k)}{k}-\frac{(\log n)^2}{2}\right)=finite\ value$$

$$\Rightarrow\Omega=\lim_{n\to\infty}\left(\log 2.\log n+\frac{(\log n)^2}{2}-\frac{(\log(2n))^2}{2}\right)$$

$$=\lim_{n\to\infty}\left(\log 2.\log n+\frac{1}{2}(\log n)^2-\frac{1}{2}[(\log 2)^2+2\log 2.\log n+(\log n)^2]\right)$$

$$= \lim_{n\to\infty}\left(\log 2.\log n + \frac{1}{2}(\log n)^2 - \frac{1}{2}(\log 2)^2 - \log 2.\log n - \frac{1}{2}(\log n)^2\right)$$

$$\Rightarrow \Omega = -\frac{1}{2}(\log 2)^2$$

Solution 2 by Pham Duc Nam-Vietnam

$$\sum_{k=n}^{2n-1}\frac{\log(k+1)}{k+1} = \sum_{k=1}^{n}\frac{\log(n+k)}{n+k} = \frac{1}{n}\sum_{k=1}^{n}\frac{\log\left(n\left(1+\frac{k}{n}\right)\right)}{1+\frac{k}{n}} =$$

$$= \log n\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}} + \frac{1}{n}\sum_{k=1}^{n}\frac{\log\left(1+\frac{k}{n}\right)}{1+\frac{k}{n}}$$

$$\Omega = \lim_{n\to\infty}\left(\log 2.\log n - \sum_{k=n}^{2n-1}\frac{\log(k+1)}{k+1}\right) =$$

$$= \lim_{n\to\infty}\left(\log 2.\log n - \log n\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}} - \frac{1}{n}\sum_{k=1}^{n}\frac{\log\left(1+\frac{k}{n}\right)}{1+\frac{k}{n}}\right) =$$

$$= -\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\log\left(1+\frac{k}{n}\right)}{1+\frac{k}{n}} = -\int_0^1\frac{\log(x+1)}{x+1}dx = -\frac{1}{2}(\log 2)^2$$