MATH

ACCENT

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PROBLEMS
ALGEBRA

A.001. If $x, y, z > 0$ then:

$$2\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)\sqrt{xy+yz+zx} \geq 3\sqrt{3(x+y)(y+z)(z+x)}$$

A.002. If $z = x + iy \in \mathbb{C}$ then:

$$\frac{|\sinh z|^4}{e^{2x}} + \frac{|\cosh z|^4}{e^{-2x}} \geq \frac{1}{2}$$

A.003. Solve for real numbers:

$$\begin{vmatrix} \cos x & \cos x & \cos 2x \\ \cos 3x & \cos 5x & \cos 4x \\ \sin 3x & \sin 5x & \sin 4x \end{vmatrix} = 0$$

A.004. If $1 \leq a \leq b$ then find:

$$\Omega(a,b) = \int_a^b \tan^{-1}\left(\frac{3x}{1-2x^2}\right) dx$$

A.005. If $z_1, z_2, z_3 \in \mathbb{C}, z_1 + z_2 + z_3 = 3 + 4i$ then:

$$2\sum_{cyc} |z_1| \leq 5 + \sum_{cyc} \left(|z_1 - z_2| + |3 + 4i - 2z_3|\right)$$

A.006. If $x_1, x_2, \ldots, x_n > 0, n \in \mathbb{N}^*$, then:

$$1 + \sum_{i=1}^{n} \frac{1}{\sqrt[3]{1+x_i}} \leq n + \frac{1}{\sqrt[3]{1+x_1+x_2+\cdots+x_n}}$$

A.007. $x_i, y_i > 0, i \in \overline{0,7},\ 512\sum_{i=0}^{7}(x_i + y_i) = 1225.$

Prove that:

$$\sum_{i=0}^{7}\frac{\sin^6\left(\frac{i\pi}{8}\right)}{x_i}+\sum_{i=0}^{7}\frac{\cos^6\left(\frac{i\pi}{8}\right)}{y_i}\geq 1$$

A.008. If $a, b, c > 0, (a+b)(b+c)(c+a) = 2\sqrt{2}$ then:

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & a^2+b^2 & a^2+c^2 & 1 \\ b^2 & a^2+b^2 & 0 & b^2+c^2 & 1 \\ c^2 & a^2+c^2 & b^2+c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}\leq 1$$

A.009. If $a, b, c > 0, abc = \frac{1}{\sqrt[4]{3}}$, then:

$$\frac{a^{120}+b^{120}+c^{120}}{a^{40}+b^{40}+c^{40}}\geq\frac{1}{(a^4+b^4+c^4)^{10}}$$

A.010. Solve for real numbers:

$$\begin{cases} x+y+z=xyz \\ \frac{x(3-x^2)}{1-3x^2}+\frac{y(3-y^2)}{1-3y^2}+\frac{z(3-z^2)}{1-3z^2}=0 \end{cases}$$

A.011. If $a, b \geq 0$ then:

$$4\sqrt{3ab}\cdot e^{\sqrt{2}}+2\sqrt{2}(a+b)e^{\sqrt{3}}\geq\left(\sqrt{a}+\sqrt{b}\right)^2\left(\sqrt{3}e^{\sqrt{2}}+\sqrt{2}e^{\sqrt{3}}\right)$$

A.012. Solve for real numbers:

$$\begin{cases} 2x^2+3y^2+z^2=7 \\ x^2+y^2+z^2=\sqrt{2}z(x+y) \end{cases}$$

A.013. If $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$ then:

$$|\sin z|^2+|\sinh z|^2+|\cos z|^2+|\cosh z|^2\geq\sinh(2x)+\cosh(2y)$$

A.014. If $x, y, z > 0, xy + yz + zx = 3xyz$ then:

$$\left(\sum_{cyc} \frac{1}{x+y}\right)\left(\sum_{cyc} \frac{1}{2x+y+z}\right) \leq \frac{9}{8}$$

A.015. For $x, y, z \in \mathbb{R}$,

$$A = \begin{pmatrix} \sin^2 x - \cos^2 y & \cos^2 z & \cos^2 z \\ \cos^2 x & \sin^2 y - \cos^2 z & \cos^2 x \\ \cos^2 y & \cos^2 y & \sin^2 z - \cos^2 x \end{pmatrix}$$

$$B = \begin{pmatrix} \cos^2 x - \sin^2 y & \sin^2 z & \sin^2 z \\ \sin^2 x & \cos^2 y - \sin^2 z & \sin^2 x \\ \sin^2 y & \sin^2 y & \cos^2 z - \sin^2 x \end{pmatrix}$$

Prove that: $\det(AB) \geq 0$.

A.016. Solve for real numbers:

$$\frac{1}{5}\binom{4}{4}x^5 + \frac{1}{4}\binom{4}{3}x^4 + \frac{1}{3}\binom{4}{2}x^3 + \frac{1}{2}\binom{4}{1}x^2 + x + \frac{1}{5} = 0$$

A.017. If $a, b, c, d > 0, ad > bc$ then:

$$\frac{(5a+3b)(7a+5b)(9a+7b)}{(5c+3d)(7c+5d)(9c+7d)} > \left(\frac{a+b}{c+d}\right)^3$$

A.018. If $a, b, c > 0, ab + bc + ca = 2$ then:

$$\sum_{cyc} \frac{a(b^4+c^4)}{b^3+b^2c+bc^2+c^3} \geq 1$$

A.019. If $a, b, c > 0$ then:

$$abc + a^2 + b^2 + c^2 + 4 \geq 2(ab + bc + ca)$$

A.020. If $x, y, z \geq 0$ then:

$$2\sum_{cyc} x^2(x^2+y^2) \geq \sum_{cyc} x(y^3+z^3) + xyz(x+y+z)$$

A.021. If $a, b, c > 0, a+b+c=3$ then

$$\frac{3(1+b)(1+c)+3(1+c)(1+a)+3(1+a)(1+b)}{a^7(1+b)(1+c)+b^7(1+c)(1+a)+c^7(1+a)(1+b)} \leq \frac{1}{a^6}+\frac{1}{b^6}+\frac{1}{c^6}$$

A.022. Prove without any software:

$$\Omega = \log_2(\log_2 e) + \log(\log \pi) + \log_\pi(\log_\pi 2)$$

$$A. \Omega < 0 \quad B. \Omega = 0 \quad C. \Omega > 0$$

A.023. If $a, b, x, y > 0$ then :

$$\frac{\left(\sqrt{ab}+\sqrt{xy}\right)\left(\frac{a+b}{2}+\frac{x+y}{2}\right)\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{\frac{x^2+y^2}{2}}\right)}{\left(\sqrt{ab}+\frac{x+y}{2}\right)\left(\frac{a+b}{2}+\sqrt{\frac{x^2+y^2}{2}}\right)\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{xy}\right)} \leq 1$$

A.024. If $x, y > 0$ then :

$$\frac{x}{x^2-x+1}+\frac{y}{y^2-y+1}+\frac{xy}{x^2y^2-xy+1} \leq \frac{x^2}{x^2-x+1}+\frac{y^2}{y^2-y+1}+\frac{1}{x^2y^2-xy+1}$$

A.025. If $x, y, z > 0, xyz = 1$ then:

$$(x-y)^4+(y-z)^4+(z-x)^4 \geq 2\left(3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)^2$$

A.026. If $a, b, c, d > 0, a+b+c+d=1$ then:

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}} \geq 4\cdot\sqrt[3]{\frac{4}{5}}$$

A.027. Solve for real numbers:

$$10^{\frac{x^3+6x^2+x}{2(x+1)}}+10^{2x\sqrt{x}}=10^{x\sqrt{x}}\left(10^{\frac{x^2+x}{2}}+10^{\frac{2x^2}{x+1}}\right)$$

A.028. If $a,b,x,y>0$ then:

$$4ab\cdot\exp\left(\frac{ax+by}{a+b}\right)\le 2ab\cdot\sqrt{e^{x+y}}+a^2e^x+b^2e^y$$

A.029. In ΔABC the following relationship holds:

$$\frac{2c+a}{2a+b}+\frac{2a+b}{2c+a}\le\frac{2(2a+b)(2b+a)(2c+b)}{(a+3b-c)(b+3c-a)(c+3a-b)}$$

A.030. If $x,y\in\mathbb{R}$; $a,b>0$ then:

$$\frac{x^2\sqrt{2}}{\sqrt{a^2+b^2}}+\frac{2y^2}{a+b}\ge\frac{2(x+y)^2}{a+b+\sqrt{2(a^2+b^2)}}+\frac{a+b}{4}\left(\frac{x\sqrt{2}}{\sqrt{a^2+b^2}}-\frac{2y}{a+b}\right)^2$$

A.031. Solve for real numbers:

$$5\left(\sqrt[5]{1-x}+\sqrt[5]{1+x}\right)=2+4\left(\sqrt[4]{1-x}+\sqrt[4]{1+x}\right)$$

A.032. If $x,y,z,t>0$ then:

$$\frac{75x+36(y+z)}{y+z+t}+\frac{75y+36(z+t)}{z+t+x}+\frac{75z+36(t+x)}{t+x+y}+\frac{75t+36(x+y)}{x+y+z}\ge 196$$

A.033. If $x,y,z>0$ then:

$$\frac{(x+y)^4}{x^4+x^2y^2+y^4}+\frac{(y+z)^4}{y^4+y^2z^2+z^4}+\frac{(z+x)^4}{z^4+z^2x^2+x^4}\le 16$$

A.034. Prove without any software:

$$\frac{2}{\sqrt[4]{\varphi}}+\sqrt{\varphi}<1+\frac{\sqrt{\varphi}+\varphi+\varphi\sqrt{\varphi}}{2},\varphi-\text{golden ratio.}$$

A.035. If $B, C \in M_2(\mathbb{R}), \det A > 0, \det B > 0, \det C > 0$,

$$\det(ABC) = 8 \text{ then: } \det(A+B+C) + \det(-A+B+C) + \det(A-B+C) + \det(A+B-C) \geq 24$$

A.036. If $a, b, c > 0$ then:

$$(3ab)^c \cdot (3bc)^a \cdot (3ca)^b \leq (a^2+b^2+c^2)^{a+b+c}$$

A.037. Solve for real numbers:

$$32(2x^{12}+x^8+x^6+x^4)+19 = 64x^2(x^8+1)$$

A.038. If $a, b, x, y > 0$ then:

$$32ab(ax+by)^4 \leq (a+b)^4(8a^2x^4+ab(x+y)^4+8b^2y^4)$$

A.039. If $1 \leq a, b, c, d, e, f \leq 2$, then:

$$(a^2+b^2+c^2)(d^2+e^2+f^2) \leq (ad+be+cf)^2 + 27$$

When equality holds?

A.040. Find all $x, y \geq 0$ such that :

$$x+y+z = \sqrt[4]{4xyz}\left(\sqrt[4]{x}+\sqrt[4]{y}\right)$$

A.041. Prove without any software:

$$\frac{\varphi\sqrt{\varphi}+1}{\sqrt{\varphi}(1+\sqrt{\varphi})} + \frac{\varphi}{1+\varphi} + \frac{2}{\sqrt{\varphi}(1+\sqrt{\varphi}+\varphi)} < 2,$$

where φ —golden ratio.

A.042. If $0 < a \leq b < \pi$ then:

$$\frac{\sin\left(\sqrt{ab}\right)}{\sin\left(\frac{a+b}{2}\right)} \geq \frac{32a^2b^2\sqrt{ab}}{(a+b)^5}$$

A.043. If $a, b, c > 0$ then:

$$e^{\left(\frac{a}{a+b+c}\right)^2} + e^{\left(\frac{b}{a+b+c}\right)^2} + e^{\left(\frac{c}{a+b+c}\right)^2} \geq 3 \cdot \sqrt[9]{e}$$

A.044. Prove without any software:

$$(\log^2 2 + \log^2 5 + \log^2 3)^2 > 3 \log 30 \cdot \log\frac{15}{2} \cdot \log\frac{6}{5} \cdot \log\frac{10}{3}$$

A.045.

$$m_h = \frac{2ab}{a+b}, \qquad m_g = \sqrt{ab}, \qquad m_a = \frac{a+b}{2}, a, b \geq 0$$

Prove that:

$$(m_h + m_g + m_a)\sqrt{\frac{a^2+b^2}{2}} \leq m_h m_g + m_g m_a + m_a m_h + \frac{a^2+b^2}{2}$$

A.046. If $a, b > e$ then:

$$(a+b)^{2\sqrt{ab}} \leq 4^{\sqrt{ab}} \cdot \left(\sqrt{ab}\right)^{a+b}$$

A.047. If $x, y, z \in [0, \infty)$ then:

$$2^x + 2^y + 2^z + 2^{x+y+z} \geq 4^{\sqrt{xy}} + 4^{\sqrt{yz}} + 4^{\sqrt{zx}} + 1$$

A.048. If $0 < a \leq 1 \leq b$ $t$hen:

$$(a+b-1)^{a+b-1} + 1 \leq a^a + b^b$$

A.049. If $a, b, c, d > 0$; $ab = cd$; $a < b, c < d$; $x, y \in [a, b]$ and $y, t \in [c, d]$, then:

$$ab(x+y+z+t)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right) \leq (a+b+c+d)^2$$

A.050. Prove without any software:

$$\frac{12\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi} < \sqrt[3]{\varphi(1+4\varphi)(4+\varphi)} < 3\varphi, \varphi - Golden\ ratio$$

A.051. If $x, y, z > 0, x + y + z = 1$ then:

$$\left(x+\frac{1}{y}\right)^5 + \left(y+\frac{1}{z}\right)^5 + \left(z+\frac{1}{x}\right)^5 \geq \frac{100.000}{81}$$

A.052. If $0 \leq x, y, z < 1$ then:

$$\frac{x^2}{\sqrt{1-x^2}} + \frac{y^2}{\sqrt{1-y^2}} + \frac{z^2}{\sqrt{1-z^2}} \geq \frac{xy}{\sqrt{1-xy}} + \frac{yz}{\sqrt{1-yz}} + \frac{zx}{\sqrt{1-zx}}$$

A.053. If $a, b, c > 0$ then

$$\sum_{cyc} \frac{2a^3 + 3a^2 + b}{(2a+1)(a+b+1)} \geq \sum_{cyc} \frac{2ab^2 + b^2 + 2ab + a}{(2a+1)(a+b+1)}$$

A.054. If $a, b, c > 0, p, q, r > 1, pq + qr + rp = pqr$ then :

$$abcpqr \leq qra^p + rpb^q + pqc^r$$

A.055. Solve for real numbers:

$$\begin{cases} x, y > 0 \\ \dfrac{(x+y)^{10}}{(2x^2+y^2)(4x^3+y^3)(16x^5+y^5)} = \dfrac{2817}{128} \\ 2^x + \log_6 y = 9 \end{cases}$$

A.056. If $n \in \mathbb{N}, n \geq 1$ then:

$$n + \frac{1}{n} \sum_{1 \leq i < j \leq n} \frac{\left(\sqrt{i} - \sqrt{j}\right)^2}{\sqrt{ij}} \geq \sqrt[2n]{n!} \cdot \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

A.057. If $x, y, z, t > 0$ then

$$xyzt(x+y+z+t)^2 \leq 2(xy+zt)(xz+yt)(xt+yz)$$

A.058. Solve for real numbers:

$$\frac{(x-1)^2}{2}+\frac{(y-2)^2}{4}+\frac{(z-3)^2}{6}+3=|x-1|+|y-2|+|z-3|$$

A.059. Solve for real numbers:

$$\begin{cases} 4x^4 = y(x^6+x^4+x^2+1) \\ 5y^5 = z(y^8+y^6+y^4+y^2+1) \\ 6z^6 = x(z^{10}+z^8+z^4+z^2+1) \end{cases}$$

A.060. Solve for real numbers:

$$\begin{cases} x+y+3=3xy \\ \dfrac{x^2-y^2}{xy-1}+\dfrac{x^2-9}{3x-1}+\dfrac{y^2-9}{3y-1}=0 \end{cases}$$

A.061. If $x, y, z \in \mathbb{R}, \sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}=3\sqrt{2}$ then:

$$x^2+y^2+z^2+6 \geq 3(x+y+z)$$

A.062. Solve for real numbers:

$$\begin{cases} x+\dfrac{9}{[x]}=\dfrac{6}{1+x-[x]} \\ z+2^z+\log_2 z = x+y\,; \quad [*]-\text{GIF.} \\ y+\dfrac{16}{[y]}=\dfrac{8}{1+y-[y]} \end{cases}$$

A.063. If $x, y, z \geq 0, x^2+y^2+z^2=3$ then:

$$\frac{x^3+1}{\sqrt{x^2-x+1}}+\frac{y^3+1}{\sqrt{y^2-y+1}}+\frac{z^3+1}{\sqrt{z^2-z+1}} \geq 6$$

A.064. If $x, y > 0$ then:

$$4(x+1)^{x+1}\cdot(y+1)^{y+1}\cdot(x+y)^{x+y} \leq x^x\cdot y^y\cdot(x+y+2)^{x+y+2}$$

A.065. If $a,b \geq e\sqrt{e}$ then:

$$\left(\left(\frac{a+2b}{3}\right)^{2a+b}\cdot\left(\frac{2a+b}{3}\right)\right)^{3ab} \leq \left(a^b\cdot b^a\right)^{(a+2b)(2a+b)}$$

A.066. If $x_1, x_2, \dots, x_n > 0, n \in \mathbb{N}^*$, then:

$$1+\sum_{i=1}^{n}\frac{1}{\sqrt[3]{1+x_i}} \leq n+\frac{1}{\sqrt[3]{1+x_1+x_2+\dots+x_n}}$$

A.067. If $a,b,c > 0, a+b+c=3$ then:

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}} \leq 3\sqrt{1+\sqrt{2}}$$

A.068. If $x,y,z > 0, x^2+y^2+z^2=3$, then:

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2}+xy+yz+zx \geq 6$$

A.069. Solve for real numbers:

$$(x+1)(x-1)\begin{vmatrix}\overline{x111} & \overline{1x11} & \overline{11x1} & \overline{111x}\\ \overline{1x11} & \overline{11x1} & \overline{111x} & \overline{x111}\\ \overline{11x1} & \overline{111x} & \overline{x111} & \overline{1x11}\\ \overline{111x} & \overline{x111} & \overline{1x11} & \overline{11x1}\end{vmatrix} + (y+3)(y-1)\begin{vmatrix}\overline{y111} & \overline{1y11} & \overline{11y1} & \overline{111y}\\ \overline{1y11} & \overline{11y1} & \overline{111y} & \overline{y111}\\ \overline{11y1} & \overline{111y} & \overline{y111} & \overline{1y11}\\ \overline{111y} & \overline{y111} & \overline{1y11} & \overline{11y1}\end{vmatrix} = 0$$

A.070. If $a,b,c > 0$, then:

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}+\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}+\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \geq 3e$$

A.071. If $a,b,c>0$, then:

$$\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}+\frac{(b^2+a^2)(c^2+a^2)}{(bc+a^2)(ba+ca)}+\frac{(c^2+b^2)(a^2+b^2)}{(ca+b^2)(cb+ab)}\geq 3$$

A.072. If $0\leq a\leq 1\leq b\ or\ 0\leq b\leq 1\leq a$ then :

$$e^{\left(\frac{a+b}{2}-\sqrt{ab}+\frac{2ab}{a+b}\right)^2}+e^{ab}\leq\sqrt[4]{e^{(a+b)^2}}+e^{\frac{4a^2b^2}{(a+b)^2}}$$

A.073. Solve for real numbers:

$$\frac{(307-x)\sqrt[5]{x-63}-(x-63)\sqrt[5]{307-x}}{\sqrt[5]{307-x}-\sqrt[5]{x-63}}=120$$

A.074. Solve for real numbers:

$$\begin{cases} x^3+y^3=516-\sqrt[3]{x}-\sqrt[3]{y}\\ y^3+z^3=20200-\sqrt[3]{y}-\sqrt[3]{z}\\ z^3+x^3=19688-\sqrt[3]{z}-\sqrt[3]{x}\end{cases}$$

A.075. If $a,b,c>0, abc=1$ then:

$$\frac{a^2+b^2}{a^7+b^7}+\frac{b^2+c^2}{b^7+c^7}+\frac{c^2+a^2}{c^7+a^7}\leq a^5+b^5+c^5$$

A.076. If $a,b,c,d>0, a+b+c+d=4$ then:

$$3^{12}\cdot a^a\cdot b^b\cdot c^c\cdot d^d\geq(4-a)^{4-a}\cdot(4-b)^{4-b}\cdot(4-c)^{4-c}\cdot(4-d)^{4-d}$$

A.077. Solve for real numbers:

$$\frac{256^{x^2}}{256^y}+\frac{256^{y^2}}{256^z}+\frac{256^{z^2}}{256^t}+\frac{256^{t^2}}{256^x}=1$$

A.078. If $x,y\geq 0, n\in\mathbb{N}$ then:

$$(x^{n+1}+y^{n+1})^{n-1}\cdot(x+y)^{n+1}\leq 2^{n-1}\cdot(x^n+y^n)^{n+1}$$

A.079. If $a,b,c\geq 0$ then:

$$a(e^{2b}+e^{-2c})+b(e^{2c}+e^{-2a})+c(e^{2a}+e^{-2b})\geq 2(a+b+c)$$

A.080. Solve for complex numbers:

$$4x^4+5x^2+4=x\left(\tan\frac{\pi}{24}\tan\frac{11\pi}{24}x^2+\tan\frac{5\pi}{24}\tan\frac{7\pi}{24}\right)$$

A.081. If $x,y,z>0$ then:

$$\frac{4x^2}{x+y}+\frac{8y^2}{y+z}+\frac{4z^2}{z+x}\geq 2x+5y+z$$

A.082. If $x,y,z>0$ then:

$$(1+xyz)\sqrt{x^2+y^2+z^2}\left(\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}\right)\geq 3\sqrt{3}xyz$$

A.083. If $0<a\leq b<\frac{\pi}{2}$ then:

$$a^2+\cos a+\log\left(\frac{\cos a}{\cos b}\right)\geq b^2+\cos b$$

A.084. If $x>0$ then:

$$e^{x+e^{-x}}+e^{-x+e^x}\geq 2\cosh x\cdot e^{\operatorname{sech} x}$$

A.085. If $a,b,c>0$ then:

$$\sqrt{a^2+5ab+7b^2}+\sqrt{b^2+5bc+7c^2}+\sqrt{c^2+5ca+7a^2}\geq\sqrt{13}(a+b+c)$$

A.086. Find all $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x)+f(yz)+9\leq f(xy)+5f(xz);\forall x,y,z\in\mathbb{R}$$

A.087. Solve for real numbers :

$$\begin{cases} x,\ \ y,\ \ z>1 \\ \log_x z+\log_y x+\log_z y=\log_{xy}(yz)+\log_{yz}(zx)+\log_{zx}(xy) \\ x+y+z=6 \end{cases}$$

A.088. If $n \in \mathbb{N}, n \geq 2$ then:

$$\sum_{k=2}^{n} \left(H_k + \sqrt[k]{k^{k-1}}\right) < \frac{(n-1)(n+4)}{2}$$

A.089. If $1 < a \leq b$ then:

$$\log_{ab}\left(1 + \sqrt{ab}\right)^2 + \log_{\frac{a+b}{2}} 2 \geq \log_{\frac{a+b}{2}}(a + b + 2)$$

A.090. If $a, b, c \in \mathbb{C}$ then:

$$\frac{|a+1|}{|b+1| + |b+c| + |c|} + \frac{|b+1|}{|c+1| + |c+a| + |a|} + \frac{|c+1|}{|a+1| + |a+b| + |b|} \geq$$

$$\geq 3 + |a| + |b| + |c|$$

A.091. Let be $A = \{a, b, c | a, b, c \in \mathbb{R}^*\}$ and $B = \{u, v, w, t | u, v, w, t \in \mathbb{R}^*\}$ such that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 3$$

$$\frac{u^2}{v^2} + \frac{v^2}{w^2} + \frac{w^2}{t^2} + \frac{t^2}{u^2} = \frac{v^2}{u^2} + \frac{w^2}{v^2} + \frac{t^2}{w^2} + \frac{u^2}{t^2} = 4$$

Find:

$$\Omega = \sum_{a,y \in A} \left|\frac{x}{y}\right| \cdot \prod_{x,y \in A} \left|\frac{x}{y}\right| + \sum_{a,y \in B} \left|\frac{x}{y}\right| \cdot \prod_{x,y \in B} \left|\frac{x}{y}\right|$$

A.092. If $a, b > 1$ then:

$$\left(a^x \cdot e^{a^{2x}} + b^x \cdot e^{b^{2x}}\right) \cdot e^{a^x \cdot b^x} \geq (a^x + b^x) \cdot e^{a^{2x} + b^{2x}}; \forall x \in \mathbb{R}$$

A.093. If $x, y, z \in \mathbb{R}, 32(x^5 + y^5 + z^5) = 3$, then:

$$\sum_{cyc} (2x^6 + x^4 + x^3 + x^2) + \frac{51}{32} \geq 2(x + y + z)$$

A.094. Solve for real numbers:

$$\log\left(\frac{yz}{x}\right)\left(\log^2 x-\log\left(\frac{zx}{y}\right)\log\left(\frac{xy}{z}\right)\right)=\log^2 y\cdot\log\left(\frac{y}{zx}\right)+\log^2 z\cdot\log\left(\frac{z}{xy}\right)$$

A.095. If $a,b>0$ then:

$$\frac{\sqrt{2^{a+b}}+\sqrt{3^{a+b}}+\sqrt{5^{a+b}}}{2^{\sqrt{ab}}+3^{\sqrt{ab}}+5^{\sqrt{ab}}}\geq\frac{\frac{1}{2^{\sqrt{ab}}}+\frac{1}{3^{\sqrt{ab}}}+\frac{1}{5^{\sqrt{ab}}}}{\frac{1}{\sqrt{2^{a+b}}}+\frac{1}{\sqrt{3^{a+b}}}+\frac{1}{\sqrt{5^{a+b}}}}$$

A.096. If $a,b>0$ then:

$$\left(a^{\sqrt{ab}}+b^{\sqrt{ab}}\right)\sqrt{(a+b)^{a+b}}\geq\left(\sqrt{a^{a+b}}+\sqrt{b^{a+b}}\right)(a+b)^{\sqrt{ab}}$$

A.097. If $a,b,c,x,y>0$ then:

$$\frac{a^{\sqrt{xy}}+b^{\sqrt{xy}}+c^{\sqrt{xy}}}{\frac{1}{\sqrt{a^{x+y}}}+\frac{1}{\sqrt{b^{x+y}}}+\frac{1}{\sqrt{c^{x+y}}}}\leq\frac{\sqrt{a^{x+y}}+\sqrt{b^{x+y}}+\sqrt{c^{x+y}}}{\frac{1}{a^{\sqrt{xy}}}+\frac{1}{b^{\sqrt{xy}}}+\frac{1}{c^{\sqrt{xy}}}}$$

A.098. If $x,y,z>0$ then:

$$8x^xy^yz^z2^{x+y+z}\geq(x+1)^{x+1}(y+1)^{y+1}(z+1)^{z+1}$$

A.099. Find $x,y,z>0$ such that:

$$\frac{6(x+y)}{\sqrt{xy}}+\frac{6(x+y+z)}{\sqrt[3]{xyz}}+\frac{6\sqrt{xy}}{x+y}+\frac{6\sqrt[3]{xyz}}{x+y+z}=35$$

A.100. Solve for complex numbers:

$$x^7+2x^6+5x^5+3x^4-16x^3-11x^2-20x-12=0$$

A.101. If $a,x>0,b,c,y,z\in\mathbb{R}$ then:

$$\frac{(a+x)^2-(b+y)^2-(c+z)^2}{a+x}\geq\frac{a^2-b^2-c^2}{a}+\frac{x^2-y^2-z^2}{x}$$

A.102. $M=\left\{\begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} \middle| a>0, b\in\mathbb{C}\right\}$. If $X,Y,Z\in M$ then:

$$Tr(XY)+Tr(YZ)+Tr(ZX)\le Tr(X^2)+Tr(Y^2)+Tr(Z^2)$$

A.103. $M=\left\{\begin{pmatrix} a & b+ic \\ b-ic & a \end{pmatrix} \middle| a>0, b,c\in\mathbb{R}, i^2=-1\right\}$. If $X,Y,Z\in M$ then:

$$Tr(XY)+Tr(YZ)+Tr(ZX)$$
$$\le\sqrt{Tr(X^2)Tr(Y^2)}+\sqrt{Tr(Y^2)Tr(Z^2)}+\sqrt{Tr(Z^2)Tr(X^2)}$$

