ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1}{x-1}-\sum_{n=0}^{\infty}\frac{1}{(n+3)^x}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Shobhit Jain-India

$$\Omega=\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1}{x-1}-\sum_{n=3}^{\infty}\frac{1}{n^x}\right)=\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1}{x-1}+1+\frac{1}{2^x}-\sum_{n=1}^{\infty}\frac{1}{n^x}\right)=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1}{x-1}-\zeta(x)\right)$$

$$Now,\ \ \eta(x)=\left(1-2^{1-x}\right)\zeta(x)$$

$$\Rightarrow\Omega=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1}{x-1}-\frac{\eta(x)}{1-2^{1-x}}\right)=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1}{x-1}-\frac{\eta(x)}{1-e^{-(x-1)ln2}}\right)$$

$$Now\ use,\ \ e^{-(x-1)ln2}=1-\frac{(x-1)ln2}{1!}+\frac{(x-1)^2ln^2 2}{2!}-O\left((x-1)^3\right)\qquad for\ x\to 1^+$$

$$\Rightarrow\Omega=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1}{x-1}-\frac{\eta(x)}{\frac{(x-1)ln2}{1!}-\frac{(x-1)^2ln^2 2}{2!}+O((x-1)^3)}\right)$$

$$=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{1-\dfrac{\eta(x)}{ln2-\frac{(x-1)}{2}ln^2 2+O((x-1)^2)}}{x-1}\right)$$

$$=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{ln2-\eta(x)-\frac{(x-1)}{2}ln^2 2+O\left((x-1)^2\right)}{(x-1)\left(ln2-\frac{(x-1)}{2}ln^2 2+O((x-1)^2)\right)}\right)$$

$$=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{\frac{ln2-\eta(x)}{x-1}-\frac{1}{2}ln^2 2+O(x-1)}{ln2-\frac{(x-1)}{2}ln^2 2+O((x-1)^2)}\right)=\frac{3}{2}+\lim_{\substack{x\to 1\\ x>1}}\left(\frac{\frac{ln2-\eta(x)}{x-1}-\frac{1}{2}ln^2 2}{ln2}\right)$$

$$=\frac{3}{2}-\frac{1}{2}ln2-\frac{1}{(ln2)}\lim_{\substack{x\to 1\\ x>1}}\left(\frac{\eta(x)-ln2}{x-1}\right)=\frac{3}{2}-\frac{1}{2}ln2-\frac{\eta'(1)}{ln2}$$

$$Now,\ \eta(x)=\frac{1}{1^x}-\frac{1}{2^x}+\frac{1}{3^x}-\frac{1}{4^x}+\ \dots\dots\quad \Rightarrow \eta(1)=ln2$$

$$and,\ \eta'(1)=\lim_{N\to\infty}\left(-\frac{ln1}{1}+\frac{ln2}{2}-\frac{ln3}{3}+\frac{ln4}{4}-\ \dots\dots-\frac{ln(2N-1)}{2N-1}+\frac{ln(2N)}{2N}\right)$$

$$=\lim_{N\to\infty}\left\{2\left(\frac{ln2}{2}+\frac{ln4}{4}+\ \dots\dots+\frac{ln(2N)}{2N}\right)-\left(\frac{ln1}{1}+\frac{ln2}{2}+\frac{ln3}{3}+\frac{ln4}{4}-\ \dots\dots+\frac{ln(2N-1)}{2N-1}+\frac{ln(2N)}{2N}\right)\right\}$$

$$=\lim_{N\to\infty}\left(2\sum_{k=1}^{N}\frac{ln(2k)}{2k}-\sum_{k=1}^{2N}\frac{ln(k)}{k}\right)=\lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{ln(2k)}{k}-\sum_{k=1}^{2N}\frac{ln(k)}{k}\right)$$

$$=\lim_{N\to\infty}\left(ln2\sum_{k=1}^{N}\frac{1}{k}+\sum_{k=1}^{N}\frac{ln(k)}{k}-\sum_{k=1}^{2N}\frac{ln(k)}{k}\right)=\lim_{N\to\infty}\left((ln2)H_N+\sum_{k=1}^{N}\frac{log(k)}{k}-\sum_{k=1}^{2N}\frac{ln(k)}{k}\right)$$

$$here,\ H_N\ is\ the\ N\_th\ \ harmonic\ number$$

$$Now,\ By\ the\ Euler-Maclaurin\ \ Summation\ \ formula$$

$$\sum_{k=1}^{N}\frac{ln(k)}{k}\sim C+\frac{(lnN)^2}{2}+O\left(\frac{lnN}{N}\right)\qquad for\ very\ large\ (N)$$

$$and,\ H_N\sim\gamma\ +lnN+O\left(\frac{1}{N}\right)\qquad for\ very\ large\ (N)$$

$$\Rightarrow\quad \eta'(1)=\lim_{N\to\infty}\left((ln2)(\gamma\ +lnN)+\frac{(lnN)^2}{2}-\frac{\left(ln(2N)\right)^2}{2}\right)$$

$$=\lim_{N\to\infty}\left((ln2)(\gamma\ +lnN)+\frac{1}{2}(lnN)^2-\frac{1}{2}\left[(ln\,2)^2+2ln2.lnN+(lnN)^2\right]\right)$$

$$\Rightarrow\quad \eta'(1)=\gamma ln2-\frac{1}{2}(ln\,2)^2$$

$$\Rightarrow\Omega=\frac{3}{2}-\frac{1}{2}ln2-\frac{1}{ln2}\left(\gamma ln2-\frac{1}{2}(ln\,2)^2\right)$$

$$\Rightarrow\Omega=\frac{3}{2}-\gamma\qquad here,\ \gamma\ is\ Euler-Mascheroni's\ constant$$