A.104. $A=\begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}, B=\begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix}, i^2=-1$. Prove that:

$$(x,y)\left(\frac{1}{2}(A+B)-2(A^{-1}+B^{-1})^{-1}\right)\begin{pmatrix} x \\ y \end{pmatrix}\ge 0, x,y\in\mathbb{R}$$

A.105. If $a, \mathrm{b}>0$, *then* :

$$\left(\sqrt{\frac{a^2+\mathrm{b}^2}{2}}\right)^5+\left(\frac{a^2+\mathrm{b}^2}{a+\mathrm{b}}-\sqrt{\frac{a^2+\mathrm{b}^2}{2}}+\sqrt{a\mathrm{b}}\right)^5\le\left(\frac{a^2+\mathrm{b}^2}{a+\mathrm{b}}\right)^5+\left(\sqrt{a\mathrm{b}}\right)^5$$

A.106. If $a,b,c>0$ then:

$$abc(a+b)^2(b+c)^2(c+a)^2\le 64\left(\frac{a+b+c}{3}\right)^9$$

A.107. Solve for real numbers:

$$\begin{cases} x+y+z=xyz \\ \dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2}=0 \end{cases}$$

A.108. Solve for real numbers:

$$\begin{cases} x+y=4 \\ 2|x-y|=(|x|+|y|)\cdot\left|\dfrac{x}{|x|}-\dfrac{y}{|y|}\right| \end{cases}$$

A.109. If $a,b,c,d,e,f>0$ then:

$$3abcdef(abc+def-5)+(ab+bc+ca)(de+ef+fd)\ge 0$$

A.110. If $x \geq 0$ then:

$$\frac{e^x}{\sqrt[3]{1+e^{-x}}}+\frac{e^{-x}}{\sqrt[3]{1+e^x}} \geq \frac{2}{\sqrt[3]{1+\operatorname{sech} x}}$$

A.111. If $a, b, c > 0, a+b+c=3$ then:

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}+\frac{ba^3}{(a^4+1)(a^2+a+1)}+\frac{cb^3}{(b^4+1)(b^2+b+1)} \leq \frac{1}{2}$$

A.112. Solve for real numbers:

$$\frac{x^2}{5}+\frac{y^2}{6}=\frac{(x+y)^2}{11}+\frac{5}{2}\left(\frac{x}{5}-\frac{y}{6}\right)^2$$

A.113. If $x, y, a, b > 0$ then:

$$\frac{2x^2}{\sqrt{ab}}+\frac{4y^2}{a+b} \geq \frac{4(x+y)^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}+\sqrt{ab}\left(\frac{x}{\sqrt{ab}}-\frac{2y}{a+b}\right)^2$$

A.114. Solve for real numbers:

$$\begin{vmatrix} 1 & 3 & x & x \\ 5 & 9 & x & x \\ x & x & 1 & 3 \\ x & x & 5 & 9 \end{vmatrix} = 0$$

A.115. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ \dfrac{x^2}{4y^2}+\dfrac{y^2}{6z^2}+\dfrac{19}{12}=\sqrt{\dfrac{x}{y}}+\sqrt[3]{\dfrac{y}{z}} \\ 2x^4+18z+54=y^3+39y^2 \end{cases}$$

A.116. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ x+y+z=2\sqrt[3]{xyz}+\dfrac{3xyz}{xy+yz+zx} \\ x^5+560x^2+x=56y^4+10z^3+56 \end{cases}$$

A.117. If $x, y, z > 0$ then:

$$x^8 + y^8 + z^8 + 15 \geq x^3 + y^3 + z^3 + 5\sqrt{3(xy + yz + zx)}$$

A.118. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ x^3 + y^3 + z^3 + 3\left(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}\right) = 12 \\ xyz = 1 \end{cases}$$

A.119. Solve for real numbers:

$$\log_x e \cdot (\log x)^{-1} + \log_{\frac{e}{x}} e \cdot \left(\log\left(\frac{e}{x}\right)\right)^{-1} = 8$$

A.120. If $a, b, c > 0, a + b + c = 3$ then:

$$\sum_{cyc} \sqrt{(a + 3b)^2 + (a + 3b)(3a + b) + (3a + b)^2} \geq 12\sqrt{3}$$

A.121. Solve for real numbers:

$$x^{32} + x^{16} + y^2 = 2\sqrt{2}x^{12}y$$

A.122. If $a, b, c, d > 0; ab = cd; a < b, c < d; x, z \in [a, b]$ and $y, t \in [c, d]$, then:

$$ab(x + y + z + t)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right) \leq (a + b + c + d)^2$$

A.123.

$$m_h = \frac{2ab}{a + b}, m_g = \sqrt{ab}, m_a = \frac{a + b}{2}, a, b > 0$$

Prove that:

$$2m_h + \sqrt{\frac{1}{2}\left(\left(m_a - m_g\right)^2 + \left(m_g - m_h\right)^2 + (m_h - m_a)^2\right)} \leq m_g + m_a$$

A.124.

$$a,b,c>0,a+b+c=3,\Omega(a)=\prod_{k=1}^{n}\left(1+\frac{k}{an^2}\right),n\in\mathbb{N},n\geq 1$$

Prove that:

$$a\Omega(a)+b\Omega(b)+c\Omega(c)\leq e^{\frac{1}{2a}}+e^{\frac{1}{2b}}+e^{\frac{1}{2c}}$$

A.125. If $a,b,c>0,a+b+c=3$ then:

$$\sum_{cyc}\left(\sqrt{a(a+2b)}+\sqrt{b(b+2a)}\right)\leq 6\sqrt{3}$$

A.126. Solve for real numbers:

$$\sqrt{x^7\cdot 2^{7x-7}}(x+2^{x-1})=x^8+2^{8x-8}$$

A.127. If $a,b,c>0$ then:

$$\frac{5}{a}+\frac{8}{b}+\frac{9}{c}\geq\frac{8}{a+b}+\frac{24}{b+c}+\frac{12}{c+a}$$

PROBLEMS GEOMETRY

G.001. If $a_1,a_2,b_1,b_2\in[0,1],x\in\mathbb{R}$ then:

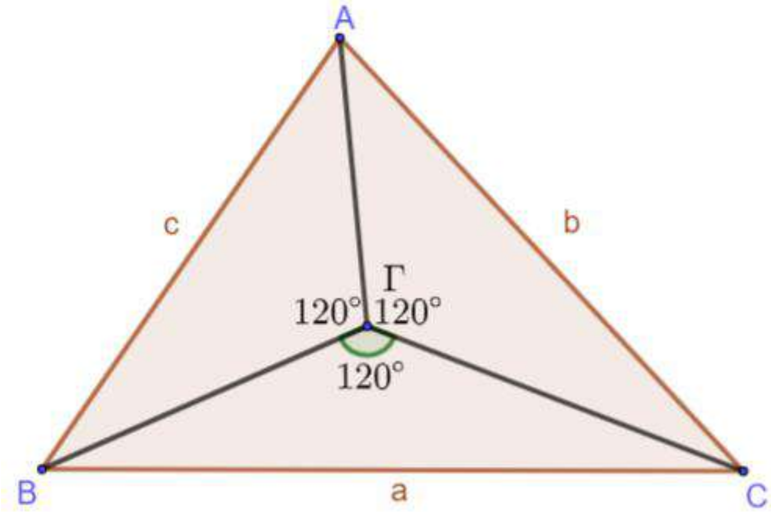
$$|a_1b_1\sin^2 x+a_2b_2\cos^2 x-(a_1\sin^2 x+a_2\cos^2 x)(b_1\sin^2 x+b_2\cos^2 x)|\leq\frac{1}{4}$$

G.002. If $a,b,c,d>0,a+b+c+d=1$ then:

$$\sum_{cyc}a^2cd\cot^{-1}(b)\geq 4abcd\cot^{-1}\left(\frac{1}{4}\right)$$

G.003. In $\Delta ABC, \Gamma$ –Toricelli's point, holds:

$$(\Gamma A^3+\Gamma B^3+\Gamma C^3)\left(\frac{1}{\Gamma A}+\frac{1}{\Gamma B}+\frac{1}{\Gamma C}\right)\geq 36r^2$$



G.004. In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\leq\frac{3\sqrt{3}RR'}{2(r+r')}$$

G.005. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}\leq 3\sqrt{3}R$$

G.006. Solve for real numbers:

$$\sqrt{\sin^4 x+\cos^2 x}+\sqrt{\sin^2 x+\cos^4 x}+\sqrt{1+\sin^2 x\cos^2 x}=3$$

G.007. In $\Delta ABC, \omega$ –Brocard's angle, holds :

$$4F\left(\cot\omega-\sqrt{3}\right)\geq\sum_{cyc}(a-b)^2+16Rr\sum_{cyc}\left(\cos^2\frac{A}{2}-\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

G.008. If $a, b, c \in \mathbb{R}; \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$, then solve for real numbers:

$$\sin x \cdot \sin y \cdot \sin z = \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}$$

G.009. Solve for real numbers:

$$\begin{cases} \sin^3 x + \cos^3 y + z^3 + 3z = 3z^2 + 2 \\ \sin^2 x + \cos^2 y + z^2 = 2z + 2 \\ \sin x + \cos y + z = 2 \end{cases}$$

G.010. If $n \in \mathbb{N}$ then in ΔABC holds :

$$\sqrt{n(n+1)}.\cos\left(A - \frac{\pi}{7}\right) + \sqrt{n(n+2)}.\cos\left(B - \frac{\pi}{7}\right) + \sqrt{(n+1)(n+2)}.\cos\left(C - \frac{\pi}{7}\right) < 3(n+1)\cos\frac{\pi}{21}$$

G.011. Solve for real numbers:

$$\frac{1}{1 + \tan^4 x} + \frac{1}{10} = \frac{2}{1 + 3\tan^2 x}$$

G.012. If in $\Delta ABC, a = m_a$ then:

$$a^2 \tan A > 2\sqrt{3}r^2$$

G.013. In ΔABC the following relationship holds:

$$\sum_{cyc} (b + c) \cdot \csc\frac{A}{2} = 24\sqrt{3}r \Leftrightarrow a = b = c$$

G.014. Find $x, y, z \in \left(0, \frac{\pi}{2}\right)$ such that:

$$\sqrt[3]{(\sec^4 x + \csc^4 x)(\sec^6 y + \csc^6 y)(\sec^8 z + \csc^8 z)} = 16$$

G.015. In ΔABC the following relationship holds:

$$\frac{2a^3 + 3b^3 + 5c^3}{2a^2 + 3b^2 + 5c^2} + \frac{3a^3 + 5b^3 + 2c^3}{3a^2 + 5b^2 + 2c^2} + \frac{5a^3 + 2b^3 + 3c^3}{5a^2 + 2b^2 + 3c^2} \geq 6\sqrt{3}r$$

G.016. If $0 \le a \le b$ then:

$$ab(4+(a+b)^2)\tan^{-1}(\sqrt{ab}) \le (1+ab)(a+b)^2\tan^{-1}\left(\frac{a+b}{2}\right)$$

G.017. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} am_a\right)^3\left(\sum_{cyc} a{m_a}^2\right)^{-\frac{1}{2}}\left(\sum_{cyc} a{m_a}^3\right)^{-\frac{1}{3}}\left(\sum_{cyc} a{m_a}^6\right)^{-\frac{1}{6}} \le 4s^2$$

G.018. If $\alpha,\beta,\gamma,\delta \in \left(0,\frac{\pi}{2}\right), 16\sin\alpha\cdot\sin\beta\cdot\sin\gamma\cdot\sin\delta = 9$ then :

$$8\sqrt{3}+\sum_{cyc}\frac{1}{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2} \le 16$$

G.019. In ΔABC the following relationship holds:

$$(a^2+b^2+c^2)(a^2\cos^2 A+b^2\cos^2 B+c^2\cos^2 C) \ge 12F^2$$

G.020. Solve for real numbers:

$$\sin^2 x\,(2\sin^2 x\cdot\sin^2 2x+4\cos^4 x+1) = \cos^2 x\,(2\cos^2 x\cdot\sin^2 2x+4\sin^4 x+1)$$

G.021. If $0 < a \le b \le c < \frac{\pi}{2}$ then:

$$\frac{5}{\tan a}+\frac{3}{\tan b}+\frac{1}{\tan c} \ge \frac{27}{\tan a+\tan b+\tan c}$$

G.022. If $a,b,c,d,e > 0$ then:

$$c+\frac{ad}{b\cot^2\frac{\pi}{20}}+\frac{bd}{c\cot^2\frac{3\pi}{20}}+\frac{d^2}{e\cot^2\frac{7\pi}{20}}+\frac{ed}{a\cot^2\frac{9\pi}{20}} \ge 5d$$

G.023. In ΔABC the following relationship holds:

$$\frac{w_a^2+w_ah_a+h_a^2}{w_a+h_a} \le \frac{3}{2}w_a$$

G.024. Solve for real numbers:

$$\sin x+\cos x+\sec x\cdot\csc x=2+\sqrt{2}$$

G.025. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_ah_a+h_a^2}{m_a+h_a}>\frac{3}{2}h_a$$

G.026. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s-a}{bc}\sqrt{\frac{2a(s-a)}{b(s-b)+c(s-c)}}\geq\sum_{cyc}\frac{s-a}{bc}$$

G.027. In $\Delta ABC, K$ –Lemoine's point, E –Exeter's point. Prove that :

$$a^2\cdot KA\cdot EA+b^2\cdot KB\cdot EB+c^2\cdot KC\cdot EC\geq\frac{3a^2b^2c^2}{a^2+b^2+c^2}$$

G.028. If in $\Delta ABC,\Delta A'B'C', 2s=2s'=3$.Prove that :

$$\prod_{cyc}\left(1+\frac{1}{\sqrt[3]{a}}\right)\left(1+\frac{1}{\sqrt[3]{a'}}\right)\geq\frac{128\sqrt{3}}{3(R+R')}$$

G.029. Solve for real numbers:

$$\sin\left(4x-\frac{\pi}{3}\right)\sin\left(6x-\frac{\pi}{3}\right)\sin\left(\frac{8\pi}{3}-10x\right)+\frac{\sqrt{3}}{8}=0$$

G.030. In ΔABC the following relationship holds:

$$\sin A+\frac{\sin B}{\sqrt{\varphi}}+\frac{\sin C}{\varphi}<\frac{1}{\varphi}+\frac{1+\sqrt{\varphi}+\varphi}{2\sqrt{\varphi}},$$ φ –golden ratio.

G.031. In any Δ ABC, the following relationship holds:

$$\frac{2\pi}{\sqrt{3}(\csc A+\csc B+\csc C)}\leq\sqrt[3]{\mu(A)\mu(B)\mu(C)}$$

G.032. If $x, y, z, u, v, w \in (-1,1)$ then:

$$(\tan^2 x + \tan^2 y + \tan^2 z)(\tan^2 u + \tan^2 v + \tan^2 w) \geq$$
$$\geq (\tan(xu) + \tan(yv) + \tan(zw))^2$$

G.033. In ΔABC holds :

$$\sum \frac{a^3}{bc + a^2} = s \iff 2s = 3\sqrt{3}R.$$

G.034. In ΔABC the following relationship holds:

$$(s-a)^5 + (s-b)^5 + (s-c)^5 + 10Rrs(a^2 + b^2 + c^2) = s^5$$

G.035. Solve for real numbers:

$$\frac{1}{1 + |\sin x|} + \frac{1}{1 + |\cos y|} = 1 + \frac{1}{1 + |\sin x + \cos y|}$$

G.036. In ΔABC the following relationship holds:

$$a \cdot \sqrt{\sin^{-1}\left(\frac{4}{5}\right)} + b \cdot \sqrt{\sin^{-1}\left(\frac{5}{13}\right)} + c \cdot \sqrt{\sin^{-1}\left(\frac{16}{64}\right)} < \frac{3R\sqrt{2\pi}}{2}$$

G.037. In acute ΔABC the following relationship holds:

$$(\tan A)^{3\tan A} \cdot (\tan B)^{3\tan B} \cdot (\tan C)^{3\tan C}$$
$$\geq (\tan A \cdot \tan B \cdot \tan C)^{\tan A \cdot \tan B \cdot \tan C}$$

G.038. If $\Delta A'B'C'$ is the pedal triangle of I $-$incenter
in ΔABC and $x, y, z > 0$ then :

$$xIA + yIB + zIC \geq 4\left(\frac{yzIA'}{y+z} + \frac{zxIB'}{z+x} + \frac{xyIC'}{x+y}\right)$$

G.039. In ΔABC the following relationship holds:

$$\frac{2bc}{(b+c)^2} + \frac{1}{2} \geq \frac{16Rr}{s^2 + r^2 + 2Rr}$$

G.040. In $\Delta ABC, \Delta A'B'C'$ the following relationship holds;

$$R^2R'F' \geq 8F(r')^3$$

G.041. $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\frac{m_a^3 \cdot (a')^2}{a^2} + \frac{m_b^3 \cdot (b')^2}{b^2} + \frac{m_c^3 \cdot (c')^2}{c^2} \geq \frac{32s^6(r')^2}{243R^5}$$

G.042. In ΔABC the following relationship holds:

$$\sqrt{(a+b)(a+c)bc} + \sqrt{(b+c)(b+a)ca} + \sqrt{(c+a)(c+b)ab} \leq 3s^2 - r^2 - 4Rr$$

G.043. Find all $x, y, z \in \left[0, \frac{\pi}{2}\right]$ such that:

$$(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z) + \cos^2 x \cdot \cos^2 y \cdot \cos^2 z = 8$$

G.044. If $0 \leq x, y, z \leq \frac{\pi}{4}$ then :

$$2\cos^2 x \,.\cos^2 y \,.\cos^2 z \leq 1 + \cos 2x \,.\cos 2y \,.\cos 2z \leq 8\cos^2 x \,.\cos^2 y \,.\cos^2 z$$

G.045. Solve for real numbers:

$$\cos x \cdot \sqrt{\tan x} = \sin^3 x + \cos^3 x$$

G.046. If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{2}{1+\tan x} + \frac{2}{1+\tan y} + \frac{1+\cot x \cdot \tan y}{1+\cot x} + \frac{1+\cot y \cdot \tan x}{1+\cot y} \geq 4$$

G.047. If $x, y, z \geq 0$ then:

$$4\left(\sin z + \sqrt[3]{x^2y} \cdot \cos z\right)\left(\sin z + \sqrt[3]{xy^2} \cdot \cos z\right) \leq 4 + (x+y)^2$$

G.048. In acute ΔABC the following relationship holds:

$$\prod_{cyc}(1+\tan A\cot B)\geq 2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}$$

G.049. In ΔABC the following relationship holds:

$$(m_am_b+m_bm_c+m_cm_a)(m_a+m_b+m_c)^2\geq 81F^2$$

G.050. K –Lemoine's point in ΔABC. Prove that:

$$\frac{m_a}{AK\cdot\sin A}+\frac{m_b}{BK\cdot\sin B}+\frac{m_c}{CK\cdot\sin C}\geq 3\sqrt{3}$$

G.051. If $x,y,z>0, xy+yz+zx=3$ then in ΔABC the following relationship holds:

$$\frac{\tan^4 A\cdot\tan^4 B}{x^3y^3}+\frac{\tan^4 B\cdot\tan^4 C}{y^3z^3}+\frac{\tan^4 C\cdot\tan^4 A}{z^3x^3}\geq 243$$

G.052. If $x,y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{1}{\frac{1}{\sin x+\sin y}+\frac{1}{\cos x+\cos y}}\leq\frac{\sqrt{2}}{2}$$

G.053. In ΔABC the following relationship holds:

$$\left(\frac{3\sin^2 A}{\cos A}+\frac{2\sin^2 B}{\cos B}+\frac{\sin^2 C}{\cos C}\right)\cos\left(\frac{\pi+2A+B}{6}\right)\geq 6\sin^2\left(\frac{\pi+2A+B}{6}\right)$$

G.054. In ΔABC let $R_A -$ be the radii of circle tangent simultaneous to AB, AC and external tangent to circumcircle of ΔABC. Prove that:

$$\frac{R_AR_B}{r_ar_b}+\frac{R_BR_C}{r_br_c}+\frac{R_CR_A}{r_cr_a}\geq\frac{64r^2}{3R^2}$$

G.055. Solve for real numbers: $\sin 5x+10\sin x=5\sin 3x$

G.056. In $\Delta ABC, r_a=3, r_b=4, r_c=5$. Find F.

G.057. In ΔABC the following relationship holds:

$$a^2+b^2+c^2 \geq 4F\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}.\sum_{cyc}\csc C.\sqrt{\frac{a+b}{c}}$$

G.058. Solve for real numbers:

$$\begin{cases} 2\sin x+2\sin y=1 \\ 2\cos x+2\cos y=\sqrt{3} \end{cases}$$

G.059. If $x,y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\tan^3 x+\frac{1}{\tan^3 x}+\tan^3 y+\frac{1}{\tan^3 y}\geq \tan x+\frac{1}{\tan x}+\tan y+\frac{1}{\tan y}$$

G.060. If $0<a\leq b<\frac{\pi}{2}$ then:

$$\sin^2 b-\sin^2 a+\cos(\sin b)-\cos(\sin a)\geq \frac{1}{12}(\sin^4 b-\sin^4 a)$$

G.061. $\vec{a}=\sqrt{x}\cdot\vec{i}+\sqrt{x-1}\cdot\vec{j}, \vec{b}=2\vec{i}+3\vec{j}, x\geq 1$. Solve for real numbers: $18+\tan\left(\sphericalangle\left(\vec{a},\vec{b}\right)\right)=13\sqrt{2}$

G.062. If $x,y,z,t\geq 0$ then :

$$x^2\cot\frac{\pi}{19}+y^2\cot\frac{2\pi}{19}+z^2\cot\frac{4\pi}{19}+t^2\tan\frac{8\pi}{19}\geq\left(x+y\sqrt{2}+2z+2t\sqrt{2}\right)^2\tan\frac{\pi}{19}$$

G.063. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}a\sqrt{bc}\right)\left(\sum_{cyc}\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\right)\leq 54\sqrt{3}\cdot R^3$$

G.064. In ΔABC holds: $4ab\cos^2\frac{C}{2}=3c^2\Rightarrow 5c^2\geq 4\sqrt{3}F+2ab$

G.065. If $0 \le x \le y \le z < 1$, then

$$(y-z)\tan^{-1}x + (z-x)\tan^{-1}y + (z-x)\tan^{-1}z < \frac{\pi}{2} - \log 2$$

G.066. Solve for real numbers:

$$\sin 2x \cdot \sin 3x \cdot \sin 11x + \sin^2 5x \cdot \sin 6x = \sin x \cdot \sin 3x \cdot \sin 8x$$

G.067. Solve for real numbers:

$$1 + 2\cos x \cdot \cos 2x \cdot \cos 5x = \cos^2 x + \cos^2 2x + \cos^2 5x$$

PROBLEMS ANALYSIS

AN.001. If $0 < a \le b < \frac{\pi}{2}$ then:

$$3\int_a^b \sin x \cdot \sinh x \, dx \le b^3 - a^3$$

AN.002. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{\tan\left(\frac{\pi - 2x}{4}\right)(1 + \sin x)}{\sin x} dx \le \frac{\sin b - \sin a}{\sin a}$$

AN.003. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\cos^{2n}\frac{\pi}{7} - 2^{1-2n} \sum_{0\le i<j\le n} \binom{n}{i}\binom{n}{j} \cos\frac{2(j-i)\pi}{7}}$$

AN.004. If $\sec\frac{\pi}{7} < a \le b$ then find:

$$\Omega(a,b) = \int_a^b \left(\tan^{-1}\left(\frac{x}{\sec\frac{\pi}{7} - x\tan\frac{\pi}{7}}\right) - \tan^{-1}\left(x\sec\frac{\pi}{7} - \tan\frac{\pi}{7}\right)\right) dx$$

AN.005. If $0 < a \leq b$ then:

$$\int_a^b \frac{1}{\sqrt{2[x]+1}} \cdot \prod_{k=1}^{[x]} \sin\left(\frac{k\pi}{2n+1}\right) dx \geq \frac{1}{2^a} - \frac{1}{2^b}, [*] - GIF.$$

AN.006. If $0 < a \leq b < 1$ then :

$$\int_a^b \int_a^b \int_a^b \left(\frac{1-xyz}{1+xyz}\right)^3 dxdydz \geq \left(\int_a^b \frac{1-x^3}{1+x^3} dx\right)^3$$

AN.007. Prove without any software:

$$\int_0^1 \int_0^1 \sqrt{1-\left(\frac{x+y}{2}\right)^2}\, dxdy > \frac{\pi}{4}$$

AN.008. Find:

$$\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x \cdot \cos 4x \cdot \cos 8x}{\sin^5\left(\frac{\pi}{4}-x\right) + \cos^5\left(\frac{\pi}{4}-x\right)} dx$$

AN.009. Find:

$$\Omega = \lim_{x\to\infty}\left(log^2 x \cdot \int_0^1 \frac{t \cdot x^t}{(1+x^t)^2} dt\right)$$

AN.010. Find:

$$\Omega = \int \frac{sinx + 4cosx}{5(e^{-x} + sinx) + 3cosx} dx$$

AN.011. $If\ 0 < a \leq b\ then$:

$$\int_a^b \int_a^b \int_a^b (x+y+z)\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right) dxdydz \geq 3(b-a)^3(a^2+ab+b^2)$$

AN.012. Find:

$$\Omega = \int_0^{\frac{\pi}{6}} \frac{\sin x \cdot \sin\left(x + \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{2\pi}{3}\right)}{\sin 3x + \cos 3x} dx$$

AN.013. If $1 < a \le b$ then find:

$$\Omega(a,b) = \int_a^b \int_a^b \int_a^b \tan^{-1}\left(\frac{x + y + z - xyz}{1 - xy - yz - zx}\right) dxdydz$$

AN.014. If $0 < a \le b$ then:

$$\int_a^b \frac{x^{19}}{\sqrt{1 + x^{30}}} dx \ge \log \sqrt[10]{\frac{2 + b^{20}}{2 + a^{20}}}$$

AN.015. If $0 < a \le b < \frac{\pi}{2}$ then :

$$\int_a^b \frac{\left(e^x - e^{\sin x}\right)(\tan x - x)}{e^{\tan x} - e^x} dx \le (b - a)\left(1 + \sin\frac{a + b}{2}\right)$$

AN.016. If $\frac{1}{\sqrt{31}} < a \le b$ then find:

$$\Omega(a,b) = \int_a^b \tan^{-1}\left(\frac{30x^3 - 10x}{31x^2 - 1}\right) dx$$

AN.017. If $0 < a \le b < \frac{\pi}{6}$ then find:

$$\Omega(a,b) = \int_a^b \frac{(1 + \tan^2 x)^2}{\cos^2 x - 3\sin^2 x} dx$$

AN.018. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{(1 - \sin^5 x)^7}{(1 - \sin^3 x)^8} dx \ge b - a$$

AN.019. If $0 < a \le b$ then:

$$\left(\int_a^b e^{2x^2}\,dx\right)\left(\int_a^b e^{-x^2}\,dx\right) \ge (b-a)\int_a^b e^{x^2}\,dx$$

AN.020. If $0 < a \le b,\ \ f:[a,b] \to (0,\infty), f$ −continuous, then :

$$3(b-a)^2\int_a^b f(x)dx + 3\left(\int_a^b \sqrt[3]{f(x)}dx\right)^3 \ge 6(b-a)\left(\int_a^b \sqrt{f(x)}dx\right)^2.$$

AN.021.

$$B(x,y) = \int_0^1 t^x(1-t)^y\,dt, x, y > 0$$

Prove that: $B(x,y)\cdot B(y,z)\cdot B(z,x) \ge B(x,x)\cdot B(y,y)\cdot B(z,z)$

AN.022. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\cot b - \cot a + \int_a^b \csc^2 x\cdot e^{\sin^2 x}\,dx \le (e-1)(b-a)$$

AN.023. If $0 < a \le b < \frac{\pi}{8}$ then find:

$$\Omega(a,b) = \int_a^b\int_a^b \frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x\cdot\tan y\cdot\tan\left(\frac{\pi}{4}-x-y\right)}\,dx\,dy$$

AN.024. If $0 < a \le b < 1$ then:

$$2\int_a^b e^{x^3}\,dx + (b-a)^2 \le 2(b-a) + e^{b^2} - e^{a^2}$$

AN.025. If $5 < a \le b$ then find:

$$\Omega(a,b) = \int_a^b \tan^{-1}\left(\frac{4x-4x^3}{x^4-6x^2+1}\right)dx$$

AN.026. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n\cdot 27^n}\sum_{i=1}^{n}\sum_{j=1}^{n}3^{i+j}\binom{3n-i-j}{n}\binom{2n-i-j}{n-i}\right)$$

AN.027. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n^2}\left(1+2\sum_{k=1}^{n}\frac{1}{2k+5}\right)^n$$

AN.028. If $0 < a \leq b < \frac{\pi}{2}$ then find:

$$\Omega(a,b) = \int_a^b \frac{3+\cos 4x}{1-\cos 4x}\,dx$$

AN.029. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n(H_n-1)}\left(\log(n!) - \sum_{k=2}^{n}\frac{\Gamma'(k)}{\Gamma(k)}\right)$$

AN.030. Find:

$$\Omega = \lim_{n\to\infty}\int_0^1 \log\left(\frac{1+\sin^n x}{1+x+x^n}\right)dx$$

AN.031. If $a \geq 0$ then:

$$\int_0^a (2x-a)\log(1+x+x^2)\,dx \geq 0$$

AN.032. If $0 < a \leq b < 1$ then:

$$\int_a^b \frac{\sin x\cdot\tan^{-1}(x^2)}{x\cdot\tan^{-1}x}\,dx \geq \cos a - \cos b$$

AN.033. If $0 < a \leq b$ find a closed form:

$$\Omega(a,b) = \int_a^b \left(\frac{x}{1 + \frac{x^2}{3 + \frac{x^2}{5 + \frac{x^2}{7 + \cdots}}}} \right) dx$$

AN.034. If $0 < a \leq b < 1$ then:

$$\exp\left(2\sqrt{2}\int_a^b x^x\,dx\right) \geq \frac{(a-\sqrt{2})(b+\sqrt{2})}{(b-\sqrt{2})(a+\sqrt{2})}$$

AN.035. Find:

$$\Omega = \lim_{n\to\infty}(n-1)!\sum_{k=0}^{n}\frac{1}{(k+1)^k(n-k+1)^{n-k}}$$

AN.036.

$$\Omega(n) = \int_0^\infty \frac{x^{n-1}(x-n)\log x}{e^x}\,dx\,, n \geq 1$$

Find:

$$\Omega = \sum_{n=1}^{\infty}\frac{1}{\Omega(n)}$$

AN.037.

$$\Omega_n(x) = \int \frac{dx}{x(1+x^n)}, n \in \mathbb{N}^*, \Omega_n(1) = \log 2$$

Find:

$$\Omega(x) = \lim_{n\to\infty}\left(n\Omega_n(x)\right), x > 0$$

AN.038. Find:

$$\Omega = \lim_{n\to\infty} \frac{n(1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + n\cdot 1)}{1\cdot n^2 + 2\cdot(n-1)^2 + 3\cdot(n-2)^2 + \cdots + n\cdot 1^2}$$

AN.039. If $1 < a \le b$ then:

$$\exp\left(2\int_a^b \frac{dx}{x\sqrt{\log x}}\right) \le \frac{(1+\log b)\log b^b}{(1+\log a)\log a^a}$$

AN.040. $0 < a \le b, f:[\mathrm{a},\mathrm{b}] \to (0,\infty), \mathrm{f} -$ "n" times differentiable. Prove that:

$$\mathrm{n}\int_{\mathrm{a}}^{\mathrm{b}} \sqrt[\mathrm{n}]{\frac{\mathrm{f}^{(\mathrm{n})}(\mathrm{x})}{\mathrm{f}(\mathrm{x})}}\,\mathrm{dx} \le \sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{\mathrm{f}^{(\mathrm{k})}(\mathrm{b})}{\mathrm{f}^{(\mathrm{k})}(\mathrm{a})}, \mathrm{f}^{(0)}(\mathrm{x}) = \mathrm{f}(\mathrm{x}), \mathrm{n} \in \mathbb{N}$$

AN.041.

$$\Omega_1(n) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left|(i-j)\left(\frac{1}{2n-i+1} - \frac{1}{2n-j+1}\right)\right|$$

$$\Omega_2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\frac{1}{2n-i+1} - \frac{1}{2n-j+1}\right|$$

Find: $\Omega = \lim_{n\to\infty} \frac{\Omega_1(n)}{\Omega_2(n)}$

AN.042. In ΔABC:

$$\omega = tan^{-1}\left(\sqrt{\frac{rr_a}{r_br_c}}\right) + tan^{-1}\left(\sqrt{\frac{rr_b}{r_cr_a}}\right) + tan^{-1}\left(\sqrt{\frac{rr_c}{r_ar_b}}\right).$$ Find:

$$\Omega = \int_0^{\omega} \frac{3sin^2x + cosx + 2}{sinx + cosx + 7}\,dx$$

AN.043. Solve for real numbers:

$$\int_1^x \frac{t \cdot \log t}{t^4 + x^2} dt = 0$$

AN.044. Find:

$$\Omega = \int_1^{21} \frac{dx}{e^{\left[2x+\frac{1}{4}\right]}}, [*] - \text{GIF}.$$

AN.045. Let $f : [0, \infty) \to R$ be a differentiable, increasing function such that f' is convex and $f(0) = 0$.For any $x, y, z \geq 0$ holds:

$$f(x) + f(y) + f(z) + f(x + y + z) \geq f\left(2\sqrt{xy}\right) + f\left(2\sqrt{yz}\right) + f\left(2\sqrt{zx}\right)$$

AN.046. Find:

$$\Omega = \lim_{n\to\infty} e^{5n+1} \tan^5 n \left(\int_0^n e^{5x}(\tan^4 x + \tan^5 x + \tan^6 x)dx\right)^{-1}$$

AN.047. If $0 < a \leq b$ then:

$$2\int_a^b \int_a^b \int_a^b \left(\frac{y+x}{y+z} + \frac{y+z}{y+x}\right) dxdydz + 2(b-a)^3 \leq 3(b+a)(b-a)^2 \log\left(\frac{b}{a}\right)$$

AN.048. Find:

$$\Omega = \lim_{n\to\infty} \frac{H_n}{n(H_{2n-1} - 2H_{n-1})}$$

AN.049. If $0 < a \leq b$ then:

$$3\int_a^b \sqrt{x^4 + x^2 + 1}\, dx \geq (b-a)\sqrt{(a^2 + ab + b^2)^2 + 3(a^2 + ab + b^2) + 9}$$

AN.050. Find:

$$\Omega(n) = \int \frac{x^{2n-1}(1 - x^2)}{e^{nx^2}} dx, n \in \mathbb{N}, n \geq 1$$

AN.051. If $f:\mathbb{R}\to\left[-\frac{5}{2},\frac{5}{2}\right], f$ −continuous, then:

$$\int_{-\frac{5}{2}}^{\frac{5}{2}}\sqrt{50-8f^2(x)}\,dx+\int_{-\frac{5}{2}}^{\frac{5}{2}}f(x)\,dx\leq\frac{75}{2}$$

AN.052. If $0\leq a\leq\frac{\pi}{12}$ then:

$$\int_0^a \sin x\,.\cos(6x)\,.\cos^6(4x)\,.\cos^{15}(2x)\,dx\leq\frac{1}{193}(1-\cos^{193}a)$$

AN.053.

$$\Omega(x)=\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{2-x}{2}\right)\sin(\pi x)\,;0<x<1$$

Solve for real numbers:

$$x^2-\frac{4x}{\Omega(x)}+\frac{1}{\pi^4}=0$$

AN.054. $x_0=1, x_1=0, x_n=(n-1)(x_{n-1}+x_{n-2}), n\geq 2, n\in\mathbb{N}$. Find:

$$\Omega=\lim_{n\to\infty}\frac{x_n}{n!}$$

AN.055.

$f,g:\mathbb{R}\to\mathbb{R}, f(1)=3, g(1)=2, xf(y)+yf(x)=2f(xy),$

$xg(y)+yg(x)=2g(xy), \forall x,y\in\mathbb{R}$. Find:

$$\Omega=\int_0^1\frac{f\left(\frac{\sinh x}{3}\right)}{f\left(\frac{\sinh x}{3}\right)+g\left(\frac{\cosh x}{2}\right)}dx$$

AN.056. Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\pi^n}\cdot\left(\frac{\pi}{e}\right)^k$$

AN.057.

$$\Omega_1 = 1 - \frac{\pi}{2} + \sum_{n=2}^{\infty}\left(-\frac{1}{\pi}\right)^n \cdot \frac{1}{n+1}; \Omega_2 = 1 - \frac{\pi}{2} + \sum_{n=2}^{\infty}\left(-\frac{1}{e}\right)^n \cdot \frac{1}{n+1}$$

$$A.\ \Omega_1 < \Omega_2, \quad B. \Omega_1 = \Omega_2, \quad C. \Omega_1 > \Omega_2$$

AN.058. Find a closed form:

$$\Omega = \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{(n^3+6n^2+11n+5)\cdot n!}{1+(n^3+6n^2+11n+6)\cdot (n!)^2}\right)$$

AN.059. $f:[0,1]\to\mathbb{R}, f$ −continuous. Prove that:

$$\int_0^1 f(x)dx = 9 \Rightarrow \int_0^1 f^2(x)dx \geq 1 + 4\int_0^1 xf(x)\,dx$$

AN.060.

$$\Omega(n) = (1+2^2)(1+2^4)(1+2^8)\cdot \ldots \cdot \left(1+2^{2^{n-1}}\right), n \geq 1$$

Find:

$$\Omega = \lim_{n\to\infty}\left(1+\frac{1}{3\Omega(n)}\right)^{2^{2^n}}$$

AN.061. Find:

$$\Omega = \lim_{n\to\infty} n\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)^{\frac{1-n}{n^2}}$$

AN.062. Find without any software:

$$\Omega = \int_1^2 \frac{\log(9x-4)}{3x^2+2}dx$$

AN.063. If $1 < a \leq b$ then:

$$2\int_a^b \int_a^b x^y \, dxdy \geq (2ab - a - b + 2)(b-a)^2$$

AN.064. $G(n)$ –Barnes G-function, $K(n) - k$ function. Find:

$$\Omega = \sum_{n=2}^{\infty} \sqrt[n]{\frac{n!}{K(n+1)\cdot G(n+2)}}$$

AN.065. Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{(2n+1)!}{(2n+1-k)!}\right)\left(\sum_{k=0}^{n}\frac{\left((2n+1)!\right)^2}{k!\left((2n+1-k)!\right)^2}\right)^{-1}$$

AN.066. Find:

$$\Omega = \lim_{n\to\infty}(1+n)^{-n}\sum_{k=1}^{n-1}\left(\binom{n}{k}\sum_{m=1}^{n}\frac{m^n}{m^k}\right)$$

AN.067. Find:

$$\Omega = \lim_{n\to\infty}\binom{2n-2}{n-1}^{-1}\sum_{k=0}^{n}\left(1-\frac{2k}{n}\binom{n}{k}\right)^2$$

AN.068.

$$\Omega(\alpha,\beta) = \int_{-1}^{1}\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx\,, \qquad \alpha,\beta > 0$$

Find a closed form and prove that: $\Omega(3,5) > \sqrt{\Omega(4,5)\cdot\Omega(3,6)}$

AN.069. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{(2n)!\cdot\left(2\sum_{k=0}^{n}\frac{1}{(n-k)!\cdot(n+k)!}-\frac{4^n}{(2n)!}\right)}$$

AN.070. Find a closed form:

$$\Omega = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\sin^{-1}\varepsilon}^{\sin^{-1}(1-\varepsilon)} \log\left((\cos x)^{\cot x} \cdot (\sin x)^{\frac{\cos x}{1+\sin x}}\right) dx$$

AN.071.

$$\Omega(n) = \sum_{k=1}^{n} e^{4\left(2-\frac{k}{n}\right)} \cdot \sum_{k=1}^{n} e^{6\left(2-\frac{k}{n}\right)} - \left(\sum_{k=1}^{n} e^{5\left(2-\frac{k}{n}\right)}\right)^2$$

Find:

$$\Omega = \lim_{n\to\infty} n^7 \cdot \Omega(n) \cdot \sin\left(\frac{1}{n^4}\right) \cdot \tan\left(\frac{1}{n^5}\right)$$

AN.072. If $1 < a \le b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{1+xy} \le (b-a)\tan^{-1}\left(\frac{b-a}{1+ab}\right)$$

AN.073. Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \frac{n(n+1)^2(2n+1)}{4^n \cdot n!}$$

AN.074. Prove without any software:

$$4e\left|1 - \int_0^1 e^{x^2}\,dx \int_0^1 e^{-x^2}\,dx\right| < (e-1)^2$$

AN.075. $f, g, h, t: \mathbb{R} \to \mathbb{R}, f, g, h, t$ –continuous, $f(1) = 11$,

$$g(0) = 2, h(0) = 3, t(0) = 4,$$

$$f(x+y+z) = g(x) + h(y) + t(z), \forall x, y, z \in \mathbb{R}$$

Solve for real numbers: $f(x) \cdot g(x) = h(x) \cdot t(x)$.

AN.076. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}\left|\left(\frac{\sin x}{x}\right)^{(k)}\right|, x\in\left(0,\frac{\pi}{2}\right), \qquad (*)^{(k)}-k^{th}\text{derivative.}$$

AN.077. Find:

$$\Omega=\lim_{n\to\infty}\frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}}$$

AN.078. Solve for real numbers:

$$e^{ix}+e^{2ix}+e^{3ix}+\frac{1}{e^{ix}}+\frac{1}{e^{2ix}}+\frac{1}{e^{3ix}}=-1$$

AN.079. Find:

$$\Omega(n)=\int_0^n \log\left(\sqrt{n+x}+\sqrt{n-x}\right)dx, n\in\mathbb{N}-\{0\}$$

AN.080. If $0<a\le b<\frac{\pi}{10}$ then:

$$\int_a^b \frac{\sin 3x\cdot\sin 5x\cdot\sin^2 x}{(1-\cos x)^2}dx<60(b-a)$$

AN.081. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^n}$$

AN.082. Find:

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\cos x\cdot\sinh x}{5+e^x\sin\left(x+\frac{\pi}{4}\right)+e^{-x}\cos\left(x+\frac{\pi}{4}\right)}dx$$

AN.083. If $a, b > 0$ then:

$$\frac{2a}{2a+3b}\log\left(1+\frac{3b}{2a}\right)+\frac{3b}{2a+3b}\log\left(1+\frac{2a}{3b}\right)\leq\log 2$$

AN.084. If $0 < a \leq b$ then:

$$\int_a^b\left(\frac{1}{x}\cdot\tan^{-1}x\right)dx\geq\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

AN.085. If $a \geq 0$ then find:

$$\Omega=\int_a^{a+1}\int_a^{a+1}\sin^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)dxdy+\int_a^{a+1}\int_a^{a+1}\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)dxdy$$

AN.086. Find without any software:

$$\Omega=\int\frac{3xe^x+2}{(2\log x+3e^x-1)(2\log x^x+3xe^x+x)}dx$$

AN.087. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n!}\int_0^\infty\frac{x^n\sin\left(x+\frac{\pi}{4}\right)}{e^x}dx$$

AN.088. Find:

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx$$

AN.089. Find:

$$\Omega=\lim_{n\to\infty}\frac{(2n-5)!}{(n+1)\cdot(2n)!\cdot B(6,2n-4)}$$

AN.090. If $0 < a \leq b \leq 1$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{x^2+y^2+z^2+3}\leq\frac{(b-a)^2}{3}\int_a^b\frac{dx}{x+x^x}$$

AN.091. If $0 < x \le y \le e \le z \le t$ then:

$$e^x + e^y + e^z + e^t \ge e^{x+y-e} + 3 \cdot \sqrt[3]{e^{z+t+e}}$$

AN.092. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{(n-k+1)H_k}{k(n-k+1)^2 + k}$$

AN.093.

$$\omega = \sum_{n=1}^{\infty} \cos^{-1}\left(\frac{n^2 - n + 1}{\sqrt{1 + (n^2 - n + 1)^2}}\right)$$

Find:

$$\Omega = \left(\int_0^{4\omega} \frac{dx}{3 + \sin x}\right)\left(\int_0^{\frac{2\omega}{\pi}} \frac{x^3}{\sqrt{1 + x + x^2}} dx\right)$$

AN.094. Let be $f: \mathbb{R} \to (0, \infty)$ continuous , $0 < a \le b$ and

$$\frac{f^2(x)}{f(y) + f(z)} + \frac{f^2(y)}{f(z) + f(x)} + \frac{f^2(z)}{f(x) + f(y)} = f(x) + f(y) + f(z);$$

$\forall x, y, z \in \mathbb{R}$. Find:

$$\Omega(a, b) = \left(\int_a^b f(x)\, dx\right)\left(\int_a^b \int_a^b \frac{dydz}{f(y) + f(z)}\right)$$

AN.095. If $0 < x, y < 1$ then:

$$\frac{x}{4\sqrt{x+y} - \sqrt{xy + y^2}} + \frac{y}{4\sqrt{x+y} - \sqrt{xy + x^2}} \le \frac{x + y}{4\sqrt{x+y} - \sqrt{2xy}}$$

AN.096. If $0 < a \le b$ then:

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2} dx \le \tan^{-1}\left(\frac{a+b}{2}\right) - \tan^{-1}\left(\sqrt{ab}\right)$$

AN.097. If $a, b > 0$ then:

$$\frac{4}{a+b} \leq \int_0^1 \frac{dx}{ax+(1-x)b} + \int_0^\infty \frac{dx}{(x+a)(x+b)} \leq \frac{2}{\sqrt{ab}}$$

AN.098. Find:

$$\Omega = \int \cosh(3x) \cdot \cosh\left(\frac{3x}{2}\right) \cdot \cosh\left(\frac{13x}{2}\right) dx$$

AN.099. If $a, b, c, d > 0, 0 \leq x \leq 1$ then:

$$\left((1-x)a+xc\right)^2 - \left((1-x)b+xd\right)^2 \geq (a^2-b^2)^{1-x}(c^2-d^2)^x$$

SOLUTIONS
ALGEBRA

A.001. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$By\ AM-GM\ inequality\ we\ have: \ (x+y+z)(xy+yz+zx) \geq 9xyz$$

$$Then\ we\ have: 9(x+y)(y+z)(z+x) = 9(x+y+z)(xy+yz+zx) - 9xyz \geq$$

$$\geq 9(x+y+z)(xy+yz+zx) - (x+y+z)(xy+yz+zx) = 8(x+y+z)(xy+yz+zx) =$$

$$= 4[(x+y)+(y+z)+(z+x)](xy+yz+zx) \overset{CBS}{\geq} 4.\frac{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2}{3}.(xy+yz+zx)$$

$$Then: \ 27(x+y)(y+z)(z+x) \geq 4\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2(xy+yz+zx)$$

$$Therefore, \ 3\sqrt{3(x+y)(y+z)(z+x)} \geq 2\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)\sqrt{xy+yz+zx}.$$

A.002. *Solution by Tapas Das-India*

$$\frac{|sinh\,z|^4}{e^{2x}}+\frac{|cosh\,z|^4}{e^{-2x}} \overset{AGM}{\geq} 2|sinh\,z|^2\cdot|cosh\,z|^2$$

We know that: $|z_1\cdot z_2| = |z_1|\cdot|z_2|; \forall z_1, z_2 \in \mathbb{C}$

$$|sinh\,z|^2\cdot|cosh\,z|^2 = |sinh\,z\cdot cosh\,z|^2 = \left|\frac{e^z-e^{-z}}{2}\cdot\frac{e^z+e^{-z}}{2}\right|$$

$$=\frac{1}{4}\left|\frac{e^{2z}-e^{-2z}}{2}\right|^2 = \frac{1}{4}|sinh(2z)|^2$$

$$sinh(2z) = sinh(2x+2iy) = sinh(2x)\cos 2y + i\,cosh(2x)\,sin(2y)$$

$$|sinh(2z)|^2 = sinh^2(2x)\,cos^2(2y) + cosh^2(2x)\,sin^2(2y) =$$

$$= (cosh^2 2x - 1)\,cos^2(2y) + cosh^2(2x)\,sin^2(2y) =$$

$$= cosh^2(2x)\,(cos^2(2y)+sin^2(2y)) - cosh^2(2y) =$$

$$= cosh^2(2x) - cosh^2(2y)$$

$$cosh(2x) = \frac{e^{2x}+e^{-2x}}{2} \geq \sqrt{e^{2x}\cdot e^{-2x}} = 1 \ and\ cos\,2y \leq 1, then$$

$$cosh^2(2x) - cos^2(2y) \geq 1 \Rightarrow |sinh(2z)| > 1$$

Therefore,

$$\frac{|sinh\,z|^4}{e^{2x}}+\frac{|cosh\,z|^4}{e^{-2x}} \geq 2\cdot\frac{1}{4}|sinh(2z)|^2 = \frac{1}{2}$$

A.003. *Solution by Ravi Prakash-New Delhi-India*

$$\Delta = \begin{vmatrix} \cos x & \cos x & \cos 2x \\ \cos 3x & \cos 5x & \cos 4x \\ \sin 3x & \sin 5x & \sin 4x \end{vmatrix}$$

If $\cos x = 0 \Rightarrow \cos 2x = -1, \cos 3x = 0, \cos 5x = 0, \cos 4x = 1$ *and*

$$\Delta = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ \pm 1 & \pm 1 & 0 \end{vmatrix} = 0.$$

Thus, $\Delta = 0$ *if* $x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$. *Assume that* $\cos x \neq 0 \Rightarrow x \in \mathbb{R}$, *if* $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

$$(2\cos x)\Delta = \begin{vmatrix} \cos x & \cos x & 2\cos x \cos 2x \\ \cos 3x & \cos 5x & 2\cos x \cos 4x \\ \sin 3x & \sin 5x & 2\cos x \sin 4x \end{vmatrix} \overset{C_3 \to C_3 - C_1 - C_2}{=}$$

$$= \begin{vmatrix} \cos x & \cos x & -2\sin 2x \sin x \\ \cos 3x & \cos 5x & 0 \\ \sin 3x & \sin 5x & 0 \end{vmatrix}$$

$$= -2\sin 2x \sin x \begin{vmatrix} \cos 3x & \cos 5x \\ \sin 3x & \sin 5x \end{vmatrix} =$$

$$= -4\sin^3 x \cos x\,; (\cos x \neq 0)$$

Now, $\Delta = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, n \in \mathbb{Z}$

Hence, $\Delta = 0 \Leftrightarrow x \in \left\{ n\pi, (2m+1)\frac{\pi}{2} \,\middle|\, m, n \in \mathbb{Z} \right\}$

A.004. *Solution by Ravi Prakash-New Delhi-India*

$$\Omega(a,b) = \int_a^b \tan^{-1}\left(\frac{3x}{1-2x^2}\right) dx = \int_a^b \tan^{-1}\left(\frac{2x+x}{1-2x \cdot x}\right) dx =$$

$$= \int_a^b [\tan^{-1}(2x) + \tan^{-1} x - \pi]\, dx = I_1 + I_2 + \pi(a-b)$$

$$I_1 = \int_a^b \tan^{-1}(2x)\, dx = \frac{1}{2}\int_{2a}^{2b} \tan^{-1} y\, dy$$

$$I_2 = \int_a^b tan^{-1} x\, dx - [x\, tan^{-1} x]_a^b - \int_a^b \frac{xdx}{1+x^2} =$$

$$= b\, tan^{-1} b - a\, tan^{-1} a - \left[\frac{1}{2} log(1+x^2)\right]_a^b =$$

$$= b\, tan^{-1} b - a\, tan^{-1} a + \frac{1}{2} log\left(\frac{1+a^2}{1+b^2}\right)$$

$$I_1 = \frac{1}{2}[2b\, tan^{-1}(2b) - 2a\, tan^{-1}(2a)] + \frac{1}{4} log\left(\frac{1+4a^2}{1+4b^2}\right)$$

Hence,

$$\Omega(a,b) = b\, tan^{-1}\left(\frac{3b}{1-2b^2}\right) - a\, tan^{-1}\left(\frac{3a}{1-2a^2}\right) - \frac{1}{4} log\left(\frac{1+4b^2}{1+4a^2}\right) - \frac{1}{2} log\left(\frac{1+b^2}{1+a^2}\right) + \pi(a-b)$$

A.005. *Solution by Ravi Prakash-New Delhi-India*

$$5 + \sum_{cyc}(|z_1 - z_2| + |3+4i-2z_3|) \ge 2\sum_{cyc}|z_1|$$

$$\Leftrightarrow 10 + 2\sum_{cuc}|z_1| \le 15 + \sum_{cyc}(|z_1 - z_2| + |3+4i-2z_3|) \Leftrightarrow$$

$$10 + 2\sum_{cyc}|z_1| \le \sum_{cyc}(|z_1 - z_2| + |3+4i-2z_3| + 5); (1)$$

Consider: $|z_1 - z_2| + |3+4i-2z_3| + 5 =$

$$= |z_1 - z_2| + |z_1 + z_2 + z_3 - 2z_3| + |z_1 + z_2 + z_3| \ge$$

$$\ge |z_1 - z_2| + |z_1 + z_2 - z_3 + z_1 + z_2 + z_3| \ge$$

$$\ge |z_1 - z_2| + 2|z_1 + z_2| \ge |z_1 - z_2| + |z_1 + z_2| + |z_1 + z_2| \ge$$

$$\ge 2|z_1| + |z_1 + z_2|$$

$$\Rightarrow \sum_{cyc}(|z_1 - z_2| + |3 + 4i - 2z_3| + 5)$$

$$\geq 2\sum_{cyc}|z_1| + |z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1| \geq$$

$$\geq 2\sum_{cyc}|z_1| + 2|z_1 + z_2 + z_3| = 2\sum_{cyc}|z_1| + 10$$

A.006. *Solution by Nikos Ntorvas-Greece*

We consider the function $f(t) = \frac{1}{\sqrt[3]{1+t}}; t \geq 0$*, then*

$$f''(t) = \frac{4}{9} \cdot \frac{1}{\sqrt[3]{(1+t)^7}} > 0; \forall t > 0$$

Then function f *os continuous on* $[0,\infty)$ *and* $f''(t) > 0$*, so* f *is strictly convex on* $[0,\infty)$

From Petrovic Inequality we have for the sequence $(x_i)_{i=\overline{1,n}}, x_i > 0$ *for* $i \in \{1,2 \dots, n\}$

$$\sum_{i=1}^{n} f(x_i) \leq f\left(\sum_{i=1}^{n} x_i\right) + (n-1)f(0)$$

$$f(0) + \sum_{i=1}^{n} \frac{1}{\sqrt[3]{1+x_i}} \leq \frac{1}{\sqrt[3]{1 + x_1 + x_2 + \cdots + x_n}} + nf(0), f(0) = 1$$

$$1 + \sum_{i=1}^{n} \frac{1}{\sqrt[3]{1+x_i}} \leq \frac{1}{\sqrt[3]{1 + x_1 + x_2 + \cdots + x_n}} + n$$

A.007. *Solution by Adrian Popa-Romania*

If $x_k, y_k > 0, k = \overline{1,n}, n \in \mathbb{N}^*, m \in \mathbb{N}; m \geq 3$ *then:*

$$\frac{y_1^m}{x_1} + \frac{y_2^m}{y_2} + \cdots + \frac{y_n^m}{x_n} \geq \frac{(y_1 + y_2 + \cdots + y_n)^m}{n^{m-2}(x_1 + x_2 + \cdots + x_n)}$$

$$Let\ y_i = sin^2\left(\frac{i\pi}{8}\right); i = \overline{0,7}, x_i = x_i, i = \overline{0,7}$$

$$y_k = cos^2\left(\frac{i\pi}{8}\right); k = \overline{8,15}, x_k = y_i, k = \overline{8,15}$$

Hence,

$$\sum_{i=0}^{7}\frac{sin^6\left(\frac{i\pi}{8}\right)}{x_i} + \sum_{i=0}^{7}\frac{cos^6\left(\frac{i\pi}{8}\right)}{y_i} = \sum_{i=0}^{7}\frac{\left(sin^2\left(\frac{i\pi}{8}\right)\right)^3}{x_i} + \sum_{i=0}^{7}\frac{\left(cos^2\left(\frac{i\pi}{8}\right)\right)^3}{y_i} \geq$$

$$\geq \frac{\sum_{i=0}^{7}\left(sin^2\left(\frac{i\pi}{8}\right) + cos^2\left(\frac{i\pi}{n}\right)\right)^3}{16^{3-2}\sum_{i=0}^{7}(x_i + y_i)} = \frac{8^3}{16\cdot\frac{1225}{512}} = \frac{32768}{2450} > 1$$

A.008. *Solution by Ravi Prakash-New Delhi-India*

$$\Delta = \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & a^2+b^2 & a^2+c^2 & 1 \\ b^2 & a^2+b^2 & 0 & b^2+c^2 & 1 \\ c^2 & a^2+c^2 & b^2+c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} =$$

$$(c_2 \to c_2 - c_1; c_3 \to c_3 - c_1; c_4 \to c_4 - c_1)$$

$$= \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & -a^2 & b^2 & c^2 & 1 \\ b^2 & a^2 & -b^2 & c^2 & 1 \\ c^2 & a^2 & b^2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} =$$

$$= a^2b^2c^2\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{vmatrix} = a^2b^2c^2\begin{vmatrix} 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 \end{vmatrix} = 8a^2b^2c^2$$

$$Now, 2\sqrt{2}(a+b)(b+c)(c+a) \geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}$$

$$abc \leq \frac{1}{2\sqrt{2}} \Rightarrow \Delta = a^2b^2c^2 \leq 1$$

A.009. *Solution by Sanong Huayrerai-Nakon Pathom-Thailand*

$$\frac{a^{120}+b^{120}+c^{120}}{a^{40}+b^{40}+c^{40}} \geq \frac{(a^{40}+b^{40}+c^{40})^3}{3^2(a^{40}+b^{40}+c^{40})}$$
$$= \frac{(a^{40}+b^{40}+c^{40})^2}{3^2} \overset{(*)}{\geq} \frac{1}{(a^4+b^4+c^4)^{10}}$$

$$(*) \Leftrightarrow \frac{(a^{40}+b^{40}+c^{40})^{10}(a^{40}+b^{40}+c^{40})^2}{3^2} \geq 1$$

$$\Leftrightarrow \frac{(a^{40}+b^{40}+c^{40})^{12}}{3^2} \geq \frac{\left(a^{\frac{120}{12}}+b^{\frac{120}{12}}+b^{\frac{120}{12}}\right)^{12}}{3^2} \geq 1$$

$$\Leftrightarrow a^{10}+b^{10}+c^{10} \geq \sqrt[12]{9} = \sqrt[6]{3}$$

$$\Leftrightarrow 3\sqrt[3]{(abc)^{10}} \geq \sqrt[6]{3} \Leftrightarrow abc \geq \frac{1}{\sqrt[4]{3}}\ true.$$

A.010. *Solution by Ravi Prakash-New Delhi-India*

If $z=0 \Rightarrow x+y=0 \Rightarrow y=-x$. *In this case, the 2nd equation is satisfied.*

Assume, $xyz \neq 0$ *and put* $x=\tan A\,; y=\tan B\,; z=\tan C\,, 0<\,, |B|, |C| < \frac{\pi}{2}$

The first equation becomes:

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C \Rightarrow \tan(A+B)=\tan(-C)$$

$$A+B=n\pi-C; n\in\mathbb{Z} \Rightarrow 3A+3B=3n\pi-3C$$

$$\Rightarrow \tan(3A+3B)=\tan(3n\pi-3C)=-\tan(3C)$$

$$\Rightarrow \frac{\tan(3A)+\tan(3B)}{1-\tan(3A)\tan(3B)}=-\tan(3C)$$

$$\Rightarrow tan(3A) + tan(3B) + tan(3C) = tan(3A)\,tan(3B)\,tan(3C)$$

$$\Rightarrow \frac{x(3-x^2)}{1-3x^2} + \frac{y(3-y^2)}{1-3y^2} + \frac{z(3-z^2)}{1-3z^2} = \frac{xyz(3-x^2)(3-y^2)(3-z^2)}{(1-3x^2)(1-3y^2)(1-3z^2)}$$

Using 2nd equation, we get:

$$(3-x^2)(3-y^2)(3-z^2) = 0; (\because xyz \neq 0)$$

Hence, $x = \pm\sqrt{3}$ *or* $y = \pm\sqrt{3}$ *or* $z = \pm\sqrt{3}$

$$S = \{(0,x,-x),(x,0,-x),(x,-x,0)\} \cup$$

$$\cup \left\{(\sqrt{3}, tan\,B\,, tan\,C), (tan\,B\,, \sqrt{3}, tan\,C), (tan\,B\,, tan\,C\,, \sqrt{3}) | B + C = n\pi - \frac{\pi}{3}, n \in \mathbb{Z}\right\} \cup$$

$$\cup \left\{(-\sqrt{3}, tan\,B\,, tan\,C), (tan\,B\,, -\sqrt{3}, tan\,C), (tan\,B\,, tan\,C\,, -\sqrt{3}) | B + C = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}\right\}$$

A.011. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$(*) \leftrightarrow \sqrt{2}.e^{\sqrt{3}}\left[2(a+b) - (\sqrt{a}+\sqrt{b})^2\right] \geq \sqrt{3}.e^{\sqrt{2}}\left[(\sqrt{a}+\sqrt{b})^2 - 4\sqrt{ab}\right]$$

$$\leftrightarrow \sqrt{2}.e^{\sqrt{3}}(\sqrt{a}-\sqrt{b})^2 \geq \sqrt{3}.e^{\sqrt{2}}(\sqrt{a}-\sqrt{b})^2 \leftrightarrow \left(\frac{e^{\sqrt{3}}}{\sqrt{3}} - \frac{e^{\sqrt{2}}}{\sqrt{2}}\right)(\sqrt{a}-\sqrt{b})^2 \geq 0 \ \ (1)$$

$$Let\ f(x) = \frac{e^x}{x}, x > 1. We\ have:\ f'(x) = \frac{(x-1)e^x}{x^2} > 0$$

$$\rightarrow f\ is\ srictly\ increasing\ on\ (1,\infty) \rightarrow \frac{e^{\sqrt{3}}}{\sqrt{3}} > \frac{e^{\sqrt{2}}}{\sqrt{2}} \rightarrow (1)\ is\ true.$$

$$4\sqrt{3ab}.e^{\sqrt{2}}+2\sqrt{2}(a+b).e^{\sqrt{3}}\geq\left(\sqrt{a}+\sqrt{b}\right)^2\left(\sqrt{3}.e^{\sqrt{2}}+\sqrt{2}.e^{\sqrt{3}}\right).$$

$$Equality\ holds\ iff\ a=b.$$

A.012. *Solution by Amir Sofi-Kosovo*

$$\begin{cases}2x^2+3y^2+z^2=7\\x^2+y^2+z^2=\sqrt{2}z(x+y)\end{cases}\Rightarrow\begin{cases}2x^2+3y^2+z^2=7\\x^2-\sqrt{2}xz+\frac{z^2}{2}+y^2-\sqrt{2}yz+\frac{z^2}{2}=0\end{cases}$$

$$\begin{cases}2x^2+3y^2+z^2=7\\\left(x-\frac{z}{\sqrt{2}}\right)^2+\left(y-\frac{z}{\sqrt{2}}\right)^2=0\end{cases}\Rightarrow\begin{cases}z^2+\frac{3}{2}z^2+z^2=7\\x=y=\frac{z}{\sqrt{2}}\end{cases}$$

$$\begin{cases}z=\sqrt{2}\\x=y=1\end{cases}\ and\ \begin{cases}z=-\sqrt{2}\\x=y=-1\end{cases}$$

$$(x,y,z)\in\left\{\left(1,1,-\sqrt{2}\right);\left(1,1,\sqrt{2}\right)\right\}$$

A.013. *Solution by Surjeet Singhania-India*

$$\sin z=\sin(x+iy)=\sin x\cosh y+i\sinh y\cos x$$

$$|\sin z|^2=\sin^2 x\cosh^2 y+\sinh^2 y\cos^2 x$$

Similarly:

$$|\sinh z|^2=\sinh^2 x\cos^2 y+\cosh^2 x\sin^2 y$$

$$|\cos z|^2=\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y$$

Hence:

$$|\sin z|^2+|\sinh z|^2+|\cos z|^2+|\cosh z|^2=$$

$$=\sinh^2 x+\cosh^2 x+\sinh^2 y+\cosh^2 y=$$

$$=\cosh(2x)+\cosh(2y)\geq\sinh(2x)+\cosh(2y)$$

A.014. *Solution by Amir Sofi-Kosovo*

$$xy + yz + zx = 3xyz \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$$

$$(x+y)\left(\frac{1}{x} + \frac{1}{y}\right) = 2\sqrt{xy} \cdot \frac{2}{\sqrt{xy}} = 2 \Rightarrow \frac{1}{x+y} \le \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{1}{4}\left(3 - \frac{1}{z}\right)$$

$$\sum_{cyc} \frac{1}{x+y} \le \frac{1}{4}\left(3 - \frac{1}{z}\right) + \frac{1}{4}\left(3 - \frac{1}{y}\right) + \frac{1}{4}\left(3 - \frac{1}{x}\right) = \frac{9}{4} - \frac{3}{4} = \frac{3}{2}$$

$$\frac{1}{2x+y+z} = \frac{1}{x+y+x+z} \le \frac{1}{4}\left(\frac{1}{x+y} + \frac{1}{x+z}\right)$$
$$\le \frac{1}{16}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{x}\right) = \frac{1}{16}\left(3 + \frac{1}{x}\right)$$

$$\sum_{cyc} \frac{1}{2x+y+z} \le \frac{1}{16}\left(3 + \frac{1}{x}\right) + \frac{1}{16}\left(3 + \frac{1}{y}\right) + \frac{1}{16}\left(3 + \frac{1}{z}\right) = \frac{9}{16} + \frac{3}{16} = \frac{3}{4}$$

Therefore,

$$\left(\sum_{cyc} \frac{1}{x+y}\right)\left(\sum_{cyc} \frac{1}{2x+y+z}\right) \le \frac{9}{8}$$

A.015. *Solution by Adrian Popa-Romania*

$$det(A) = \begin{vmatrix} sin^2 x - cos^2 y & cos^2 z & cos^2 z \\ cos^2 x & sin^2 y - cos^2 z & cos^2 x \\ cos^2 y & cos^2 y & sin^2 z - cos^2 x \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ cos^2 x & sin^2 y - cos^2 z & cos^2 x \\ cos^2 y & cos^2 y & sin^2 z - cos^2 x \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ cos^2 x & sin^2 y - cos^2 z - cos^2 x & 0 \\ cos^2 y & 0 & sin^2 z - cos^2 x - cos^2 y \end{vmatrix} =$$

$$= (\sin^2 y - \cos^2 z - \cos^2 x)(\sin^2 z - \cos^2 x - \cos^2 y) =$$

$$= (1 - \cos^2 x - \cos^2 y - \cos^2 z)(1 - \cos^2 z - \cos^2 x - \cos^2 y) =$$

$$= (1 - \cos^2 x - \cos^2 y - \cos^2 z)^2 \geq 0$$

$$\det(B) = \begin{vmatrix} \cos^2 x - \sin^2 y & \sin^2 z & \sin^2 z \\ \sin^2 x & \cos^2 y - \sin^2 z & \sin^2 x \\ \sin^2 y & \sin^2 y & \cos^2 z - \sin^2 x \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \sin^2 x & \cos^2 y - \sin^2 z & \sin^2 x \\ \sin^2 y & \sin^2 y & \cos^2 z - \sin^2 x \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \sin^2 x & \cos^2 y - \sin^2 z - \sin^2 x & 0 \\ \sin^2 y & 0 & \cos^2 z - \sin^2 x - \sin^2 y \end{vmatrix} =$$

$$= (1 - \sin^2 y - \sin^2 z - \sin^2 x)(\cos^2 z - \sin^2 x - \sin^2 y) =$$

$$= (1 - \sin^2 y - \sin^2 z - \sin^2 x)(1 - \sin^2 z - \sin^2 x - \sin^2 y) =$$

$$= (1 - \sin^2 x - \sin^2 y - \sin^2 z)^2 \geq 0$$

So, $\det(AB) = \det(A) \cdot \det(B) \geq 0$

A.016. *Solution by Ravi Prakash-New Delhi-India*

The given equation can be written as:

$$\frac{1}{5}x^5 + x^4 + 2x^3 + 2x^2 + x + \frac{1}{5} = 0$$

$$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0 \Leftrightarrow (x+1)^5 = 0 \Leftrightarrow x = -1$$

So, the only real solution is $x = -1$.

A.017. *Solution by Ravi Prakash-New Delhi-India*

For $k \geq 4$, $[(k+1)a + (k-1)b](c+d) - [(k+1)c + (k-1)d](a+b) =$

$$= (k+1)ac + (k-1)bc + (k+1)ad + (k-1)bd -$$

$$-[(k+1)ac + (k-1)da + (k+1)bc + (k-1)bd] = 2(ad - bc) > 0$$

Thus,

$$\frac{(k+1)a + (k-1)b}{(k+1)c + (k-1)d} > \frac{a+b}{c+d}$$

Taking $k = 4,6,8$, *we get:*

$$\frac{5a+3b}{5c+3d} > \frac{a+b}{c+d}, \frac{7a+5b}{7c+5d} > \frac{a+b}{c+d}, \frac{9a+7b}{9c+7d} > \frac{a+b}{c+d}$$

Multiplying these three inequalities, we get:

$$\frac{(5a+3b)(7a+5b)(9a+7b)}{(5c+3d)(7c+5d)(9c+7d)} > \left(\frac{a+b}{c+d}\right)^3$$

A.018. *Solution by Tapas Das-India*

$$\frac{b^4+c^4}{b^3+b^2c+bc^2+c^3} \ge \frac{b^4+c^4}{(b+c)(b^2+c^2)} \ge \frac{(b^2+c^2)^2}{2(b+c)(b^2+c^2)}$$

$$\because \frac{b^4+c^4}{2} \ge \left(\frac{b^2+c^2}{2}\right)^2$$

$$\frac{b^4+c^4}{b^3+b^2c+bc^2+c^3} \ge \frac{(b+c)^2}{4(b+c)}$$

$$\because \frac{b^2+c^2}{2} \ge \left(\frac{b+c}{2}\right)^2$$

$$\frac{b^4+c^4}{b^3+b^2c+bc^2+c^3} \ge \frac{b+c}{4} \Leftrightarrow \frac{a(b^4+c^4)}{b^3+b^2c+bc^2+c^3} \ge \frac{a(b+c)}{4}$$

Therefore,

$$\sum_{cyc} \frac{a(b^4+c^4)}{b^3+b^2c+bc^2+c^3} \ge \sum_{cyc} \frac{a(b+c)}{4} = \frac{2(ab+bc+ca)}{4} = 1$$

A.019. *Solution by Tran Quoc Thinh-Vietnam*

According to the Dirichlet's principle, we have 3 numbers $a-2; b-2; c-2$ *has at least 2 numbers with the same sign. Suppose that* $(a-2)(b-2)\ge 0$ *it follows*

$$ab-2(a+b)+4\ge 0,\qquad ab+4\ge 2(a+b)$$

$$abc+4c\ge 2(bc+ca), abc\ge 2(bc+ca)\ -4c; (1)$$

Need to prove that:

$$a^2+b^2+c^2+4-4c\ge 2ab; (2)$$

$(a-b)^2+(c-2)^2\ge 0$ *which is true.From (1) and (2) we get:*

$$abc+a^2+b^2+c^2+4\ge 2(ab+bc+ca)$$

Equality holds if $a=b=c=2$.

A.020. *Solution by Tapas Das-India*

$$2\sum_{cyc}x^2(x^2+y^2)=2\left(\sum_{cyc}x^4+\sum_{cyc}x^2y^2\right)=$$

$$=(x^4+x^2y^2)+(y^4+y^2z^2)+(z^4+z^2x^2)+(y^2z^2+z^4)+(z^2x^2+x^4)+(x^2y^2+y^4)$$

$$\overset{AGM}{\ge} 2\sqrt{x^4y^2z^2}+2\sqrt{y^4y^2z^2}+2\sqrt{z^4z^2z^2}+2\sqrt{y^2z^2z^4}+2\sqrt{z^2x^2x^4}+2\sqrt{x^2y^2y^4}=$$

$$=x(y^3+z^3)+y(z^3+x^3)+z(x^3+y^3)+(x^3y+xy^3)+(y^3z+yz^3)+(z^3x+zx^3)$$

$$\overset{AGM}{\ge} x(y^3+z^3)+y(z^3+x^3)+z(x^3+y^3)+2x^2y^2+2y^2z^2+2z^2x^2=$$

$$=\sum_{cyc}x(y^3+z^3)+2\sum_{cyc}x^2y^2\ge\sum_{cyc}x(y^3+z^3)+2\sum_{cyc}xy\cdot yz\ge$$

$$\geq \sum_{cyc} x(y^3+z^3)+2xyz(x+y+z)$$

A.021. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$\text{The desired inequality is succesively equivalent to}$$

$$(a+b+c)[(1+b)(1+c)+(1+c)(1+a)+(1+a)(1+b)] \leq \left(\frac{1}{a^6}+\frac{1}{b^6}+\frac{1}{c^6}\right)\left(\sum\nolimits_{cyc} a^7(1+b)(1+c)\right)$$

$$\leftrightarrow \sum\nolimits_{cyc}(b+c)(1+b)(1+c) \leq \sum\nolimits_{cyc} a^7\left(\frac{1}{b^6}+\frac{1}{c^6}\right)(1+b)(1+c)$$

$$\leftrightarrow\ 0 \leq \sum\nolimits_{cyc}\left(\frac{a^7}{b^6}-b+\frac{a^7}{c^6}-c\right)(1+b)(1+c)$$

$$\leftrightarrow\ 0 \leq \sum\nolimits_{cyc}\left(\frac{a^7-b^7}{b^6}\right)(1+b)(1+c)-\sum\nolimits_{cyc}\left(\frac{c^7-a^7}{c^6}\right)(1+b)(1+c)$$

$$\leftrightarrow\ 0 \leq \sum\nolimits_{cyc}\left(\frac{a^7-b^7}{b^6}\right)(1+b)(1+c)-\sum\nolimits_{cyc}\left(\frac{a^7-b^7}{a^6}\right)(1+c)(1+a)$$

$$\leftrightarrow\ 0 \leq \sum\nolimits_{cyc}(1+c)(a^7-b^7)\left(\frac{1+b}{b^6}-\frac{1+a}{a^6}\right)$$

$$\text{Which is true because } a^7-b^7 \text{ and}$$

$$\frac{1+b}{b^6}-\frac{1+a}{a^6} \text{ have the same sign.}$$

$$\left(\therefore \text{If } a \geq b \text{ we have}:\ a^7 \geq b^7 \text{ and } \frac{1+b}{b^6} \geq \frac{1+a}{a^6}\right)$$

$$\text{So the proof is completed. Equality holds iff } a=b=c=1.$$

A.022. *Solution by Ravi Prakash-New Delhi-India*

$$\Omega = log_2(log_2 e)+log(log\,\pi)+log_\pi(log_\pi 2) =$$

$$= -\log_2\left(\frac{1}{\log 2}\right) + \log(\log \pi) + \log_\pi(\log 2) - \log_\pi(\log \pi) =$$

$$= \frac{\log(\log 2)}{\log \pi} - \frac{\log(\log 2)}{\log 2} + \log(\log \pi) - \frac{\log(\log \pi)}{\log \pi} =$$

$$= \log(\log 2)\left[\frac{1}{\log \pi} - \frac{1}{\log 2}\right] + \log(\log \pi)\left[1 - \frac{1}{\log \pi}\right]$$

Because $\log(\log 2) < 0, \log 2 < \log \pi, \log \pi > 1$ *we get* $\Omega > 0$.

A.023. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Let\ a' = \sqrt{\frac{a^2+b^2}{2}},\ \ b' = \frac{a+b}{2},\ \ c' = \sqrt{ab},\ \ x' = \sqrt{\frac{x^2+y^2}{2}},\ \ y' = \frac{x+y}{2},\ \ z' = \sqrt{xy}.$$

$$By\ QM - AM - GM\ inequalities\ we\ have:$$

$$a' \ge b' \ge c'\ \ and\ \ x' \ge y' \ge z'\ \ (1)$$

$$The\ desired\ inequality\ is\ succesively\ equivalent\ to$$

$$(c'+z')(b'+y')(a'+x') \le (c'+y')(b'+x')(a'+z')$$

$$a'b'z' + c'a'y' + b'c'x' + x'y'c' + z'x'b' + y'z'a' \le a'b'y' + c'a'x' + b'c'z' + x'y'a' + z'x'c' + y'z'b'$$

$$From\ (1)\ we\ have:\ \begin{cases} a'b' \ge c'a' \ge b'c' \\ z' \le y' \le x' \end{cases}\ \ and\ \ \begin{cases} x'y' \ge z'x' \ge y'z' \\ c' \le b' \le a' \end{cases}$$

$$Then\ by\ Rearrangement\ inequality\ we\ have$$

$$\begin{cases} a'b'.z' + c'a'.y' + b'c'.x' \le a'b'.y' + c'a'.x' + b'c'.z' \\ x'y'.c' + z'x'.b' + y'z'.a' \le x'y'.a' + z'x'.c' + y'z'.b' \end{cases}$$

$$Summing\ these\ inequalities\ yields\ the\ required\ inequality.$$

$$Equality\ holds\ iff\ a' = b' = c'\ and\ x' = y' = z'\ \leftrightarrow$$

$$a = b \text{ and } x = y.$$

A.024. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$\frac{x}{x^2-x+1}+\frac{y}{y^2-y+1}+\frac{xy}{x^2y^2-xy+1}\overset{(*)}{\leq}\frac{x^2}{x^2-x+1}+\frac{y^2}{y^2-y+1}+\frac{1}{x^2y^2-xy+1}$$

$$Lemma: \ If\ a,b,c>0 \text{ such that } abc=1 \text{ then}:$$

$$\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1}\geq 1 \ (1)$$

$$Let\ A=a^2+a,\ B=b^2+b,\ C=c^2+c.\ We\ have:\ (1)\leftrightarrow$$

$$\sum_{cyc}(A+1)(B+1)\geq (A+1)(B+1)(C+1)$$

$$\leftrightarrow\ A+B+C+2\geq ABC\ \leftrightarrow$$

$$(a^2+b^2+c^2)+(a+b+c)+2\geq (a+1)(b+1)(c+1)$$

$$\leftrightarrow\ a^2+b^2+c^2\geq ab+bc+ca \text{ which is true}.$$

$$So\ the\ proof\ of\ lemma\ is\ completed.$$

$$Now\ Let\ z=\frac{1}{xy}.\ We\ have:\ (*)\leftrightarrow\frac{x}{x^2-x+1}+\frac{y}{y^2-y+1}+\frac{z}{z^2-z+1}\leq\frac{x^2}{x^2-x+1}+\frac{y^2}{y^2-y+1}+\frac{z^2}{z^2-z+1}$$

$$\leftrightarrow\ 3\leq\left(\frac{x^2-x}{x^2-x+1}+1\right)+\left(\frac{y^2-y}{y^2-y+1}+1\right)+\left(\frac{z^2-z}{z^2-z+1}+1\right)$$

$$Since\ \frac{x^2-x}{x^2-x+1}+1=\frac{(2x^2-2x+1)(x^2+x+1)}{(x^2-x+1)(x^2+x+1)}=\frac{2x^4+(x^2-x+1)}{(x^2-x+1)(x^2+x+1)}=\frac{2x^4}{x^4+x^2+1}+\frac{1}{x^2+x+1}$$

$$Then\ we\ have:\ \ (*)\ \leftrightarrow$$

$$3 \le 2\left(\frac{x^4}{x^4+x^2+1}+\frac{y^4}{y^4+y^2+1}+\frac{z^4}{z^4+z^2+1}\right)+\left(\frac{1}{x^2+x+1}+\frac{1}{y^2+y+1}+\frac{1}{z^2+z+1}\right)$$

$$Since\ xyz=1\ then\ by\ the\ lemma\ we\ have:$$

$$\frac{1}{x^2+x+1}+\frac{1}{y^2+y+1}+\frac{1}{z^2+z+1} \ge 1\ \ (2)$$

$$Also\ taking\ in\ the\ lemma\ (a,b,c)=\left(\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}\right)\ we\ get:$$

$$\frac{x^4}{x^4+x^2+1}+\frac{y^4}{y^4+y^2+1}+\frac{z^4}{z^4+z^2+1} \ge 1\ \ (3)$$

$$From\ (2)\ and\ (3)\ yields\ the\ required\ inequality.$$

$$Equality\ holds\ iff\ \ x=y=1.$$

A.025. *Solution by Asmat Qatea-Afghanistan*

$$(x-y)^4+(y-z)^4+(z-x)^4 = \frac{((x-y)^2)^2}{1}+\frac{((y-z)^2)^2}{1}+\frac{((z-x)^2)^2}{1} \overset{Bergstrom}{\ge}$$

$$\ge \frac{4(x^2+y^2+z^2-xy-yz-zx)^2}{3} = \frac{4\big((x+y+z)^2-3(xy+yz+zx)\big)^2}{3} \overset{AM-GM}{\ge}$$

$$\ge \frac{4\left(\left(3\sqrt[3]{xyz}\right)^2-3(xy+yz+zx)\right)^2}{3} = \frac{4\big(9-3(xy+yz+zx)\big)^2}{3} =$$

$$= 12\left(3-\frac{xy+yz+zx}{xyz}\cdot xyz\right)^2 = 12\left(3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)^2$$

$$(x-y)^4+(y-z)^4+(z-x)^4 \geq 2\left(3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)^2$$

A.026. *Solution by Asmat Qatea-Afghanistan*

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}} \overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq} 4\cdot\sqrt[4]{\frac{1}{\sqrt[3]{(1+a)(1+b)(1+c)(1+d)}}} \overset{AM-GM}{\geq} 4\cdot\sqrt[3]{\frac{1}{\frac{a+b+c+d+4}{4}}}$$

$$=4\cdot\sqrt[3]{\frac{4}{5}}$$

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}} \geq 4\cdot\sqrt[3]{\frac{4}{5}}$$

A.027. *Solution by Amir Sofi-Kosovo*

$$10^{\frac{x^3+6x^2+x}{2(x+1)}}+10^{2x\sqrt{x}}=10^{x\sqrt{x}}\left(10^{\frac{x^2+x}{2}}+10^{\frac{2x^2}{x+1}}\right)$$

$$Let\ 10^{\frac{x^2+x}{2}}=a;\ 10^{\frac{2x^2}{x+1}}=b, 10^{x\sqrt{x}}=c; a,b,c>0, then$$

$$ab+c^2=c(a+b) \Leftrightarrow (a-c)(b-c)=0$$

$$a=c \Rightarrow \frac{x^2+x}{2}=x\sqrt{x} \Leftrightarrow x^2-2x\sqrt{x}+x=0 \Leftrightarrow \left(x-\sqrt{x}\right)^2=0 \Leftrightarrow x=0, x=1$$

$$b=c \Rightarrow \frac{2x^2}{x+1}=x\sqrt{x} \Leftrightarrow 2x^2-x^2\sqrt{x}-x\sqrt{x}=0 \Leftrightarrow x\sqrt{x}\left(2\sqrt{x}-x-1\right)=0$$

$x\sqrt{x}=0\ or\ \left(\sqrt{x}-1\right)^2=0 \Leftrightarrow x=0, x=1.$ *So,* $x\in\{0,1\}$.

A.028. *Solution by Nguyen Van Canh-Ben Tre-Vietnam*

$$4ab.e^{\left(\frac{ax+by}{a+b}\right)} \le 2ab.\sqrt{e^{x+y}} + a^2e^x + b^2e^y;$$

$$\Leftrightarrow 4ab.e^{\left(\frac{ax+by}{a+b}\right)} \le 2ab.e^{\frac{1}{2}x}.e^{\frac{1}{2}y} + a^2e^x + b^2e^y;$$

$$\Leftrightarrow 4ab.e^{\left(\frac{ax+by}{a+b}\right)} \le \left(ae^{\frac{1}{2}x} + be^{\frac{1}{2}y}\right)^2;$$

$$\Leftrightarrow 2\sqrt{ab}.e^{\frac{1}{2}\left(\frac{ax+by}{a+b}\right)} \le ae^{\frac{1}{2}x} + be^{\frac{1}{2}y};$$

$$\Leftrightarrow 2.e^{\frac{1}{2}\left(\frac{ax+by}{a+b}\right)} \le \sqrt{\frac{a}{b}}.e^{\frac{1}{2}x} + \sqrt{\frac{b}{a}}.e^{\frac{1}{2}y};$$

Let $f(t) = e^t \Rightarrow f''(t) = e^t > 0$. *Using Jensen's inequalitywe have:*

$$\sqrt{\frac{a}{b}}.e^{\frac{1}{2}x} + \sqrt{\frac{b}{a}}.e^{\frac{1}{2}y} = \sqrt{\frac{a}{b}}.f\left(\frac{1}{2}x\right) + \sqrt{\frac{b}{a}}.f\left(\frac{1}{2}y\right)$$

$$\ge \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right).f\left(\frac{\sqrt{\frac{a}{b}}.\frac{1}{2}x + \sqrt{\frac{b}{a}}.\frac{1}{2}y}{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}}\right)$$

$$= \frac{a+b}{\sqrt{ab}}.f\left(\frac{1}{2}\left(\frac{ax+by}{a+b}\right)\right) = \frac{a+b}{\sqrt{ab}}.e^{\frac{1}{2}\left(\frac{ax+by}{a+b}\right)} \overset{Cauchy}{\ge} 2.e^{\frac{1}{2}\left(\frac{ax+by}{a+b}\right)}$$

A.029. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Let\ \ a' = 2c + a,\ \ b' = 2a + b,\ \ c' = 2b + c.$$

$$We\ have: a' + b' = 3a + b + 2c$$
$$= 2a + (a + c) + b + c \overset{a+c>b}{>} 2a + 2b + c$$
$$> c'\ \ (and\ analogs)$$

$$Then\ a', b', c'\ can\ be\ the\ side\ lengths\ of\ a\ triangle\ \Delta'\ with$$

$$semiperimeter\ s', inradius\ r', circumradius\ R'.$$

$$We\ have: 2s' = a' + b' + c' = 3(a+b+c)$$

$$Then\ \ a + 3b - c = 3(a+b+c) - 2(2c+a) = 2s' - 2a' = 2(s'-a')\ \ (And\ analogs)$$

$$So\ we\ need\ to\ prove\ that:$$

$$\frac{a'}{b'} + \frac{b'}{a'} \le \frac{2a'b'c'}{2(s'-a').2(s'-b').2(s'-c')} = \frac{2.4s'r'R'}{8s'r'^2} = \frac{R'}{r'}$$

$$Which\ is\ Bandila's\ inequality\ in\ triangle\ \Delta'.$$

$$Therefore,\quad \frac{2c+a}{2a+b} + \frac{2a+b}{2c+a} \le \frac{2(2a+b)(2b+c)(2c+a)}{(a+3b-c)(b+3c-a)(c+3a-b)}.$$

A.030. *Solution by Vivek Kumar-India*

$$Let: u = \sqrt{\frac{a^2+b^2}{2}}\ and\ v = \frac{a+b}{2}, then:$$

$$\frac{x^2}{u} + \frac{y^2}{v} \ge \frac{2(x+y)^2}{2v+2u} + \frac{v}{2}\left(\frac{x}{u} - \frac{y}{v}\right)^2 \Leftrightarrow$$

$$2uv(u+v)(vx^2+uy^2) \ge 2u^2v^2(x+y)^2 + v(u+v)(vx-uy)^2 \Leftrightarrow$$

$$2u(u+v)(vx^2+uy^2) \ge 2u^2v(x+y)^2 + (u+v)(vx-uy)^2 \Leftrightarrow$$

$$2u[(u+v)(vx^2+uy^2) - uv(x+y)^2] \ge (u+v)(vx-uy)^2 \Leftrightarrow$$

$$2u(uvx^2 + u^2y^2 + v^2x^2 + uvy^2 - uvx^2 - uvy^2 - 2uvxy) \ge$$

$$\ge (u+v)(vx-uy)^2 \Leftrightarrow$$

$$2u(v^2x^2 + u^2y^2 - 2uvxy) \ge (u+v)(vx-uy)^2 \Leftrightarrow$$

$$2u(vx-uy)^2 \ge (u+v)(vx-uy)^2 \Leftrightarrow$$

$$(u-v)(vx-uy)^2 \ge 0\ \textit{true as}\ u \ge v$$

A.031. *Solution by Ravi Prakash-New Delhi-India*

The equation is defined for $-1 \leq x \leq 1$.

Let $f(x) = 5\left[\sqrt[5]{1+x} + \sqrt[5]{1-x}\right] - 4\left[\sqrt[4]{1+x} + \sqrt[4]{1-x}\right]; -1 \leq x \leq 1$

$$f'(x) = \frac{1}{\sqrt[5]{(1+x)^4}} - \frac{1}{\sqrt[5]{(1-x)^4}} - \frac{1}{\sqrt[4]{(1+x)^3}} + \frac{1}{\sqrt[4]{(1-x)^3}} =$$

$$= \frac{\sqrt[4]{(1+x)^3} - \sqrt[5]{(1+x)^4}}{\sqrt[20]{(1+x)^{31}}} + \frac{\sqrt[5]{(1-x)^4} - \sqrt[4]{(1-x)^3}}{\sqrt[20]{(1-x)^{31}}}$$

For $-1 < x < 0;\ 0 < 1 + x < 1;\ 1 < 1 - x < 2 \Rightarrow \sqrt[4]{(1+x)^3} < \sqrt[5]{(1+x)^4}$ *and*

$\sqrt[5]{(1-x)^4} > \sqrt[4]{(1-x)^3}$ *as* $\frac{3}{4} < \frac{4}{5}$. *Thus,* $f'(x) < 0$ *for* $-1 < x < 0$.

For $0 < x < 1; 1 < 1 + x < 2; 1 < 1 - x < 1$

$\sqrt[4]{(1+x)^3} < \sqrt[5]{(1+x)^4}$ *and* $\sqrt[4]{(1-x)^3} > \sqrt[5]{(1-x)^4}$ *as* $\frac{3}{4} < \frac{4}{5} \Rightarrow$

$f'(x) < 0$ *for* $0 < x < 1$. *Also,* $f'(x) < 0$ *for* $0 < x < 1$.

Thus, $f(x) < f(0) = 2$ *for* $-1 \leq x < 0$ *and* $f'(x) < f(0) = 2$ *for* $0 < x \leq 1$.

Therefore,

$f(x) = f(0) = 2$ *has only one solution,* $x = 0$.

A.032. *Solution by George Florin Şerban-Romania*

$$\sum_{cyc} \frac{75x + 36(y+z)}{y+z+t} = 39\sum_{cyc} \frac{x}{y+z+t} + 36\sum_{cyc} \frac{x+y+z}{y+z+t} \overset{(1)}{\geq} 196$$

$$\sum_{cyc} \frac{x+y+z}{y+z+t} \overset{AM-GM}{\geq} 4 \cdot \sqrt[4]{\prod_{cyc} \frac{x+y+z}{y+z+t}} = 4$$

We prove that:

$$\sum_{cyc}\frac{x}{y+z+t}\geq\frac{4}{3};(2)\Leftrightarrow\sum_{cyc}\frac{x+y+z+t}{y+z+t}\geq\frac{16}{3}\Leftrightarrow$$

$$\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{1}{y+z+t}\right)\overset{AM-HM}{\geq}\left(\sum_{cyc}x\right)\left(\frac{16}{\sum(y+z+t)}\right)=\frac{16}{3}$$

Hence, (2) *is true. From (1),(2) it follows thet:*

$$\sum_{cyc}\frac{75x+36(y+z)}{y+z+t}\geq 39\cdot\frac{4}{3}+36\cdot 4=196$$

A.033. *Solution by Theodoros Sampas-Greece*

$$\frac{(x+y)^4}{x^4+x^2y^2+y^4}\overset{y=kx;k>0}{=}\frac{(1+k)^4}{1+k^2+k^4}\leq\frac{16}{3}\Leftrightarrow$$

$$13k^4-12k^3-2k^2-12k+13\geq 0$$
$$\Leftrightarrow (k-1)^2(13k^2+14k+13)\geq 0(true)$$

Equality holds for $k=1\Leftrightarrow x=y$. *So, we get:*

$$\frac{(x+y)^4}{x^4+x^2y^2+y^4}\leq\frac{16}{3}\ (and\ analogs)$$

Therefore,

$$\sum_{cyc}\frac{(x+y)^4}{x^4+x^2y^2+y^4}\leq 16$$

Equality holds for $x=y=z$.

A.034. *Solution by George Florin Şerban-Romania*

$$\varphi=\frac{1+\sqrt{5}}{2},\varphi^2=\varphi+1$$

$$1+\frac{\sqrt{\varphi}+\varphi+\varphi\sqrt{\varphi}}{2}=1+\frac{(1+\varphi)\sqrt{\varphi}+\varphi}{2}=1+\frac{\varphi^2\sqrt{\varphi}+\varphi}{2}\overset{AGM}{>}$$

$$>1+\sqrt{\varphi^2\sqrt{\varphi}\varphi}=1+\varphi\sqrt{\varphi}\sqrt[4]{\varphi}\overset{(1)}{>}\frac{2}{\sqrt[4]{\varphi}}+\sqrt{\varphi}$$

$$(1)\Leftrightarrow\sqrt[4]{\varphi}+\varphi\sqrt{\varphi}\sqrt{\varphi}>2+\sqrt[4]{\varphi^3}$$

Let $\sqrt[4]{\varphi}=x$, *then* $x^8=\varphi^2=\varphi+1=x^4+1$. *Hence,*

$$x+x^8\overset{(2)}{>}2+x^3\Leftrightarrow x^4+x+1>x^3+2\Leftrightarrow$$

$(x^3+1)(x-1)>0$ *true because* $x>0\Rightarrow(2)\Rightarrow(1)$ *it's true.*

Therefore,

$$\frac{2}{\sqrt[4]{\varphi}}+\sqrt{\varphi}<1+\frac{\sqrt{\varphi}+\varphi+\varphi\sqrt{\varphi}}{2}$$

A.035. *Solution by George Florin Şerban-Romania*

Lemma. $det(X+Y)+det(X-Y)=2(det\,X+det\,Y),\forall X,Y\in M_2(\mathbb{R})$

$$det(A+B+C)+det(-A+B+C)+det(A-B+C)+det(A+B-C)=$$

$$=2(det(B+C)+det\,A)+2(det\,A+det(B-C))=$$

$$=2\,det(B+C)+2\,det\,A+2\,det(B-C)=$$

$$=4\,det\,A+2(det(B+C)+det(B-C))=$$

$$=4\,det\,A+4\,det\,B+4\,det\,C\overset{AM-GM}{\geq}4\cdot3\sqrt[3]{det\,A\cdot det\,B\cdot det\,C}=12\sqrt[3]{2}=24$$

A.036. *Solution by Ruxandra Daniela Tonila-Romania*

We have $a^2+b^2+c^2\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{3}\overset{AM-GM}{\geq}3\sqrt[3]{a^2b^2c^2}$

So it is enough to prove that:

$$\left(3\sqrt[3]{a^2b^2c^2}\right)^{a+b+c} \geq (3ab)^c \cdot (3bc)^2 \cdot (3ca)^b$$

$$\Leftrightarrow (abc)^{\frac{2}{3}(a+b+c)} \geq (ab)^c \cdot (bc)^a \cdot (ca)^b$$

$$\Leftrightarrow \frac{(abc)^{a+b+c}}{(abc)^{\frac{a+b+c}{3}}} \geq \frac{(abc)^{a+b+c}}{a^a \cdot b^b \cdot c^c} \Leftrightarrow a^a \cdot b^b \cdot c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Let $f: (0, +\infty) \to \mathbb{R}, f(x) = x \log x$ *with* $f'(x) = \log x + 1,$

$f''(x) > \frac{1}{x} > 0, \forall x > 0 \Rightarrow f$ *convex function. Then*

$$f\left(\frac{a+b+c}{3}\right) \overset{Jensen}{\leq} \frac{f(a)+f(b)+f(c)}{3}$$

$$\frac{a+b+c}{3} \log \frac{a+b+c}{3} \leq \frac{a \log a + b \log b + c \log c}{3}$$

$$3 \log \left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3}} \leq \log(a^a \cdot b^b \cdot c^c)$$

$$3 \log\left(\sqrt[3]{abc}\right)^{\frac{a+b+c}{3}} \overset{AM-GM}{\leq} 3 \log \left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3}}$$

$$\Rightarrow \log(abc)^{\frac{a+b+c}{3}} \leq \log(a^a \cdot b^b \cdot c^c) \Leftrightarrow a^a \cdot b^b \cdot c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Therefore,

$$(3ab)^c \cdot (3bc)^a \cdot (3ca)^b \leq (a^2+b^2+c^2)^{a+b+c}, \forall a, b, c > 0$$

A.037. *Solution by Ravi Prakash-New Delhi-India*

$$32(2x^{12} + x^8 + x^6 + x^4) + 19 = 64x^2(x^8 + 1)$$

$$64x^{12} - 64x^{10} + 32x^8 + 32x^6 + 32x^4 - 64x^2 + 19 = 0 \Leftrightarrow$$

$$(2x^6)^2 - 2(2x^2)^5 + 2(2x^2)^4 + 4(2x^2)^3 + 8(2x^2)2 - 32(2x^2) + 19 = 0$$

$$\text{Put } 2x^2 = y \Rightarrow y^6 - 2y^5 + 2y^4 + 4y^3 + 8y^2 - 32y + 19 = 0$$

$$(y^2 - 2y + 1)y^4 + (y^2 - 2y + 1)y^2 + 6(y^2 - 2y + 1)y + 19(y^2 - 2y + 1) = 0$$

$$(y-1)^2(y^4 + y^2 + 6y + 19) = 0, \qquad (y-1)^2(y^4 + (y+3)^2 + 10) = 0$$

$$\text{As } y \in \mathbb{R}, y^4 + (y+3)^2 + 10 > 0 \Rightarrow (y-1)^2 = 0 \Rightarrow y = 1 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}.$$

A.038. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$(a+b)^4(8a^2x^4 + ab(x+y)^4 + 8b^2y^4) \overset{AGM}{\geq}$$

$$\geq (a+b)^4\left(8a^2x^4 + ab\left(2\sqrt{xy}\right)^4 + 8b^2y^4\right) =$$

$$= 8(a+b)^2(a^2 + 2ab + b^2)(a^2x^4 + 2abx^2y^2 + b^2y^4) \overset{CBS}{\geq}$$

$$\geq 8(a+b)^2(a^2x^2 + 2abxy + b^2y^2)^2 \overset{AGM}{\geq}$$

$$\geq 8 \cdot 4ab(ax+by)^4 = 32ab(ax+by)^4$$

Therefore,

$$32ab(ax+by)^4 \leq (a+b)^4(8a^2x^4 + ab(x+y)^4 + 8b^2y^4)$$

Equality holds for $a = b$.

A.039. *Solution by Ravi Prakash-New Delhi-India*

$$\text{Let} \begin{cases} \vec{x} = a\vec{i} + b\vec{j} + c\vec{k} \\ \vec{y} = d\vec{i} + e\vec{j} + f\vec{k} \end{cases} \Rightarrow \begin{cases} |\vec{x}|^2 = a^2 + b^2 + c^2 \\ |\vec{y}|^2 = d^2 + e^2 + f^2 \end{cases}$$

$$\vec{x} \cdot \vec{j} = ad + be + cf$$

$$|\vec{x}|^2 \cdot |\vec{y}|^2 - (\vec{x} \cdot \vec{y})^2 = (\vec{x} \times \vec{y})^2$$

$$(a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix}^2 =$$

$$= (bf - ce)^2 + (cd - af)^2 + (ae - bd)^2$$

But $1 \le b, c, e, f \le 2$, *then* $1 \le bf, ce \le 4$ *and* $-3 \le bf - ce \le 3$,

$|bf - ce| \le 3$, *then* $(bf - ce)^2 \le 9$

Similarly, $(cd - af)^2 \le 9$ *and* $(ae - bd)^2 \le 9$. *Thus,*

$$(a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2 \le 27$$

Therefore,

$$(a^2 + b^2 + c^2)(d^2 + e^2 + f^2) \le (ad + be + cf)^2 + 27$$

A.040. *Solution by Soumava Chakraborty-Kolkata-India*

$$\forall\, m, n \ge 0, (m - n)^2 \ge 0 \Rightarrow 4mn \le (m + n)^2 \Rightarrow \sqrt{4mn} \le m + n$$

$$\Rightarrow \sqrt[4]{4mn} \overset{(*)}{\le} \sqrt{m + n} \text{ and } \sqrt{4mn} \le m + n \Rightarrow \sqrt{mn} \overset{(**)}{\le} \frac{m + n}{2}$$

$$Now, \sqrt[4]{4xyz}\left(\sqrt[4]{x} + \sqrt[4]{y}\right)$$

$$= \sqrt[4]{(2y)(2z)}.\sqrt{x} + \sqrt[4]{(2z)(2x)}.\sqrt{y} \overset{via\,(*)}{\le} \sqrt{y + z}.\sqrt{x}$$

$$+ \sqrt{z + x}.\sqrt{y} \overset{via\,(**)}{\le} \frac{y + z + x}{2} + \frac{z + x + y}{2} = x + y + z$$

$$\therefore \sqrt[4]{4xyz}\left(\sqrt[4]{x} + \sqrt[4]{y}\right) \le x + y + z \text{ with equality iff } 2y = 2z \text{ and } 2z$$

$$= 2x \text{ and } y + z \overset{(i)}{=} x \text{ and } z + x \overset{(i)}{=} y \Rightarrow iff\ x = y$$

$$= z \text{ and } y + z = y - z\ \big(from\ (i) \text{ and } (ii)\big)$$

$$\Rightarrow iff\ x = y = z \text{ and } z = 0 \Rightarrow iff\ x = y = z = 0 \text{ and}$$

$$\because \text{given equation is that equality case}$$

$$\therefore \boxed{x = y = z = 0} \text{ is sole } sol^n \text{ to given equation } (ans)$$

A.041. *Solution by Hikmat Mammadov-Azerbaijan*

$$\varphi^2 = \varphi + 1 \Leftrightarrow \frac{1}{\varphi} = \varphi - 1 \Leftrightarrow \varphi^3 = \varphi^2 + 1 = 2\varphi + 1$$

$$\frac{\varphi\sqrt{\varphi} + 1}{\sqrt{\varphi}\left(1 + \sqrt{\varphi}\right)} = \varphi\sqrt{\varphi} - 1; \frac{\varphi}{1 + \varphi} = \varphi - 1$$

$$\frac{2}{\sqrt{\varphi}\left(1+\sqrt{\varphi}+\varphi\right)}=\ \varphi+\sqrt{\varphi}(\varphi-1)-2$$

$$\frac{\varphi\sqrt{\varphi}+1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)}+\frac{\varphi}{1+\varphi}+\frac{2}{\sqrt{\varphi}\left(1+\sqrt{\varphi}+\varphi\right)}$$
$$=\varphi\sqrt{\varphi}-1+\varphi-1+\sqrt{\varphi}(\varphi-1)-2=$$

$$=2\varphi-4+(2\varphi-1)\sqrt{\varphi}\Rightarrow 2\varphi-1=\sqrt{5}=\sqrt{5}-3+\sqrt{5\varphi}$$

Now, we will prove that:

$$\sqrt{5}-3+\sqrt{5\varphi}>2\Leftrightarrow\sqrt{\varphi}>\sqrt{5}-1\Leftrightarrow\varphi>6-2\sqrt{5}$$

$$\frac{1}{2}+\frac{\sqrt{5}}{2}>6-2\sqrt{5}\Rightarrow 1+\sqrt{5}>12-4\sqrt{5}$$

$$\sqrt{5}>\frac{11}{5}\Rightarrow 5>\frac{121}{25}\Leftrightarrow\frac{125}{25}>\frac{121}{25}(true.)$$

A.042. *Solution by Tapas Das-India*

Let $f(x)=\sin x-x$*, then* $f'(x)=\cos x-1<0,\forall x\in(0,\pi)\Rightarrow$
f *−decreasing*

Let $g(t)=\sin t$ *and* $t_1,t_1\in(0,\pi),t_2>t_1$*, hence*

$$g(t_2)-g(t_1)=\sin t_2-\sin t_1=2\cos\left(\frac{t_1+t_2}{2}\right)\sin\left(\frac{t_2-t_1}{2}\right)>0$$

$$g(t_2)>g(t_1)\Rightarrow g\text{ −increasing}$$

$$\frac{a+b}{2}\geq\sqrt{ab}\Rightarrow\sin\left(\frac{a+b}{2}\right)\geq\sin\left(\sqrt{ab}\right)$$

$$\sin x<x\Rightarrow\sin\left(\frac{a+b}{2}\right)<\frac{a+b}{2}\ and\ \sin\left(\sqrt{ab}\right)<\sqrt{ab},hence:$$

$$\frac{\sin\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}\leq\frac{\frac{a+b}{2}}{\sqrt{ab}}\ or$$

$$\frac{\sin\sqrt{ab}}{\sin\left(\frac{a+b}{2}\right)} \geq \frac{\sqrt{ab}}{\frac{a+b}{2}} = \frac{(a+b)^4\sqrt{ab}}{(a+b)^4(a+b)} \geq \frac{2\sqrt{ab}\left(2\sqrt{ba}\right)^4}{(a+b)^5}$$

Therefore,

$$\frac{\sin\left(\sqrt{ab}\right)}{\sin\left(\frac{a+b}{2}\right)} \geq \frac{32a^2b^2\sqrt{ab}}{(a+b)^5}$$

A.043. *Solution by Ravi Prakash-New Delhi-India*

For $a, b, c > 0, 3(a^2+b^2+c^2) - (a+b+c)^2 =$

$$= 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$$
$$= (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$$

Equality holds when $a = b = c$.

$$E = \frac{a^2+b^2+c^2}{(a+b+c)^2} \geq \frac{1}{3}$$

Now, for $a, b, c > 0$*, we have:*

$$e^{\frac{a^2}{(a+b+c)^2}} + e^{\frac{b^2}{(a+b+c)^2}} + e^{\frac{c^2}{(a+b+c)^2}} \geq 3e^{\frac{E}{3}} = 3e^{\frac{1}{a}}$$

Equality holds when $a = b = c$.

A.044. *Solution by Adrian Popa-Romania*

$$\begin{cases} \log 2 + \log 5 > \log 10 > \log 3 \\ \log 2 + \log 3 = \log 6 > \log 5 \\ \log 3 + \log 5 = \log 15 > \log 2 \end{cases}$$

Let $a = \log 2\,; b = \log 3\,; c = \log 5$. *We see that* a, b, c *can be the sides of a triangle.*

We must show:

$$(a^2+b^2+c^2)^2 > 3(a+b+c)(b+c-a)(a+b-c)(a+c-b)$$

$$(a^2+b^2+c^2)^2 > 3\cdot 2s[2(s-a)\cdot 2(s-b)\cdot 2(s-c)]$$

$$(a^2+b^2+c^2)^2 \ge 48s(s-a)(s-b)(s-c) \Leftrightarrow$$

$$(a^2+b^2+c^2)^2 \ge 48F^2 \Leftrightarrow a^2+b^2+c^2 \ge 4\sqrt{3}F(Ionescu-Weitzenbock)$$

A.045. *Solution by Soumava Chakraborty-Kolkata-India*

$$If\ a=b=0, then, m_h=\frac{2ab}{a+b}\ is\ undefined$$
$$\Rightarrow \boxed{a=b=0\ isn't\ a\ possibility}\ and\ if\ one\ of\ a,b = 0\ (say\ a=0\ and\ b>0), then: m_h=m_g = 0\ and\ m_a=\frac{a+b}{2}=\frac{b}{2}$$

$$\Rightarrow (m_h+m_g+m_a)\sqrt{\frac{a^2+b^2}{2}} \le m_hm_g+m_gm_a+m_am_h+\frac{a^2+b^2}{2}$$

$$\Leftrightarrow \frac{b}{2}.\sqrt{\frac{b^2}{2}} \le \frac{b^2}{2} \Leftrightarrow \frac{b^2}{4}.\frac{b^2}{2} \le \frac{b^4}{4} \Leftrightarrow \frac{1}{2} \le 1 \rightarrow true$$

$$\therefore \boxed{(m_h+m_g+m_a)\sqrt{\frac{a^2+b^2}{2}} \le m_hm_g+m_gm_a+m_am_h+\frac{a^2+b^2}{2}\ if\ exactly\ one\ of\ a,b=0 \rightarrow (1)}$$

$$We\ now\ shift\ our\ attention\ to\ the\ case\ when\ a,b > 0\ and\ denoting\ m_h\ by\ H, m_g\ by\ G, m_a\ by\ A\ and\ \sqrt{\frac{a^2+b^2}{2}}\ by\ Q,$$

$$we\ notice\ that: AH=\frac{a+b}{2}.\frac{2ab}{a+b}=ab=G^2 \Rightarrow H \overset{(i)}{=} \frac{G^2}{A}$$

$$and\ Q^2+G^2=\frac{a^2+b^2}{2}+ab=2.\frac{(a+b)^2}{4}=2A^2$$

$$\Rightarrow Q^2 \overset{(ii)}{=} 2A^2-G^2\ and\ A^2-G^2=\frac{(a+b)^2}{4}-ab$$

$$=\frac{(a-b)^2}{4} \ge 0 \Rightarrow A^2 \ge G^2 \Rightarrow A \ge G$$

$$\Rightarrow \frac{A}{G} \overset{(iii)}{\ge} 1\ (G \ne 0\ as\ a,b>0)$$

$$Now, via\ for\ mentioned\ substitutions,$$

$$(m_h + m_g + m_a)\sqrt{\frac{a^2+b^2}{2}} \le m_h m_g + m_g m_a + m_a m_h + \frac{a^2+b^2}{2}$$

$$\Leftrightarrow (H+G+A)Q \le HG + GA + AH + Q^2$$

$$\overset{via\ (i),(ii)}{\Leftrightarrow} \left(\frac{G^2}{A} + G + A\right).\sqrt{2A^2 - G^2} \le \frac{G^2}{A}.G + GA + A.\frac{G^2}{A} + 2A^2 - G^2$$

$$\Leftrightarrow (G^2 + AG + A^2).\sqrt{2A^2 - G^2} \le G^3 + GA^2 + 2A^3$$

$$\Leftrightarrow (G^3 + GA^2 + 2A^3)^2 \ge (2A^2 - G^2)(G^2 + AG + A^2)^2$$

$$\overset{expanding\ and\ re-arranging}{\Leftrightarrow} 2A^6 - 4A^4G^2 + 2A^3G^3 + 3A^2G^4 + 2AG^5 + 2G^6 \ge 0 \Leftrightarrow 2t^6 - 4t^4 + 2t^3 + 3t^2 + 2t + 2 \ge 0 \ \left(t = \frac{A}{G}\right)$$

$$\Leftrightarrow (t-1)(2t^5 + 2(t-1)t^3 + 3t + 5) + 7 \ge 0 \to$$

$$true \because t = \frac{A}{G} \overset{via\ (iii)}{\ge} 1$$

$$\Rightarrow \boxed{(m_h + m_g + m_a)\sqrt{\frac{a^2+b^2}{2}} \le m_h m_g + m_g m_a + m_a m_h + \frac{a^2+b^2}{2}\ \forall\, a,b > 0} \to (2)$$

$$\therefore combining\ (1)\ and\ (2), (m_h + m_g + m_a)\sqrt{\frac{a^2+b^2}{2}} \le m_h m_g + m_g m_a + m_a m_h + \frac{a^2+b^2}{2}\ \forall\, a,b \ge 0\ (QED)$$

A.046. *Solution by Ravi Prakash-New Delhi-India*

$$(a+b)^{2\sqrt{ab}} \le 4^{\sqrt{ab}}\left(\sqrt{ab}\right)^{a+b}, \qquad (a+b)^{2\sqrt{ab}} \le 2^{2\sqrt{ab}}(ab)^{\frac{a+b}{2}}$$

$$\left(\frac{a+b}{2}\right)^{2\sqrt{ab}} \le (ab)^{\frac{a+b}{2}}, \qquad \left(\frac{a+b}{2}\right)^{\frac{2}{a+b}} \le \left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}; (1)$$

$$Let\ f(x) = x^{\frac{1}{x}}, x \ge e \Rightarrow log\ f(x) = \frac{1}{x} log\ x$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x^2}(1 - \log x) < 0, \forall x > e$$

f —is a decreasing function on $[e, \infty)$

As $a, b \geq e, \frac{a+b}{2} \geq \sqrt{ab} \geq e \Rightarrow f\left(\frac{a+b}{2}\right) \leq f(\sqrt{ab})$

$$(a+b)^{2\sqrt{ab}} \leq 4^{\sqrt{ab}} \cdot \left(\sqrt{ab}\right)^{a+b}$$

A.047. *Solution by Vivek Kumar-India*

We have: $2^x - 1 \geq 0, 2^y - 1 \geq 0, 2^z - 1 \geq 0$, so

$$(2^x - 1)(2^y - 1)(2^z - 1) \geq 0$$

$$2^x + 2^y + 2^z + 2^{x+y+z} - 2^{x+y} - 2^{y+z} - 2^{z+x} - 1 \geq 0$$

$$2^x + 2^y + 2^z + 2^{x+y+z} \geq 2^{x+y} + 2^{y+z} + 2^{z+x} + 1$$

Also, we have:

$$2^{x+y} \geq 2^{2\sqrt{xy}} = 4^{\sqrt{xy}}$$

$$2^{y+z} \geq 2^{2\sqrt{yz}} = 4^{\sqrt{yz}}$$

$$2^{z+x} \geq 2^{2\sqrt{zx}} = 4^{\sqrt{zx}}$$

Equality holds for $x = y = z = 0$. Therefore,

$$2^x + 2^y + 2^z + 2^{x+y+z} \geq 4^{\sqrt{xy}} + 4^{\sqrt{yz}} + 4^{\sqrt{zx}} + 1$$

A.048. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$If\ a = b = 1\ we\ have:\ (a+b-1)^{a+b-1} + 1 = a^a + b^b.\ Assume\ now\ that\ a \neq b.$$

$$Let\ f(x) = x^x,\ \ x > 0.\ We\ have:$$

$$f''(x) = x^{x-1} + (\log x + 1)^2 . x^x > 0\ then\ f\ is\ convex\ on\ (0, \infty).$$

$$Let\ \ t = \frac{b-1}{b-a} \in [0,1].\ We\ have: 1 = t.a + (1-t).b\ \ and\ \ a+b-1 = (1-t).a + t.b.$$

$$Then\ by\ Jensen's\ inequality\ we\ have:$$

$$1 = f(1) = f(t.a + (1-t).b) \le t.f(a) + (1-t).f(b) = t.a^a + (1-t).b^b$$

$$And:\ \ (a+b-1)^{a+b-1} = f(a+b-1) = f\big((1-t).a + t.b\big) \le (1-t).a^a + t.b^b$$

$$Summing\ up\ the\ two\ inequalities, we\ obtain:$$

$$(a+b-1)^{a+b-1} + 1 \le a^a + b^b,\ \ the\ desired\ result.$$

A.049.

$$x \in [a,b] \Rightarrow (x-a)(x-b) \le 0 \Rightarrow x^2 - (a+b)x + ab \le 0$$

$$x^2 + ab \le (a+b)x \Rightarrow x + \frac{ab}{x} \le a+b; (1)$$

Analogous,

$$z + \frac{ab}{z} \le a+b; (2)$$

$$y \in [c,d] \Rightarrow (y-c)(y-d) \le 0 \Rightarrow y^2 - (c+d)y + cd \le 0$$

$$y^2 + cd \le (c+d)y \Rightarrow y + \frac{cd}{y} \le c+d; (3)$$

Analogous,

$$t + \frac{cd}{t} \le c+d; (4)$$

$$x+y+z+t+ab\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right) \overset{AGM}{\ge} 2\sqrt{(x+y+z+t)ab\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}; (5)$$

By adding (1),(2),(3) and (4), it follows that

$$x+y+z+t+ab\left(\frac{1}{x}+\frac{1}{z}\right)+cd\left(\frac{1}{y}+\frac{1}{t}\right)\leq 2(a+b+c+d);(6)$$

By (5) and (6):

$$2\sqrt{(x+y+z+t)ab\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}\leq 2(a+b+c+d)$$

$$ab(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\leq (a+b+c+d)^2$$

Equality holds for $x=t=z=t; ab=2(a+b)$.

A.050. *Solution by Ruxandra Daniela Tonilă-Romania*

Using: $x^2+xy+y^2\geq\frac{3}{4}(x+y)^2;\forall x,y>0$*, for* $x=1$ *and* $y=\sqrt{\varphi}$*, we get:*

$$1+\sqrt{\varphi}+\varphi\geq\frac{3}{4}\left(\sqrt{\varphi}+1\right)^2\Leftrightarrow$$

$$\frac{1}{1+\varphi+\sqrt{\varphi}}\leq\frac{4}{3\left(\sqrt{\varphi}+1\right)^2}\overset{AGM}{\leq}\frac{1}{3\sqrt{\varphi}}\Leftrightarrow\frac{12\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}\leq\frac{12\sqrt{\varphi}}{3\sqrt{\varphi}}=4$$

We must to prove:

$$\sqrt[3]{\varphi(1+4\varphi)(4+\varphi)}>4\Leftrightarrow(\varphi+4\varphi^2)(4+\varphi)>64$$

Since φ *–is roof of the eqution* $x^2-x-1=0$*, we have* $\varphi^2=\varphi+1\Rightarrow$

$$(5\varphi+4)(4+\varphi)>64\Leftrightarrow 5\varphi^2+24\varphi+16>64\Leftrightarrow$$

$$29\varphi+21>64\Leftrightarrow\varphi>\frac{43}{29}$$

But $\varphi=\frac{1+\sqrt{5}}{2}>\frac{3}{2}>\frac{43}{29}$

Therefore, $\sqrt[3]{\varphi(1+4\varphi)(4+\varphi)}>4\geq\frac{12\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}$

We have: $1+4\varphi \overset{CBS}{\leq} \sqrt{(1^2+4^2)(1^2+\varphi^2)} = \sqrt{17(\varphi^2+1)}$ *and*

$4+\varphi \overset{CBS}{\leq} \sqrt{(4^2+1^2)(1^2+\varphi^2)} = \sqrt{17(\varphi^2+1)}$. *Hence,*

$$\sqrt[3]{\varphi(1+4\varphi)(4+\varphi)} \leq \sqrt[3]{17\varphi(\varphi^2+1)}$$

Now, we must to prove $\sqrt[3]{17\varphi(\varphi^2+1)} < 3\varphi \Leftrightarrow$

$$17\varphi(\varphi^2+1) < 27\varphi^3 \Leftrightarrow 10\varphi^2 > 17 \Leftrightarrow \varphi^2 > \frac{17}{10} \Leftrightarrow \varphi > 0{,}7 \; true\ from\ \varphi > 0{,}5$$

Therefore,

$$\frac{12\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi} < \sqrt[3]{\varphi(1+4\varphi)(4+\varphi)} < 3\varphi$$

A.051.

$$\sum_{cyc}\left(x+\frac{1}{y}\right)^5 \overset{AGM}{\geq} 3\cdot\sqrt[3]{\prod_{cyc}\left(x+\frac{1}{y}\right)^5} = 3\left(\prod_{cyc}\left(x+\frac{1}{y}\right)\right)^{\frac{5}{3}} =$$

$$= 3\left(\prod_{cyc}\left(x+9\cdot\frac{1}{9y}\right)\right)^{\frac{5}{3}} \overset{AGM}{\geq} 3\left(10\prod_{cyc}\sqrt[10]{x\cdot\left(\frac{1}{9y}\right)^9}\right)^{\frac{5}{3}} =$$

$$= 3\cdot 10^{\frac{5}{3}\cdot 3}\left(\prod_{cyc}\frac{x^{\frac{1}{10}}}{9^{\frac{9}{10}}\cdot y^{\frac{9}{10}}}\right)^{\frac{5}{3}} = 3\cdot 10^5\cdot\frac{1}{9^{\frac{9}{10}\cdot 3\cdot\frac{5}{3}}}\left(\prod_{cyc}\frac{1}{x^{\frac{8}{10}}}\right)^{\frac{5}{3}} \overset{AGM}{\geq}$$

$$\geq 3\cdot 10^5\cdot\frac{1}{9^{\frac{9}{2}}}\cdot\frac{1}{\left(\frac{x+y+z}{3}\right)^{3\cdot\frac{8}{10}\cdot\frac{5}{3}}} = 3\cdot 10^5\cdot\frac{1}{3^9}\cdot\frac{1}{\left(\frac{1}{3}\right)^4} = 3\cdot 10^4\cdot\frac{1}{3^5} =$$

$$= \frac{10^5}{81} = \frac{100.000}{81}$$

Equality holds for $x = y = z = \frac{1}{3}$.

A.052. *Solution by Tapas Das-India*

$$\frac{x^2}{\sqrt{1-x^2}}+\frac{y^2}{\sqrt{1-y^2}}+\frac{z^2}{\sqrt{1-z^2}}=$$

$$=\frac{1}{2}\left(\frac{x^2}{\sqrt{1-x^2}}+\frac{y^2}{\sqrt{1-y^2}}\right)+\frac{1}{2}\left(\frac{y^2}{\sqrt{1-y^2}}+\frac{z^2}{\sqrt{1-z^2}}\right)+\frac{1}{2}\left(\frac{z^2}{\sqrt{1-z^2}}+\frac{x^2}{\sqrt{1-x^2}}\right)\geq$$

$$\geq\frac{1}{2}\cdot\frac{2xy}{\sqrt[4]{(1-x^2)(1-y^2)}}+\frac{1}{2}\cdot\frac{2yz}{\sqrt[4]{(1-y^2)(1-z^2)}}+\frac{1}{2}\cdot\frac{2zx}{\sqrt[4]{(1-z^2)(1-x^2)}}=$$

$$=\frac{xy}{\sqrt[4]{(1-x^2)(1-y^2)}}+\frac{yz}{\sqrt[4]{(1-y^2)(1-z^2)}}+\frac{zx}{\sqrt[4]{(1-z^2)(1-x^2)}}=$$

$$\geq\frac{xy}{\sqrt{1-xy}}+\frac{yz}{\sqrt{1-yz}}+\frac{zx}{\sqrt{1-zx}}$$

$$\because (1-x^2)(1-y^2)-(1-xy)^2=2xy-x^2-y^2=-(x-y)^2<0\ (and\ analogs)$$

A.053. *Solution by Mohamed Amine Ben ajiba-Tanger-Morocco*

$$\sum_{cyc}\frac{2a^3+3a^2+b}{(2a+1)(a+b+1)}\geq\sum_{cyc}\frac{2ab^2+b^2+2ab+a}{(2a+1)(a+b+1)};\ (*)$$

$$We\ have:\ (2a^3+3a^2+b)-(2ab^2+b^2+2ab+a)=2a(a^2-b^2)+3a(a-b)+b(a-b)-(a-b)=$$

$$=(a-b)[2a(a+b)+3a+b-1]=(a-b)[2a(a+b+1)+(a+b+1)-2]=$$

$$= (a-b)(2a+1)(a+b+1) - 2(a-b)$$
$$= (a-b)(2a+1)(a+b+1)$$
$$- 2[(2a+1) - (a+b+1)]$$

$$Then: \ (*) \leftrightarrow \sum_{cyc} \frac{(2a^3+3a^2+b) - (2ab^2+b^2+2ab+a)}{(2a+1)(a+b+1)} \geq 0$$

$$\leftrightarrow \sum_{cyc} \frac{(a-b)(2a+1)(a+b+1) - 2(2a+1) + 2(a+b+1)}{(2a+1)(a+b+1)} \geq 0$$

$$\leftrightarrow \sum_{cyc} \left((a-b) - \frac{2}{a+b+1} + \frac{2}{2a+1} \right) \geq 0$$

$$\leftrightarrow \sum_{cyc} \frac{2}{2a+1} \geq \sum_{cyc} \frac{2}{a+b+1} \leftrightarrow$$

$$\sum_{cyc} \left(\frac{1}{2a+1} + \frac{1}{2b+1} \right) \geq \sum_{cyc} \frac{2}{a+b+1}$$

$$Which\ is\ true\ from\ CBS\ inequality$$

$$\therefore \frac{1}{2a+1} + \frac{1}{2b+1} \geq \frac{(1+1)^2}{(2a+1)+(2b+1)} = \frac{2}{a+b+1}\ (And\ analogs)$$

$$Equality\ holds\ if\ a=b=c.$$

A.054. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Since\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1\ then\ by\ Weighted\ AM-GM\ we\ have:$$

$$\frac{1}{p}.a^p + \frac{1}{q}.b^q + \frac{1}{r}.c^r \geq (a^p)^{\frac{1}{p}}.(b^q)^{\frac{1}{q}}.(c^r)^{\frac{1}{r}} = abc.$$

$$Multiplying\ the\ both\ sides\ by\ pqr\ we\ get:$$

$$qra^p + rpb^q + pqc^r \geq abcpqr$$

A.055. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$By\ Radon's\ inequality:$$

$$■\ 2x^2+y^2=\frac{x^2}{\frac{1}{2}}+\frac{y^2}{1}\ge\frac{(x+y)^2}{\frac{1}{2}+1}\ \Rightarrow\ \frac{(x+y)^2}{2x^2+y^2}\le\frac{3}{2}$$

$$■\ 4x^3+y^3=\frac{x^3}{\left(\frac{1}{2}\right)^2}+\frac{y^3}{1^2}\ge\frac{(x+y)^3}{\left(\frac{1}{2}+1\right)^2}\ \Rightarrow\ \frac{(x+y)^3}{4x^3+y^3}\le\left(\frac{3}{2}\right)^2$$

$$■\ 16x^5+y^5=\frac{x^5}{\left(\frac{1}{2}\right)^4}+\frac{y^5}{1^4}\ge\frac{(x+y)^5}{\left(\frac{1}{2}+1\right)^4}\ \Rightarrow\ \frac{(x+y)^5}{16x^5+y^5}\le\left(\frac{3}{2}\right)^4$$

$$Multiplying\ these\ inequalities\ we\ get:$$

$$\frac{(x+y)^{10}}{(2x^2+y^2)(4x^3+y^3)(16x^5+y^5)}\le\left(\frac{3}{2}\right)^7$$

$$=\frac{2817}{128}.\ Equality\ holds\ iff\ \frac{x}{\frac{1}{2}}=\frac{y}{1}\ \Leftrightarrow\ y=2x.$$

$$Now\ we\ have:\ \ 2^x+log_6 y=9\ \Rightarrow\ 2^x+log_6(2x)=9.$$

$$The\ function\ x\to 2^x+log_6(2x)\ is\ strictly\ increasing\ on\ (0,\infty)\ and$$

$$2^3+log_6(2.3)=9.$$

$$Therefore,\qquad x=3\ \ and\ \ y=6.$$

A.056. *Solution by Ruxandra Daniela Tonilă-Romania*

$$n+\frac{1}{n}\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}=n+\frac{1}{n}\sum_{1\le i<j\le n}\frac{i+j-2\sqrt{ij}}{\sqrt{ij}}=$$

$$= n + \frac{1}{n} \sum_{1 \le i < j \le n} \left(\sqrt{\frac{i}{j}} + \sqrt{\frac{j}{i}} - 2 \right) = n + \frac{1}{n} \left[\sum_{1 \le i < j \le n} \left(\sqrt{\frac{i}{j}} + \sqrt{\frac{j}{i}} \right) - 2\binom{n}{2} \right] =$$

$$n - \frac{1}{n} \cdot \frac{2n(n-1)}{2} + \frac{1}{n} \cdot \sum_{1 \le i < j \le n} \left(\sqrt{\frac{i}{j}} + \sqrt{\frac{j}{i}} \right) =$$

$$= 1 + \frac{1}{n} \cdot \sum_{1 \le i < j \le n} \left(\sqrt{\frac{i}{j}} + \sqrt{\frac{j}{i}} \right) = \frac{1}{n} \left[n + \sum_{1 \le i < j \le n} \left(\sqrt{\frac{i}{j}} + \sqrt{\frac{j}{i}} \right) \right] =$$

$$= \frac{1}{n} \left[\sum_{i=1}^{n} \sqrt{\frac{i}{i}} + \sum_{1 \le i < j \le n} \left(\sqrt{\frac{i}{j}} + \sqrt{\frac{j}{i}} \right) \right] = \frac{1}{n} \left[\sum_{k=1}^{n} \sqrt{k} \cdot \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right] \overset{AGM}{\ge}$$

$$\ge \sqrt[n]{\prod_{k=1}^{n} \sqrt{k}} \cdot \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \sqrt[2n]{n!} \cdot \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

Therefore,

$$n + \frac{1}{n} \sum_{1 \le i < j \le n} \frac{\left(\sqrt{i} - \sqrt{j}\right)^2}{\sqrt{ij}} \ge \sqrt[2n]{n!} \cdot \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

A.057. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$xyzt(x+y+z+t)^2 \le 2(xy+zt)(xz+yt)(xt+yz); (*)$$

$$RHS_{(*)} = 2(x^2yz + xy^2t + xz^2t + yzt^2)(xt+yz)$$
$$= 2xyzt(x^2+y^2+z^2+t^2)$$
$$+ 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2)$$

$$\to (*) \leftrightarrow xyzt(x+y+z+t)^2$$
$$\le 2xyzt(x^2+y^2+z^2+t^2)$$
$$+ 2(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2)$$

$$\leftrightarrow 2xyzt(xy+xz+xt+yz+yt+zt) \le xyzt(x^2+y^2+z^2+t^2) + 2(x^2y^2z^2+x^2y^2t^2+x^2z^2t^2+y^2z^2t^2)$$

$$\leftrightarrow xyzt[(x-y)^2+(z-t)^2]+x^2z^2(y-t)^2+x^2t^2(y-z)^2 + y^2z^2(x-t)^2+y^2t^2(x-z)^2 \ge 0 \text{ which is true.}$$

$$Therefore,$$

$$xyzt(x+y+z+t)^2 \le 2(xy+zt)(xz+yt)(xt+yz)$$

$$\text{with equality iff } x=y=z=t.$$

A.058. *Solution by Adrian Popa-Romania*

$$\text{Let } |x-1|=a, |y-2|=b, |z-3|=c; a,b,c \ge 0$$

$$\frac{a^2}{2}+\frac{b^2}{4}+\frac{c^2}{6}+3=a+b+c$$

$$6a^2+3b^2+2c^2+36=12a+12b+12c$$

$$6(a-1)^2+3(b-2)^2+2(c-3)^2=0 \Rightarrow a=1; b=2; c=3.$$

$$|x-1|=1 \Rightarrow x=0, x=2$$

$$|y-2|=0 \Rightarrow y=0, y=4$$

$$|z-3|=3 \Rightarrow z=0, z=6.$$

$$S=\{(x,y,z)|x\in\{0,2\}, y\in\{0,4\}, z\in\{0,6\}\}$$

A.059. *Solution by Ravi Prakash-New Delhi-India*

$$y=\frac{4x^4}{x^6+x^4+x^2+1} \ge 0$$

$$z=\frac{5y^5}{y^8+y^6+y^4+y^2+1} \ge 0$$

$$x=\frac{6z^6}{z^{10}+z^8+z^6+z^4+z^2+1} \ge 0$$

If $x = 0$, *then* $y = 0$ *and* $z = 0$.

Assume $x > 0$,

$$y = \frac{4x^4}{x^6 + x^4 + x^2 + 1} \le \frac{4x^4}{4(x^6 \cdot x^4 \cdot x^2 \cdot 1)^{\frac{1}{4}}} \Rightarrow y \le \frac{x^4}{x^3} = x$$

Similarly, $z \le y$ *and* $x \le z$.

$$x \le y \le z \le x \Rightarrow x = y = z$$

$$y = \frac{4x^4}{x^6 + x^4 + x^2 + 1} \Rightarrow 4x^3 = x^6 + x^4 + x^2 + 1$$

$$\left(x^3 + \frac{1}{x^3} - 2\right) + \left(x + \frac{1}{x} - 2\right) = 0$$

$$\left(x^{\frac{3}{2}} - \frac{1}{x^{\frac{3}{2}}}\right)^2 + \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = 0 \Rightarrow x^{\frac{3}{2}} - \frac{1}{x^{\frac{3}{2}}} = 0 \text{ and } \sqrt{x} - \frac{1}{\sqrt{x}} = 0$$

$$\Rightarrow x^3 = 1 \text{ and } x = 1. S = \{(0,0,0), (1,1,1)\}$$

A.060. *Solution by Asmat Qatea-Afghanistan*

$$\begin{cases} x + y + 3 = 3xy; (*) \\ \dfrac{x^2 - y^2}{xy - 1} + \dfrac{x^2 - 9}{3x - 1} + \dfrac{y^2 - 9}{3y - 1} = 0; (**) \end{cases}$$

$$(*) \Rightarrow x + y = 3(xy - 1) \Rightarrow x = \frac{y + 3}{3y - 1}, y = \frac{x + 3}{3x - 1}$$

$$(**) \Rightarrow \frac{(x - y)(x + y)}{xy - 1} + \frac{(x - 3)(x + 3)}{3x - 1} + \frac{(y + 3)(y - 3)}{3y - 1} = 0$$

$$3(x - y) + \frac{(x - 3)(x + 3)}{3x - 1} + x(y - 3) = 0$$

$$y(x - 3) + \frac{(x - 3)(x + 3)}{3x - 1} = 0$$

$$(x-3)2\left(\frac{x+3}{3x-2}\right)=0\Rightarrow\begin{cases}x=3\\x=-3\end{cases}\Rightarrow\begin{cases}y=\frac{3}{4}\\y=0\end{cases}$$

A.061. *Solution by Asmat Qatea-Afghanistan*

$$(1+1+1)(x^2+1+y^2+1+z^2+1)\overset{Holder}{\geq}\left(\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\right)^2$$
$$=\left(3\sqrt{2}\right)^2$$

$$x^2+y^2+z^2+3\geq 6\Rightarrow x^2+y^2+z^2+6\geq 9\overset{(?)}{\geq}3(x+y+z)$$

$$x+y+z\overset{?}{\leq}3;(1)$$

We have: $\sqrt{x^2+1}\overset{RMS-AM}{\geq}\frac{x+1}{\sqrt{2}}$

$$\sum_{cyc}\sqrt{x^2+1}\geq\frac{x+y+z+3}{\sqrt{2}}\Rightarrow 3\sqrt{2}\geq\frac{x+y+z+3}{\sqrt{2}}$$

$$x+y+z\leq 3\Rightarrow(1)\ is\ true.$$

A.062. *Solution by Ravi Prakash-New Delhi-India*

$$x+\frac{9}{[x]}=\frac{6}{1+x-[x]};(1)$$

As $x\geq[x],1+x-[x]\geq 1\Rightarrow\frac{6}{1+x-[x]}\leq 6\Rightarrow x+\frac{9}{[x]}\leq 6;(2)$

Also, $\frac{6}{1+x-[x]}>0\Rightarrow x+\frac{9}{[x]}>0\Rightarrow x>0.$

$$[x]=0,\forall x\in[0,1)$$

$\because x\geq 1.$ *As* $[x]\leq x,\frac{9}{[x]}\geq\frac{9}{x}$. *Thus,* $x+\frac{9}{[x]}\geq x+\frac{9}{x}\geq 6;(3)$

From (1),(2),(3) we get $x+\frac{9}{[x]}=\frac{6}{1+x-[x]}\Rightarrow x=[x].$

From (1), $x+\frac{9}{x}=6\Rightarrow x=3.$ *Similarly,*

$$y+\frac{16}{[y]}=\frac{8}{1+y-[y]}\Rightarrow y=4$$

Now, $z+2^z+log_2 z=x+y\Rightarrow z+2^z+log_2 z=7;(4)\Rightarrow z>0$ *and if* $1<z<2,$

we have $z+2^z+log_2 z\le 3\neq 7.$

For $z=2, z+2^z+log_2 z=7.$ *For* $z>2, z+2^z+log_2 z>7.$

Therefore, $x=3, y=4, z=2.$

A.063. *Solution by Ravi Prakash-New Delhi-India*

We show that if $x\ge 1$, *then* $\frac{x^3+1}{\sqrt{x^2-x+1}}\ge x^2+1\Leftrightarrow$

$$\frac{(x+1)(x^2-x+1)}{\sqrt{x^2-x+1}}\ge x^2+1\Leftrightarrow (x+1)^2(x^2-x+1)\ge (x^2+1)^2$$
$$\Leftrightarrow$$

$(x+1)(x^3+1)\ge x^4+2x^2+1\Rightarrow^3+x\ge 2x^2$, *which is true as AM-GM inequality.Thus,*

$$\frac{x^3+1}{\sqrt{x^2-x+1}}+\frac{y^3+1}{\sqrt{y^2-y+1}}+\frac{z^3+1}{\sqrt{z^2-z+1}}\ge x^2+y^2+z^2+3=6$$

A.064. *Solution by Adrian Popa-Romania*

$$4(x+1)^{x+1}\cdot(y+1)^{y+1}\cdot(x+y)^{x+y}\le x^x\cdot y^y\cdot(x+y+2)^{x+y+2}$$

$$\left(\frac{x+1}{x}\right)^x\cdot\left(\frac{y+1}{y}\right)^y\cdot(x+1)\cdot(y+1)\le\left(\frac{x+y+2}{x+y}\right)^{x+y}\cdot\left(\frac{x+y+2}{2}\right)^2$$

Now,

$$\left(\frac{x+1}{x}\right)^x\cdot\left(\frac{y+1}{y}\right)^y\overset{AGM}{\le}\left(\frac{x\cdot\frac{x+1}{x}+y\cdot\frac{y+1}{y}}{x+y}\right)^{x+y}=\left(\frac{x+y+2}{x+y}\right)^{x+y};(1)$$

$$(x+1)(y+1) \overset{AGM}{\leq} \left(\frac{x+1+y+1}{2}\right)^2 = \left(\frac{x+y+2}{2}\right)^2 ;(2)$$

By multiply (1) and (2) it follows

$$4(x+1)^{x+1} \cdot (y+1)^{y+1} \cdot (x+y)^{x+y} \leq x^x \cdot y^y \cdot (x+y+2)^{x+y+2}$$

A.065. *Solution by Soumava Chakraborty-Kolkata-India*

$$\left(\left(\frac{a+2b}{3}\right)^{2a+b} . \left(\frac{2a+b}{3}\right)^{a+2b}\right)^{3ab} \leq \left(a^b . b^a\right)^{(2a+b)(a+2b)}$$

$$\Leftrightarrow ln\left(\left(\left(\frac{a+2b}{3}\right)^{2a+b} . \left(\frac{2a+b}{3}\right)^{a+2b}\right)^{3ab}\right)$$

$$\leq ln\left(\left(a^b . b^a\right)^{(2a+b)(a+2b)}\right)$$

$$\Leftrightarrow 3ab\left((2a+b)ln\left(\frac{a+2b}{3}\right) + (a+2b)ln\left(\frac{2a+b}{3}\right)\right)$$

$$\leq (2a+b)(a+2b)(blna + alnb)$$

$$\Leftrightarrow \frac{lnm}{m} + \frac{lnn}{n} \overset{(*)}{\leq} \frac{lna}{a} + \frac{lnb}{b} \left(m = \frac{a+2b}{3}, n = \frac{2a+b}{3}\right)$$

$\boxed{Case\ 1}$ $a \geq b$ and then : $a + 2b \geq 3b \Rightarrow m \geq b$ and also : $3a \geq 2a + b$

$\Rightarrow a \geq n$ and moreover : $2a + b \geq a + 2b \Rightarrow n \geq m$

$\therefore$ we can choose c_1, c_2 such that :

$b \leq c_1 \leq m$ and $n \leq c_2 \leq a$ $\therefore$ we arrive at the following chain

: $\boxed{\boxed{e\sqrt{e} \leq b \leq c_1 \leq m \leq n \leq c_2 \leq a}}$

$$Now, (*) \Leftrightarrow \frac{lnm}{m} - \frac{lnb}{b} \leq \frac{lna}{a} - \frac{lnn}{n} \overset{via\ MVT}{\Leftrightarrow} (m-b)F'(c_1)$$

$$\leq (a-n)F'(c_2) \left(where\ F(x) = \frac{lnx}{x}\ \forall\, x \in \left[e\sqrt{e}, \infty\right)\right.$$

$$\left.\Rightarrow F'(x) = \frac{1-lnx}{x^2}\right) \Leftrightarrow \xi\left(\frac{1-lnc_2}{c_2^2} - \frac{1-lnc_1}{c_1^2}\right) \geq 0$$

$$\left(\because m+n=a+b\Rightarrow m-b=a-n=\xi\ (say)\right)\overset{\because\xi\geq 0}{\Leftrightarrow}\frac{1-lnc_2}{c_2^2}-\frac{1-lnc_1}{c_1^2}$$

$$\geq 0\Leftrightarrow\frac{lnc_1-1}{c_1^2}\geq\frac{lnc_2-1}{c_2^2}$$

$$\Leftrightarrow\frac{c_2^2}{lnc_2-1}\overset{(i)}{\geq}\frac{c_1^2}{lnc_1-1}\left(\because lnc_1-1,lnc_2-1\geq\frac{3}{2}>0\right.$$

$$\left.\because c_1,c_2\geq e\sqrt{e}\right)$$

$$Now, let\ f(x)=\frac{x^2}{lnx-1}\ \forall x\in[e\sqrt{e},\infty)\therefore f'(x)=\frac{x(2lnx-3)}{(lnx-1)^2}$$

$$\geq 0\left(\because x\geq e\sqrt{e}\Rightarrow lnx\geq\frac{3}{2}\right)\Rightarrow f(x)\ is\uparrow on\ [e\sqrt{e},\infty)\therefore as\ c_2$$

$$\geq c_1\geq e\sqrt{e}, so, f(c_2)\geq f(c_1)\Rightarrow(i)\Rightarrow(*)\ is\ true$$

$$\therefore\boxed{\forall a\geq b\geq e\sqrt{e}:\left(\left(\frac{a+2b}{3}\right)^{2a+b}\cdot\left(\frac{2a+b}{3}\right)^{a+2b}\right)^{3ab}\leq\left(a^b\cdot b^a\right)^{(2a+b)(a+2b)}}$$

$\boxed{Case\ 2}$ $a\leq b\ and\ then: a+2b\leq 3b\Rightarrow m\leq b\ and\ also: 3a\leq 2a+b$

$$\Rightarrow a\leq n\ and\ moreover: 2a+b\leq a+2b\Rightarrow n\leq m$$

$$\therefore we\ can\ choose\ c_1,c_2\ such\ that:$$

$$m\leq c_1\leq b\ and\ a\leq c_2\leq n\therefore we\ arrive\ at\ the\ following\ chain$$

$$:\boxed{\boxed{e\sqrt{e}\leq a\leq c_2\leq n\leq m\leq c_1\leq b}}$$

$$Now, (*)\Leftrightarrow\frac{lnb}{b}-\frac{lnm}{m}\geq\frac{lnn}{n}-\frac{lna}{a}\overset{via\ MVT}{\Leftrightarrow}(b-m)F'(c_1)$$

$$\geq(n-a)F'(c_2)\left(where\ F(x)=\frac{lnx}{x}\ \forall x\in[e\sqrt{e},\infty)\right.$$

$$\left.\Rightarrow F'(x)=\frac{1-lnx}{x^2}\right)\Leftrightarrow\xi\left(\frac{1-lnc_1}{c_1^2}-\frac{1-lnc_2}{c_2^2}\right)\geq 0$$

$$\left(\because m+n=a+b\Rightarrow b-m=n-a=\xi\ (say)\right)\overset{\because\xi\geq 0}{\Leftrightarrow}\frac{1-lnc_1}{c_1^2}-\frac{1-lnc_2}{c_2^2}$$

$$\geq 0\Leftrightarrow\frac{lnc_2-1}{c_2^2}\geq\frac{lnc_1-1}{c_1^2}$$

$$\Leftrightarrow\frac{c_2^2}{lnc_2-1}\overset{(ii)}{\leq}\frac{c_1^2}{lnc_1-1}\left(\because lnc_1-1,lnc_2-1\geq\frac{3}{2}>0\right.$$

$$\left.\because c_1,c_2\geq e\sqrt{e}\right)$$

$$Now, let\ f(x)=\frac{x^2}{lnx-1}\ \forall\ x\in\ \left[e\sqrt{e},\infty\right)\ \therefore\ f'(x)=\frac{x(2lnx-3)}{(lnx-1)^2}$$
$$\geq 0\left(\because\ x\geq e\sqrt{e}\Rightarrow lnx\geq\frac{3}{2}\right)\Rightarrow f(x)\ is\uparrow on\ \left[e\sqrt{e},\infty\right)\ \therefore\ as\ c_1$$
$$\geq c_2\geq e\sqrt{e}, so, f(c_1)\geq f(c_2)\Rightarrow(ii)\Rightarrow(*)\ is\ true$$

$$\therefore\ \boxed{\forall\ b\geq a\geq e\sqrt{e}:\left(\left(\frac{a+2b}{3}\right)^{2a+b}.\left(\frac{2a+b}{3}\right)^{a+2b}\right)^{3ab}\leq\left(a^b.b^a\right)^{(2a+b)(a+2b)}}$$

$$\therefore\ combining\ cases\ (1),(2),\forall\ a,b\geq e\sqrt{e}, then$$
$$:\left(\left(\frac{a+2b}{3}\right)^{2a+b}.\left(\frac{2a+b}{3}\right)^{a+2b}\right)^{3ab}$$
$$\leq\left(a^b.b^a\right)^{(2a+b)(a+2b)}\ (QED)$$

A.066. *Solution by Khaled Abd Imouti-Damascus-Syria*

$$E(n): 1+\sum_{i=1}^{n}\frac{1}{\sqrt[3]{1+x_i}}\leq n+\frac{1}{\sqrt[3]{1+x_1+x_2+\cdots+x_n}}$$

$$E(1): 1+\frac{1}{\sqrt[3]{1+x_1}}\leq 1+\frac{1}{\sqrt[3]{1+x_1}}\ (true.)$$

Suppose that above issue is true for $n\geq 1$ *and let us prove is true for* $n+1$.

$$E(n+1): 1+\sum_{i=1}^{n+1}\frac{1}{\sqrt[3]{1+x_i}}\leq n+1+\frac{1}{\sqrt[3]{1+x_1+x_2+\cdots+x_n+x_{n+1}}}$$

$$1+\sum_{i=1}^{n+1}\frac{1}{\sqrt[3]{1+x_i}}=\left(1+\sum_{i=1}^{n}\frac{1}{\sqrt[3]{1+x_i}}\right)+\frac{1}{\sqrt[3]{1+x_{n+1}}}\leq$$

$$\leq n+\frac{1}{\sqrt[3]{1+x_1+x_2+\cdots+x_n}}+\frac{1}{\sqrt[3]{1+x_{n+1}}}\overset{(1)}{\leq} n+1$$
$$+\frac{1}{\sqrt[3]{1+x_1+x_2+\cdots+x_n+x_{n+1}}}$$

Let $a=x_1+x_2+\cdots+x_n$ *and* $b=x_{n+1}$, *then:*

$$(1) \Leftrightarrow \frac{1}{\sqrt[3]{1+a}} + \frac{1}{\sqrt[3]{1+b}} \leq \frac{1}{\sqrt[3]{1+a+b}} + 1$$

Let be the function: $f(a) = \frac{1}{\sqrt[3]{1+a}} + \frac{1}{\sqrt[3]{1+b}} - \frac{1}{\sqrt[3]{1+a+b}}; a, b \geq 0, f(0) = 1$

$$\lim_{a \to \infty} f(a) = \frac{1}{\sqrt[3]{1+b}} \leq 1$$

a	0		$+\infty$
$f'(a)$		$- - - - - - - - - -$	
$f(a)$	1	$\searrow$	$\frac{1}{\sqrt[3]{1+b}}$

$$f'(a) = \frac{1}{3}\left(-\frac{1}{\sqrt[3]{(1+a)^4}} + \frac{1}{\sqrt[3]{(1+a+b)^4}}\right) < 0 \Rightarrow (2) \Rightarrow (1)(true.)$$

$$1 + \sum_{i=1}^{n} \frac{1}{\sqrt[3]{1+x_i}} \leq n + \frac{1}{\sqrt[3]{1+x_1+x_2+\cdots+x_n}}$$

A.067. *Solution by Tapas Das-India*

$$\sqrt{a+\sqrt{b+\sqrt{c}}} + \sqrt{b+\sqrt{c+\sqrt{a}}} + \sqrt{c+\sqrt{a+\sqrt{b}}} \leq$$

$$\leq \sqrt{3}\left[a+b+c+\left(\sqrt{b+\sqrt{c}} + \sqrt{c+\sqrt{a}} + \sqrt{a+\sqrt{b}}\right)\right]^{\frac{1}{2}}$$

$$= \sqrt{3}\left[3+\left(\sqrt{b+\sqrt{c}} + \sqrt{c+\sqrt{a}} + \sqrt{a+\sqrt{b}}\right)\right]^{\frac{1}{2}}$$

$(\because a+b+c=3)$ *(1)*

Now, $\sqrt{b+\sqrt{c}} + \sqrt{c+\sqrt{a}} + \sqrt{a+\sqrt{b}} \leq$

$$\leq \sqrt{3}\left[(a+b+c)+\left(\sqrt{c}+\sqrt{a}+\sqrt{b}\right)\right]^{\frac{1}{2}}=\sqrt{3}\left[3+\left(\sqrt{c}+\sqrt{a}+\sqrt{b}\right)\right]^{\frac{1}{2}}$$

$$\leq \sqrt{3}\left[3+\sqrt{3}(a+b+c)^{\frac{1}{2}}\right]^{\frac{1}{2}} \leq \sqrt{3}\left[3+\sqrt{3}\cdot\sqrt{3}\right]^{\frac{1}{2}}$$

$$\leq \sqrt{3}(6)^{\frac{1}{2}}=\sqrt{3}\cdot\sqrt{3}\cdot\sqrt{2}=3\sqrt{2}$$

From (1)

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}} \leq$$

$$\leq \sqrt{3}\left[3+3\sqrt{2}\right]^{\frac{1}{2}}=\sqrt{3}\cdot\sqrt{3}\left(1+\sqrt{2}\right)^{\frac{1}{2}}=3\sqrt{1+\sqrt{2}}$$

Note:

$$\frac{x_1^m+x_2^m+\cdots+x_n^m}{n} \leq \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^m$$

when m was between 0 *and* 1*.*

A.068.

$$\frac{x^4+y^4}{x^2+y^2} \geq x^2-xy+y^2 \Leftrightarrow x^4+y^4 \geq (x^2-xy+y^2)(x^2+y^2) \Leftrightarrow$$

$$x^4+y64 \geq x^4+x^2y^2-x^3y-xy^3+x^2y^2+y^4$$

$$x^3y+xy^3-2x^2y^2 \geq 0 \Leftrightarrow xy(x^2-2xy+y^2) \geq 0$$

$$\frac{x^4+y^4}{x^2+y^2} \geq x^2-xy+y^2; \ \ (1)$$

Analogous:

$$\frac{y^4+z^4}{y^2+z^2} \geq y^2-yz+z^2; \ \ (2)$$

$$\frac{z^4+x^4}{z^2+x^2} \geq z^2 - zx + x^2; \quad (3)$$

By adding (1),(2) and (3), we get:

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2} \geq 2(x^2+y^2+z^2)-(xy+yz+zx)$$

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2}+xy+yz+zx \geq 2(x^2+y62+z^2)$$
$$= 2\cdot 3 = 6$$

Equality holds for $x=y=z=1$.

A.069.

$$(x+3)(x-1)\,det\begin{pmatrix}1000 & 100 & 10 & 1\\ 100 & 10 & 1 & 1000\\ 10 & 1 & 1000 & 100\\ 1 & 1000 & 100 & 10\end{pmatrix}\begin{pmatrix}x & 1 & 1 & 1\\ 1 & x & 1 & 1\\ 1 & 1 & x & 1\\ 1 & 1 & 1 & x\end{pmatrix}+$$

$$+(y+3)(y-1)\begin{pmatrix}1000 & 100 & 10 & 1\\ 100 & 10 & 1 & 1000\\ 10 & 1 & 1000 & 100\\ 1 & 1000 & 100 & 10\end{pmatrix}\begin{pmatrix}y & 1 & 1 & 1\\ 1 & y & 1 & 1\\ 1 & 1 & y & 1\\ 1 & 1 & 1 & y\end{pmatrix}=0$$

$$\left((x+3)(x-1)(x+3)(x-1)^3+(y+3)(y-1)(y+3)(y-1)^3\right)$$
$$\cdot\begin{vmatrix}1000 & 100 & 10 & 1\\ 100 & 10 & 1 & 1000\\ 10 & 1 & 1000 & 100\\ 1 & 1000 & 100 & 10\end{vmatrix}=0$$

$$\left((x+3)(x-1)^2\right)^2+\left((y+3)(y-1)^2\right)^2=0$$

$$x-1=y-1=0 \Rightarrow x=y=1.$$

A.070. *Let be* $f:(0,\infty)\to\mathbb{R}, f(x)=x-e^{x-1}, f'(x)=1-e^{x-1}$

$$f'(x)=0 \Rightarrow 1-e^{x-1}=0 \Leftrightarrow e^{x-1}=1 \Leftrightarrow x-1=0 \Leftrightarrow x=1$$

x	0		1		∞
$f'(x)$			0		
$f(x)$		$\nearrow$	M	$\searrow$	

$$M = \max_{x>0} f(x) = f(1) = 1 - e^{1-1} = 1 - 1 = 0$$

$$\Rightarrow f(x) \le 0 \Rightarrow x \le e^{x-1} \Rightarrow x \le \frac{1}{e}e^x \Rightarrow \frac{e^{x^2}}{x^2} \ge e$$

For $x = \frac{c}{a} \Rightarrow \frac{e^{\left(\frac{c}{a}\right)^2}}{\left(\frac{c}{a}\right)^2} \ge e \Rightarrow \left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} \ge e; (1)$

Analogous:

$$\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} \ge e;\ \ (2)$$

$$\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge e;\ \ (3)$$

By adding (1), (2) and (3), we get:

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2} + \left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2} + \left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2} \ge 3e$$

Equality holds for $a = b = c$.

A.071. $(a^2 + c^2)(b^2 + c^2) = a^2b^2 + (a^2 + b^2)c^2 + c^4 \overset{AGM}{\ge}$

$$\ge a^2b^2 + 2abc^2 + c^4 = (ab + c^2)^2$$

$$(a^2 + c^2)(b^2 + c^2) \ge (ab + c^2)^2; (1)$$

$$(a^2 + c^2)(b^2 + c^2) = a^2b^2 + (a^2 + b^2)c^2 + c^4 \overset{AGM}{\ge}$$

$$\ge 2abc^2 + a^2c^2 + b^2c^2 = (ac + bc)^2; (2)$$

By (1) and (1), we get:

$$\left((a^2+c^2)(b^2+c^2)\right)^2 \geq (ab+c^2)^2(ac+bc)^2$$

$$(a^2+c^2)(b^2+c^2) \geq (ab+c^2)(ab+bc)$$

$$\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} \geq 1, \qquad \sum_{cyc} \frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)} \geq 3$$

Equality holds for $a=b=c$.

A.072. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Let\ x=\frac{a+b}{2}, \qquad y=\sqrt{ab}, \qquad z=\frac{2ab}{a+b}.$$

$$We\ have\ \ x\geq y\geq z\ \ with\ equality\ holds\ when\ a=b.$$

$$If\ a=b=1\ we\ have\ the\ equality.$$

$$Assume\ that\ a\neq 1\ or\ b\neq 1\ then\ a\neq b\ and\ x>y>z.$$

$$Let\ f(t)=e^{t^2}, t\geq 0.$$

$$We\ have:\ f''(t)=2(2t^2+1)e^{t^2}>0\ then\ f\ is\ convex\ on\ (0,\infty)$$

$$The\ problem\ becomes\ to\ prove:$$

$$f(x-y+z)+f(y)\leq f(x)+f(z)$$

$$By\ Jensen's\ inequality, we\ have:$$

$$f(y)=f\left(\frac{y-z}{x-z}.x+\frac{x-y}{x-z}.z\right)\leq \frac{y-z}{x-z}.f(x)+\frac{x-y}{x-z}.f(z)$$

$$\left(\because \frac{y-z}{x-z},\frac{x-y}{x-z}>0\ \&\ \frac{y-z}{x-z}+\frac{x-y}{x-z}=1\right)$$

$$f(x-y+z)=f\left(\frac{x-y}{x-z}.x+\frac{y-z}{x-z}.z\right)\leq \frac{x-y}{x-z}.f(x)+\frac{y-z}{x-z}.f(z)$$

$$Summing\ up\ these\ inequality, we\ obtain:\ f(x-y+z)+f(y) \leq f(x)+f(z)$$

$$Therefore,\qquad e^{\left(\frac{a+b}{2}-\sqrt{ab}+\frac{2ab}{a+b}\right)^2}+e^{ab}\le\sqrt[4]{e^{(a+b)^2}}+e^{\frac{4a^2b^2}{(a+b)^2}}.$$

A.073. *Solution by Fayssal Abdelli-Bejaia-Algerie*

$$Let\ a=\sqrt[5]{307-x};\ b=\sqrt[5]{x-63}\Rightarrow\frac{a^5b-ab^5}{a-b}=120\Rightarrow\frac{ab(a^4-b^4)}{a-b}=120$$

$$\Rightarrow\frac{ab(a-b)}{a-b}(a^3+a^2b+ab^2+b^3)=120$$

$$ab(a^3+a^2b+ab^2+b^3)=120$$

$$ab((a+b)^3-2a^2b-2ab^2)=120$$

$$ab[(a+b)^3-2ab(a+b)]=120$$

$$ab[(a+b)[(a+b)^2-2ab]=120$$

$$\Rightarrow ab(a+b)(a^2+b^2)=120=(1\cdot 3)(1+3)(1^2+3^2)\Rightarrow(a,b)\in\{(1,3);(3,1)\}$$

$$\begin{cases}a=1\\b=3\end{cases}\Rightarrow\begin{cases}\sqrt[5]{307-x}=1\\\sqrt[5]{x-63}=3\end{cases}\Rightarrow x=306$$

$$\begin{cases}a=3\\b=1\end{cases}\Rightarrow\begin{cases}\sqrt[5]{307-x}=3\\\sqrt[5]{x-63}=1\end{cases}\Rightarrow x=64$$

A.074. *Solution by Ravi Prakash-New Delhi-India*

Adding all the equations, we get $2(x^3+y^3+z^3)=40404-\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\Rightarrow$

$$x^3+y^3+z^3=20202-\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right);(1)$$

From this equation and second equation, we get $x^3=2-\sqrt[3]{x}^3+\sqrt[3]{x}=2;(2)$

For $x\le 0, LHS\le 0\ne RHS$. *For* $0<x<1, LHS<2<RHS$.

For $x=1, LHS=2=RHS$.

For $x > 1, LHS > 2 = RHS$.

So, the only solution for (2) is $x = 1$. *From (1) and the third equation, we get*

$$y^3 = 514 - \sqrt[3]{y} \Rightarrow y^3 + \sqrt[3]{y} = 514; (3)$$

For $y \leq 0, LHS_{(3)} \leq 0 \neq RHS.$

For $0 < y < 8, LHS_{(3)} < 514 = RHS$.

For $y = 8, LHS_{(3)} = 514 = RHS$.

For $y > 8, LHS_{(3)} > 514 = RHS$.

Thus, the only solution of (3) is $y = 8$. *Lastly, from (1) and the first equation,*

$z^3 = 19686 - \sqrt[3]{z}$. *As above, it can shown,* $z = 27$.

Therefore: $S = \{x = 1, y = 8. z = 27\}$

A.075. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$If\ a \geq b \ \rightarrow a^5 \geq b^5 \ and\ a^2 \geq b^2$$

$$\rightarrow 2(a^7 + b^7) \overset{Chebyshev}{\geq} (a^5 + b^5)(a^2 + b^2)$$

$$\rightarrow \frac{a^2 + b^2}{a^7 + b^7} \leq \frac{2}{a^5 + b^5} \overset{AM-GM}{\leq} \frac{1}{\sqrt{a^5 b^5}} = \sqrt{c^5} \ (And\ analogs)$$

$$\rightarrow \sum_{cyc} \frac{a^2 + b^2}{a^7 + b^7} \leq \sum_{cyc} \sqrt{a^5} \overset{CBS}{\leq} \sqrt{3 \sum_{cyc} a^5} \overset{?}{\leq} \sum_{cyc} a^5 \leftrightarrow 3 \leq \sum_{cyc} a^5$$

$$Which\ is\ true\ from\ AM - GM, \quad \sum_{cyc} a^5 \geq 3\sqrt[3]{(abc)^5} = 3$$

$$Therefore, \quad \sum_{cyc} \frac{a^2 + b^2}{a^7 + b^7} \leq \sum_{cyc} a^5.$$

A.076. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$(*) \leftrightarrow 3^{12}a^ab^bc^cd^d \geq (b+c+d)^{b+c+d}.(c+d+a)^{c+d+a}.(d+a+b)^{d+a+b}.(a+b+c)^{a+b+c}$$

$$\leftrightarrow 3^{12} \geq \prod_{a,b,c,d}\left(\frac{(a+b+c)(a+b+d)(a+c+d)}{a}\right)^a$$

$$\leftrightarrow 3^3 \overset{(1)}{\geq} \sqrt[a+b+c+d]{\prod_{a,b,c,d}\left(\frac{(a+b+c)(a+b+d)(a+c+d)}{a}\right)^a}$$

$$By\ Weighted\ AM-GM, we\ have:\ RHS_{(1)} \leq \frac{1}{a+b+c+d}\sum_{a,b,c,d} a.\frac{(a+b+c)(a+c+d)(a+b+d)}{a} =$$

$$=\frac{1}{4}\sum_{a,b,c,d}(a+b+c)(a+c+d)(a+b+d)$$

$$By\ Maclaurin's\ inequality, we\ know\ that:$$

$$\frac{xyz+xyt+xzt+yzt}{4} \leq \left(\frac{x+y+z+t}{4}\right)^3, \forall x,y,z,t>0$$

$$\to \frac{1}{4}\sum_{a,b,c,d}(a+b+c)(a+c+d)(a+b+d) \leq \left(\frac{(a+b+c)+(a+c+d)+(a+b+d)+(b+c+d)}{4}\right)^3 =$$

$$=\left(\frac{3}{4}(a+b+c+d)\right)^3 = 3^3 \ \to\ RHS_{(1)} \leq 3^3 \ \to\ (1)\ is\ true.$$

$$Therefore,\quad 3^{12}a^ab^bc^cd^d \geq (4-a)^{4-a}.(4-b)^{4-b}.(4-c)^{4-c}.(4-d)^{4-d}.$$

A.077. *Solution by Ravi Prakash-New Delhi-India*

$$\frac{1}{4}=\frac{1}{4}\left(256^{x^2-y}+256^{y^2-z}+256^{z^2-t}+256^{t^2-x}\right)$$

$$\frac{1}{256}\geq 256^{x^2+y^2+z^2+t^2-x-y-z-t}$$

$$x^2+y^2+z^2+t^2-x-y-z-t\leq -1\Leftrightarrow$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2+\left(t-\frac{1}{2}\right)^2\leq 0$$

$$\Rightarrow x=y=z=t=\frac{1}{2}$$

A.078. *Solution by Ravi Prakash-New Delhi-India*

For $n=0$ *we take* $x>0,y>0$ *then,*

$$LHS=(x+y)^{-1}(x+y)^1\leq\frac{1}{2}\cdot 2\Leftrightarrow 1\leq 1\ true.$$

We now take $n\geq 1,x\geq 0,y\geq 0$. *If* $x=0$, *the inequality becomes*

$y^{n^2-1}y^{n+1}\leq 2^{n-1}y^{n(n+1)}$ *which is true. Similarly for* $y=0$.

So, we assume that $0<y\leq x,n\geq 1$. *We can rewrite the inequality as*

$$x^{n^2-1}(1+t^{n+1})^{n-1}(1+t)^{n+1}x^{n+1}\leq 2^{n-1}x^{n(n+1)}(1+t^n)^{n+1}$$

$$\frac{(1+t^{n+1})^{n-1}(1+t)^{n+1}}{(1+t^n)^{n+1}}\leq 2^{n-1};(i),0<t=\frac{y}{x}\leq 1$$

$$Let:f(t)=\frac{(1+t^{n+1})^{n-1}(1+t)^{n+1}}{(1+t^n)^{n+1}},0\leq t\leq 1$$

$$\log f(t)=(n-1)\log(1+t^{n+1})+(n+1)\log(1+t)-(n+1)\log(1+t^n)$$

$$\frac{f'(t)}{f(t)}=\frac{(n-1)(n+1)t^n}{1+t^{n+1}}+\frac{n+1}{1+t}-\frac{n(n+1)t^{n-1}}{1+t^n}=$$

$$= (n+1)\left[\frac{(n-1)t^n}{1+t^{n+1}} - \frac{nt^{n-1}}{1+t^n} + \frac{n+1}{1+t}\right] =$$

$$= (n+1)\cdot\left[\frac{(n-1)t^n(1+t^n) - nt^{n-1}(1+t^{n+1})}{(1+t^{n+1})(1+t^n)} - \frac{1}{1+t}\right] =$$

$$= (n+1)\cdot\left[\frac{(n-1)t^n + (n-1)t^{2n} - nt^{n-1} - nt^{2n}}{(1+t^{n+1})(1+t^n)} + \frac{1}{1+t}\right] =$$

$$= \frac{(n+1)\{[(n-1)t^n - t^{2n} - nt^{n-1}](1+t) + (1+t^{n+1})(1+t^n)\}}{(1+t^{n+1})(1+t^n)(1+t)} =$$

$$= \frac{(n+1)[nt^{n-1}(t^2-1) + (1-t^{2n})]}{(1+t^{n+1})(1+t^n)(1+t)} =$$

$$= \frac{(n+1)[(1-t^2)(1+t^2+t^4+\dots+t^{2n-2} - nt^{n-1})}{(1+t^{n+1})(1+t^n)(1+t)}$$

But: $1+t^2+t^4+\dots+t^{2n-2} \geq n(t^2\cdot t^4\cdot t^6\cdot\ldots\cdot t^{2n-2})^{\frac{1}{n}} = nt^{n-1}$

Thus, $\frac{f'(t)}{f(t)} \geq 0, \forall t \in (0,1) \Rightarrow f$ *−is maximum when* $t = 1$.

$$\Rightarrow f(t) \leq \frac{2^{n-1}2^{n+1}}{2^{n+1}} = 2^{n-1}$$

A.079. *Solution by Ravi Prakash-New Delhi-India*

Let $f(x) = e^x - x - 1, x \in \mathbb{R}$, *then* $f'(x) = e^x - 1$,

$f'(x) < 0$ *if* $x > 0$; $f'(x) = 0$ *if* $x = 0$ *and* $f'(x) > 0$ *if* $x > 0$.

$f(x) \geq f(0)$ *for* $x \leq 0$ *and* $f(x) \geq f(0)$

for $x \geq 0 \Rightarrow f(x) \geq 0, \forall x \in \mathbb{R}$

Hence, $e^x \geq x + 1, \forall x \in \mathbb{R}$.

Now, $a(e^{2b} + e^{-2c}) + b(e^{2c} + e^{-2a}) + c(e^{2a} + e^{-2b}) \geq$

$\geq a(1+2b+1-2c) + b(1+2c+1-2a) + c(1+2a+1-2b) =$

$$= 2(a+b+c) + 2a(b-c) + 2b(c-a) + 2c(a-b) = 2(a+b+c)$$

A.080. *Solution by Ravi Prakash-New Delhi-India*

$$tan\frac{11\pi}{24} = tan\left(\frac{12\pi}{24} - \frac{\pi}{24}\right) = cot\frac{\pi}{24}$$

$$tan\frac{7\pi}{24} = tan\left(\frac{12\pi}{24} - \frac{5\pi}{24}\right) = cot\frac{5\pi}{24}$$

Hence, $tan\frac{\pi}{24} \cdot tan\frac{11\pi}{24} = 1$ *and* $tan\frac{5\pi}{24} \cdot tan\frac{5\pi}{24} = 1$. *The equation becomes as:*

$$4x^4 - x^3 + 5x^2 - x + 4 = 0; (x^2 \neq 0) \Rightarrow 4x^2 - x + 5 - \frac{1}{x} + \frac{4}{x^2} = 0$$

$$\Leftrightarrow$$

$$4\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) + 5 = 0; \left(y = x + \frac{1}{x} \Rightarrow y^2 = x^2 + \frac{1}{x^2} - 2\right)$$

$$\Rightarrow 4y^2 - y - 3 = 0 \Rightarrow y \in \left\{1, -\frac{3}{4}\right\} \Rightarrow x + \frac{1}{x} \in \left\{1, -\frac{3}{4}\right\}$$

$x + \frac{1}{x} = 1 \Rightarrow x^2 - x + 1 = 0 \Rightarrow x \in \{-\omega, -\omega^2\}$, *where* $\omega \neq 1$ *is cube root of unity.*

$$x + \frac{1}{x} = -\frac{3}{4} \Rightarrow 4x^2 + 3x + 4 = 0 \Rightarrow x_{3,4} = \frac{-3 \pm i\sqrt{55}}{8}$$

A.081. *Solution by Khaled Abd Imouti-Damascus-Syria*

$$\frac{4x^2}{x+y} + \frac{8y^2}{y+z} + \frac{4z^2}{z+x} = 4\left(\frac{x^2}{x+y} + \frac{2y^2}{y+z} + \frac{z^2}{z+x}\right) \geq$$

$$\geq 4 \cdot \frac{(x+y+y+z)^2}{(2(x+y+z)+y+z)} \overset{?}{\geq} 2x + 5y + z$$

$$(2x+4y+2z)^2 \overset{?}{\geq} (2x+2y+2z)(2x+5y+z)$$

$$\Leftrightarrow 6y^2 + 2z^2 + 2xy + 4yz + 2xz \geq$$

Equality holds for $x = y = z = 1$.

A.082. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$\frac{x}{1+x} + \frac{y}{1+y} + \frac{z}{1+z} \geq 3\sqrt[3]{\frac{x}{1+x} \cdot \frac{y}{1+y} \cdot \frac{z}{1+z}}$$
$$\geq \frac{3\sqrt[3]{xyz}}{\frac{(1+x)+(1+y)+(1+z)}{3}} = \frac{\sqrt[3]{xyz}}{1+\frac{x+y+z}{3}}; (1)$$

By QM-AM inequality, we have:

$$\sqrt{x^2+y^2+z^2} \geq \sqrt{3} \cdot \frac{x+y+z}{3}; (2)$$

$$1 + xyz = 1^3 + \sqrt[3]{(xyz)^3} \geq \frac{\left(1+\sqrt[3]{xyz}\right)^3}{2^2} \geq \frac{4 \cdot 1 \cdot \sqrt[3]{xyz}\left(1+\sqrt[3]{xyz}\right)}{4}$$
$$= \sqrt[3]{xyz}\left(1+\sqrt[3]{xyz}\right); (3)$$

Multiplying (1),(2) and (3), we obtain:

$$(1+xyz)\sqrt{x^2+y^2+z^2}\left(\frac{x}{1+x} + \frac{y}{1+y} + \frac{z}{1+z}\right)$$
$$\geq 3\sqrt{3}\sqrt[3]{(xyz)^2}\left(1+\sqrt[3]{xyz}\right) \cdot \frac{\frac{x+y+z}{3}}{1+\frac{x+y+z}{3}}$$

So it suffices to prove that:

$$\frac{\frac{x+y+z}{3}}{1+\frac{x+y+z}{3}} \geq \frac{\sqrt[3]{xyz}}{1+\sqrt[3]{xyz}} \Leftrightarrow \frac{x+y+z}{3} \geq \sqrt[3]{xyz}; (AM-GM)$$

Equality holds for $x = y = z = 1$.

A.083. *Solution by Ruxandra Daniela Tonilă-Romania*

$$a^2 + \cos a + \log\left(\frac{\cos a}{\cos b}\right) \geq b^2 + \cos b \Leftrightarrow$$

$$a^2+\cos a+\log(\cos a)\geq b^2+\cos b+\log(\cos b)$$

We must to prove that:

$$\int_a^b (2x-\sin x-\tan x)\,dx\leq 0$$

Let be the function $f(x)=2x-\tan x-\sin x$*, then*

$f'(x)=2-\frac{1}{\cos^2 x}-\cos x$. *We have:*

$$\frac{1}{\cos^2 x}+\cos x\geq\frac{1}{\cos^2 x}+\cos^2 x\overset{AGM}{\geq}2;\forall x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow$$

$$2-\cos x-\frac{1}{\cos^2 x}\leq 0\Leftrightarrow f'(x)\leq 0;\forall x\in\left(0,\frac{\pi}{2}\right), then\ f(x)\ decreasing. Namely,$$

$$f(x)\leq f(0)=0;\forall x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow\int_a^b f(x)\,dx\leq 0;0<a\leq b<\frac{\pi}{2}$$

$$Therefore,\qquad a^2+\cos a+\log\left(\frac{\cos a}{\cos b}\right)\geq b^2+\cos b$$

A.084. *Solution by Tapas Das-India*

It is well known $e^x>x$. *Need to prove that:*

$$e^{x+e^{-x}}+e^{-x+e^x}\geq 2\cosh x\cdot \operatorname{sech} x\Leftrightarrow$$

$$e^{x+e^{-x}}+e^{-x+e^x}\geq 2$$

$$Now,\ e^{x+e^{-x}}+e^{-x+e^x}\geq 2\sqrt{e^{x+e^{-x}}\cdot e^{-x+e^x}}=2\sqrt{e^{e^{-x}+e^x}}=2e^{\frac{e^{-x}+e^x}{2}}=2e^{\cosh x}$$

$e^{x+e^{-x}}+e^{-x+e^x}\geq 2\cosh x\geq 2$, *where* $\cosh x\geq 1$ *and*

$$e^{\cosh x}\geq\cosh x$$

Let $f(y)=e^y-y; y>0,$ *then* $f'(y)=e^y-1>0; \forall y\in(2,3),$ *namely*

$f(y)$ *increasing function and then* $f(y)>f(0)$*or* $e^y>y$.

A.085.

$$3(a-b)^2+(7a+19b)^2$$
$$=3(a^2-2ab+b^2)+49a^2+361b^2+266ab=$$
$$=52a^2+260ab+364b^2=52(a^2+5ab+7b^2)$$
$$a^2+5ab+7b^2=\frac{1}{52}(3(a-b)^2+(7a+19b)^2)\geq\frac{1}{52}(7a+19b)^2$$
$$\sqrt{a^2+5ab+7b^2}\geq\frac{1}{\sqrt{52}}(7a+19b)$$
$$\sum_{cyc}\sqrt{a^2+5ab+7b^2}\geq\frac{1}{2\sqrt{13}}\left(7\sum_{cyc}a+19\sum_{cyc}b\right)=$$
$$=\frac{1}{2\sqrt{13}}\big(7(a+b+c)+19(a+b+c)\big)=\frac{26}{2\sqrt{13}}(a+b+c)$$
$$=\sqrt{13}(a+b+c)$$

Equality holds for $a=b=c$.

A.086. *Solution by Ravi Prakash-New Delhi-India*

$$f(x)f(yz)+9\leq f(xy)+5f(xz);(1)$$

Put $x=y=z=1$, *then* $f^2(1)+9\leq 6f(1)\Rightarrow(f(1)-3)^2\leq 0\Rightarrow f(1)=3$

Put $y=z=1$, *then* $f(x)f(1)+9\leq 6f(x)\Rightarrow 3f(x)+9\leq 6f(x)$

$$\Rightarrow f(x)\geq 3;(2)$$

Next, put $x=y=z=0$ *in (1)*

$$f^2(0)+9\leq 6f(0)\Rightarrow(f(0)-3)^2\leq 0\Rightarrow f(0)=3$$

In (1) put $y = z = 0$*, we get:* $f(x)f(0) + 9 \leq 6f(0)$

$$3f(x) + 9 \leq 18 \Rightarrow f(3) \leq 3; (3)$$

From (2) and (3), we get: $f(x) = 3, \forall x \in \mathbb{R}$

A.087. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$\begin{cases} x,\ \ y,\ \ z > 1 \\ log_x z + log_y x + log_z y = log_{xy}(yz) + log_{yz}(zx) + log_{zx}(xy)\ \ (1) \\ x + y + z = 6\ \ (2) \end{cases}$$

$$Let\ \ a = log_y x\,,\ \ b = log_z y\,,\ \ c = log_x z\,,\ \ a,b,c > 0\ \ \ and\ \ \ abc = 1.$$

$$We\ have:\ log_{xy}(yz) = \frac{log\,y + log\,z}{log\,x + log\,y} = \frac{1 + log_y z}{log_y x + 1}$$
$$= \frac{1 + ca}{a + 1}\ \ (and\ analogs)$$

$$Then:\ (1)\ \leftrightarrow\ a + b + c = \frac{1 + ca}{a + 1} + \frac{1 + ab}{b + 1} + \frac{1 + bc}{c + 1}$$
$$\leftrightarrow \left(a - \frac{1 + ab}{b + 1}\right) + \left(b - \frac{1 + bc}{c + 1}\right) + \left(c - \frac{1 + ca}{a + 1}\right) = 0$$

$$\leftrightarrow\ \frac{a - 1}{b + 1} + \frac{b - 1}{c + 1} + \frac{c - 1}{a + 1} = 0$$
$$\leftrightarrow\ (c + 1)(a^2 - 1) + (a + 1)(b^2 - 1) + (b + 1)(c^2 - 1) = 0$$

$$\leftrightarrow\ (ca^2 + ab^2 + bc^2) + (a^2 + b^2 + c^2) = (a + b + c) + 3$$

$$By\ AM - GM\ inequality\ we\ have:\ \ ca^2 + ab^2 + bc^2 \geq 3abc = 3$$

$$And\ by\ CBS\ \&\ AM - GM\ inequalities\ we\ have:\ a^2 + b^2 + c^2$$
$$\geq \frac{(a + b + c)^2}{3} \geq \sqrt[3]{abc}(a + b + c) = a + b + c.$$

$$Then:\ (ca^2 + ab^2 + bc^2) + (a^2 + b^2 + c^2) \geq (a + b + c) + 3$$

$$with\ equality\ holds\ iff\ a = b = c = 1.$$

$$Thus,\ \ \ x = y = z\ \ and\ from\ (2)\ we\ get:\ \ x = y = z = 2.$$

A.088. *Solution by Michael Sterghiou-Greece*

$$\sum_{k=2}^{n}\left(H_k+\sqrt[k]{k^{k-1}}\right)<\frac{(n-1)(n+4)}{2};(1)$$

(1) holds for $n=2$ *as* $H_2+\sqrt{2}<3$ *or* $\frac{1}{2}+\sqrt{2}<3$ *which is true.*

Let (1) holds for $n=n,$ *that is*

$$\sum_{k=2}^{n}\left(H_k+\sqrt[k]{k^{k-1}}\right)<\frac{(n-1)(n+4)}{2}$$

We will show that (1) holds for $n+1$ *or*

$$\sum_{k=2}^{n+1}\left(H_k+\sqrt[k]{k^{k-1}}\right)<\frac{n(n+5)}{2};(3)$$

$$(3)\Rightarrow\left(\sum_{k=2}^{n}H_k\right)+\left(\sum_{k=2}^{n}\sqrt[k]{k^{k-1}}\right)+\frac{1}{n+1}+(n+1)^{\frac{n}{n+1}}<\frac{(n-1)(n+4)}{2}+n+2$$

So, as (2) holds by the induction assumption, it is enough to prove that

$$\frac{1}{n+1}+(n+1)^{\frac{n}{n+1}}<(n+2);(4)$$

But $(n+1)^{\frac{n}{n+1}}<n+1$ *so we get the stronger inequality* $\frac{1}{n+1}+n+1<n+2$ *or* $-\frac{n}{n+1}<0,$ *which is true. Therefore, by induction (1) holds.*

A.089. *Solution by Bedri Hajrizi-Mitrovica-Kosovo*

$$log_{ab}\left(1+\sqrt{ab}\right)^2+log_{\frac{a+b}{2}}2\ge log_{\frac{a+b}{2}}(a+b+2)$$

$$log_{ab}\left(1+\sqrt{ab}\right)^2\ge log_{\frac{a+b}{2}}\left(\frac{a+b+2}{2}\right)$$

$$Let\ f(x) = log_x(1+x), x > 1\ then$$

$$f'(x) = \left(\frac{log(1+x)}{log\,x}\right)' = \frac{log\,\frac{x^x}{(1+x)^{1+x}}}{x(1+x)\,log^2 x} < 0$$

Hence, f −decreasing function, then $f(x_1) \le f(x_2); 1 < x_2 \le x_1$.

Therefore,

$$f\left(\frac{a+b}{2}\right) \le f(\sqrt{ab}).$$

A.090.

$$1 = |1| = |1 + a - (a+b) + b| \le |1+a| + |a+b| + |b|$$

$$\frac{1}{|1+a| + |a+b| + |b|} \le 1 \Rightarrow \frac{|c+1|}{|1+a| + |a+b| + |b|} \le |c+1| \le |c| + 1; (1)$$

Analogous:

$$\frac{|a+1|}{|b+1| + |b+c| + |c|} \le |a+1| \le |a| + 1; (2)$$

$$\frac{|b+1|}{|c+1| + |c+a| + |a|} \le |b+1| \le |b| + 1; (3)$$

By adding (1),(2) and (3), we get:

$$\frac{|a+1|}{|b+1| + |b+c| + |c|} + \frac{|b+1|}{|c+1| + |c+a| + |a|} + \frac{|c+1|}{|a+1| + |a+b| + |b|} \ge$$

$$\ge 3 + |a| + |b| + |c|$$

Equality holds for $a = b = c$.

A.091.

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=3;(1)$$

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=3;(2)$$

By adding (1) and (2), we get:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=6$$

$$\left(\frac{a}{b}\right)^2-2+\left(\frac{b}{a}\right)^2+\left(\frac{c}{a}\right)^2-2+\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2-2+\left(\frac{c}{b}\right)^2=0$$

$$\left(\frac{a}{b}-\frac{b}{a}\right)^2+\left(\frac{b}{c}-\frac{c}{b}\right)^2+\left(\frac{c}{a}-\frac{a}{c}\right)^2=0\Leftrightarrow$$

$$\frac{a}{b}=\frac{b}{a},\frac{b}{c}=\frac{c}{b},\frac{c}{a}=\frac{a}{c}\Rightarrow a^2=b^2=c^2\Rightarrow |a|=|b|=|c|$$

$$\Rightarrow\left|\frac{x}{y}\right|=1;\forall x,y\in A\Rightarrow\sum_{a,y\in A}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in A}\left|\frac{x}{y}\right|=9\cdot 1=9$$

Analogous:

$$\left(\frac{u}{v}-\frac{v}{u}\right)^2+\left(\frac{v}{w}-\frac{w}{v}\right)^2+\left(\frac{w}{t}-\frac{t}{w}\right)^{22}+\left(\frac{t}{u}-\frac{u}{t}\right)^2=0$$

$$\Rightarrow\frac{u}{v}=\frac{v}{u},\frac{v}{w}=\frac{w}{v},\frac{w}{t}=\frac{t}{w},\frac{t}{u}=\frac{u}{t}\Rightarrow u^2=v^2=t^2=w^2\Rightarrow$$

$$|u|=|v|=|w|=|t|\Rightarrow\left|\frac{x}{y}\right|=1;\forall x,y\in B$$

$$\Rightarrow\sum_{a,y\in B}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in B}\left|\frac{x}{y}\right|=4\cdot 4\cdot 1=16$$

Therefore,

$$\Omega=\sum_{a,y\in A}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in A}\left|\frac{x}{y}\right|+\sum_{a,y\in B}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in B}\left|\frac{x}{y}\right|=9+16=25$$

A.092. *Let* $f:\mathbb{R}\to(0,\infty), f(x)=e^x$,

then $f'(x)=e^x, f''(x)=e^x>0, f$ *−convexe.*

By Jensen's inequality, we have:

$$\frac{a^x}{a^x+b^x}f(a^{2x})+\frac{b^x}{a^x+b^x}f(b^{2x})\geq f\left(\frac{a^x}{a^x+b^x}\cdot a^{2x}+\frac{b^x}{a^x+b^x}\cdot b^{2x}\right)$$

$$\frac{a^x f(a^{2x})+b^x f(b^{2x})}{a^x+b^x}\geq f\left(\frac{a^{3x}+b^{3x}}{a^x+b^x}\right)$$

$$\frac{a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}}{a^x+b^x}\geq e^{\frac{(a^x+b^x)(a^{2x}-a^xb^x+b^{2x})}{a^x+b^x}}$$

$$a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}\geq(a^x+b^x)\cdot e^{a^{2x}+b^{2x}}\cdot e^{-a^xb^x}$$

Therefore,

$$\left(a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}\right)\cdot e^{a^x\cdot b^x}\geq(a^x+b^x)\cdot e^{a^{2x}+b^{2x}};\forall x\in\mathbb{R}$$

Equality holds for $x=0$.

A.093. $2x^6-2x^5+x^4+x^3+x^2-2x+2=$

$$=2x^6-2x^5+2x^4-x^4+x^3-x^2+2x^2-2x+2=$$

$$=2x64\,(x^2-x+1)-x^2(x^2-x+1)+2(x^2-x+1)=$$

$$=(x^2-x+1)(2x^4-x^2+2)=$$

$$=\left(\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\right)\left(2\left(x^2-\frac{1}{4}\right)^2+\frac{15}{8}\right)\geq\frac{1}{4}\cdot\frac{15}{8}=\frac{15}{32}$$

Hence,

$$2x^6+x^4+x^3+x^2+2\geq\frac{15}{32}+2x^5+2x$$

$$2x^6+x^4+x^3+x^2+\frac{19}{32}\geq 2x^5+2x$$

$$\sum_{cyc}(2x^6+x^4+x^3+x^2)+\frac{57}{32}\geq 2\sum_{cyc}x^5+2(x+y+z)=$$

$$=2\cdot\frac{3}{32}+2(x+y+z)$$

$$\sum_{cyc}(2x^6+x^4+x^3+x^2)+\frac{57-6}{32}\geq 2(x+y+z)$$

$$\sum_{cyc}(2x^6+x^4+x^3+x^2)+\frac{51}{32}\geq 2(x+y+z)$$

Equality holds for $x=y=z=\frac{1}{2}$.

A.094. *Solution by George Florin Șerban-Romania*

$$Let\ log\,x=a; log\,y=b; log\,z=c; x,y,z>0$$

$$\Rightarrow (b+c-a)[a^2-(a+c-b)(a+b-c)]$$
$$=b^2(b-a-c)+c^2(c-a-b)$$

$$(b+c-a)(b-c)^2=b^3-ab^2-b^2c+c^3-ac^2-bc^2\Leftrightarrow$$

$$(b+c-a)(b^2-2bc+c^2)=b^3-ab^2-b^2c+c^3-ac^2-bc^2\Leftrightarrow$$

$$2abc=0\Rightarrow a=0\ or\ b=0\ or\ c=0.$$

If $a=0\Rightarrow log\,x=0\Rightarrow x=1\Rightarrow(1;y;z)$ *–solution.*

If $b=0\Rightarrow log\,y=0\Rightarrow y=1\Rightarrow(x;1;z)$ *–solution.*

If $c=0\Rightarrow log\,z=0\Rightarrow(x;y;1)$ *–solution.*

A.095. *Solution by Tapas Das-India*

$$Let\ f(x)=(2^x+3^x+5^x)(2^{-x}+3^{-x}+5^{-x})=$$

$$= 3 + \left(\frac{2}{3}\right)^x + \left(\frac{2}{5}\right)^x + \left(\frac{3}{2}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{5}{3}\right)^x + \left(\frac{5}{2}\right)^x$$

$$f'(x) = \left(\frac{2}{3}\right)^x log\left(\frac{2}{3}\right) + \left(\frac{2}{5}\right)^x log\left(\frac{2}{5}\right) + \left(\frac{3}{2}\right)^x log\left(\frac{3}{2}\right) + \left(\frac{3}{5}\right)^x log\left(\frac{3}{5}\right) + \left(\frac{5}{3}\right)^x log\left(\frac{5}{3}\right) + \left(\frac{5}{2}\right)^x log\left(\frac{5}{2}\right) =$$

$$= -\left(\frac{2}{3}\right)^x log\left(\frac{3}{2}\right) - \left(\frac{2}{5}\right)^x log\left(\frac{5}{2}\right) + \left(\frac{3}{2}\right)^x log\left(\frac{3}{2}\right) - \left(\frac{3}{5}\right)^x log\left(\frac{5}{3}\right) + \left(\frac{5}{3}\right)^x log\left(\frac{5}{3}\right) + \left(\frac{5}{2}\right)^x log\left(\frac{5}{2}\right) =$$

$$= \left[\left(\frac{3}{2}\right)^x - \left(\frac{2}{3}\right)^x\right] log\left(\frac{3}{2}\right) + \left[\left(\frac{5}{2}\right)^x - \left(\frac{2}{5}\right)^x\right] log\left(\frac{5}{2}\right) + \left[\left(\frac{5}{3}\right)^x - \left(\frac{3}{5}\right)^x\right] log\left(\frac{5}{3}\right) \geq 0$$

Because $\left(\frac{3}{2}\right)^x \geq \left(\frac{2}{3}\right)^x, \left(\frac{5}{3}\right)^x \geq \left(\frac{3}{5}\right)^x, \left(\frac{5}{2}\right)^x \geq \left(\frac{2}{5}\right)^x$ *for all* $x \geq 0$.

So, f *−increasing function and from* $\frac{a+b}{2} \geq \sqrt{ab}$ *we get:*

$$f\left(\frac{a+b}{2}\right) \geq f\left(\sqrt{ab}\right) \Leftrightarrow$$

$$\left(\sqrt{2^{a+b}} + \sqrt{3^{a+b}} + \sqrt{5^{a+b}}\right)\left(\frac{1}{\sqrt{2^{a+b}}} + \frac{1}{\sqrt{3^{a+b}}} + \frac{1}{\sqrt{5^{a+b}}}\right) \geq$$

$$\geq \left(2^{\sqrt{ab}} + 3^{\sqrt{ab}} + 5^{\sqrt{ab}}\right)\left(\frac{1}{2^{\sqrt{ab}}} + \frac{1}{3^{\sqrt{ab}}} + \frac{1}{5^{\sqrt{ab}}}\right)$$

Therefore,

$$\frac{\sqrt{2^{a+b}} + \sqrt{3^{a+b}} + \sqrt{5^{a+b}}}{2^{\sqrt{ab}} + 3^{\sqrt{ab}} + 5^{\sqrt{ab}}} \geq \frac{\frac{1}{2^{\sqrt{ab}}} + \frac{1}{3^{\sqrt{ab}}} + \frac{1}{5^{\sqrt{ab}}}}{\frac{1}{\sqrt{2^{a+b}}} + \frac{1}{\sqrt{3^{a+b}}} + \frac{1}{\sqrt{5^{a+b}}}}$$

A.096. *Solution by Ravi Prakash-New Delhi-India*

$$\left(a^{\sqrt{ab}}+b^{\sqrt{ab}}\right)\sqrt{(a+b)^{a+b}}\geq\left(\sqrt{a^{a+b}}+\sqrt{b^{a+b}}\right)(a+b)^{\sqrt{ab}}\Leftrightarrow$$

$$\left(\frac{a}{a+b}\right)^{\sqrt{ab}}+\left(\frac{b}{a+b}\right)^{\sqrt{ab}}\geq\left(\frac{a}{a+b}\right)^{\frac{a+b}{2}}+\left(\frac{b}{a+b}\right)^{\frac{a+b}{2}};(1)$$

$$As\ 0<\frac{a}{a+b},\frac{b}{a+b}<1\ and\ \frac{a+b}{2}\geq\sqrt{ab}, we\ get:$$

$$\left(\frac{a}{a+b}\right)^{\sqrt{ab}}\geq\left(\frac{a}{a+b}\right)^{\frac{a+b}{2}};(2)\ and\ \left(\frac{b}{a+b}\right)^{\sqrt{ab}}\geq\left(\frac{b}{a+b}\right)^{\frac{a+b}{2}};(3)$$

By adding (2) and (3), we get (1).

A.097. *Let be the function:*

$$f:(0,\infty)\to\mathbb{R}; f(x)=(a^x+b^x+c^x)(a^{-x}+b^{-x}+c^{-x})$$

$$f(x)=3+\sum_{cyc}\left(\left(\frac{a}{b}\right)^x+\left(\frac{b}{a}\right)^x\right)$$

$$f'(x)=\sum_{cyc}\left(\left(\frac{a}{b}\right)^x log\left(\frac{a}{b}\right)+\left(\frac{b}{a}\right)^x log\left(\frac{b}{a}\right)\right)$$

$$f''(x)=\sum_{cyc}\left(\left(\frac{a}{b}\right)^x log^2\left(\frac{a}{b}\right)+\left(\frac{b}{a}\right)^x log^2\left(\frac{b}{a}\right)\right)>0$$

$$\Rightarrow f'\ -increasing\Rightarrow min\, f'(x)=\lim_{x\to 0_+}f'(x)=0\Rightarrow f'(x)>0$$

$$\Rightarrow f\ -increasing$$

$$\sqrt{xy}\leq\frac{x+y}{2}\Rightarrow f\left(\sqrt{xy}\right)\leq f\left(\frac{x+y}{2}\right)$$

$$\left(a^{\sqrt{xy}}+b^{\sqrt{xy}}+c^{\sqrt{xy}}\right)\left(\frac{1}{a^{\sqrt{xy}}}+\frac{1}{b^{\sqrt{xy}}}+\frac{1}{c^{\sqrt{xy}}}\right)\leq$$

$$\leq \left(\sqrt{a^{x+y}}+\sqrt{b^{x+y}}+\sqrt{c^{x+y}}\right)\left(\frac{1}{\sqrt{a^{x+y}}}+\frac{1}{\sqrt{b^{x+y}}}+\frac{1}{\sqrt{c^{x+y}}}\right)$$

Therefore,

$$\frac{a^{\sqrt{xy}}+b^{\sqrt{xy}}+c^{\sqrt{xy}}}{\frac{1}{\sqrt{a^{x+y}}}+\frac{1}{\sqrt{b^{x+y}}}+\frac{1}{\sqrt{c^{x+y}}}} \leq \frac{\sqrt{a^{x+y}}+\sqrt{b^{x+y}}+\sqrt{c^{x+y}}}{\frac{1}{a^{\sqrt{xy}}}+\frac{1}{b^{\sqrt{xy}}}+\frac{1}{c^{\sqrt{xy}}}}$$

A.098. *Solution by Ravi Prakash-New Delhi-India*

$$(x+1)^{x+1} \leq 2x^x 2^x, x > 0$$

Let $f(x) = (x+1)\,log(x+1) - x\,log\,x - x\,log\,2 - log\,2$*, then*

$$f'(x) = 1 + log(x+1) - 1 - log\,x - log\,2 = log\left(1+\frac{1}{x}\right) - log\,2$$

$$f'(x) = \begin{cases} > 0; if\ 0 < x < 1 \\ = 0; if\ x = 1 \\ < 0; if\ x > 1 \end{cases}$$

Thus, f *is increasing on* $[0,1]$ *and decreasing on* $[1,\infty)$.

For $0 < x \leq 1$*, then* $f(x) \leq f(1)$ *and for* $x \geq 1, f(x) \leq f(1)$.

Thus, $f(x) \leq f(1); \forall x > 0$ *and hence,*

$$(x+1)\,log(x+1) - x\,log\,x - x\,log\,2 - log\,2 \leq 0; \forall x > 0 \Leftrightarrow$$

$$(x+1)^{x+1} \leq 2x^x 2^x, x > 0$$

Therefore,

$$8x^x y^y z^z 2^{x+y+z} \geq (x+1)^{x+1}(y+1)^{y+1}(z+1)^{z+1}$$

A.099. *Solution by George Florin Șerban-Romania*

$$Let's\ prove\ that: \frac{x+y}{\sqrt{xy}} + \frac{\sqrt{xy}}{x+y} \geq \frac{5}{2}, let\ x+y = s, xy = p$$

$$\frac{s}{\sqrt{p}}+\frac{\sqrt{p}}{s}\geq\frac{5}{2}\Leftrightarrow 2s^2+2p-5s\sqrt{p}\geq 0\Leftrightarrow\left(s-2\sqrt{p}\right)\left(2s-\sqrt{p}\right)\geq 0$$

true, because $s\overset{AGM}{\geq}2\sqrt{s}$ *and* $2s-\sqrt{p}\geq 4\sqrt{p}-\sqrt{p}=3\sqrt{p}\geq 0$

Equality holds for $s=2\sqrt{p}\Leftrightarrow x=y$.

$$Let'sprove\ that:\frac{x+y+z}{\sqrt[3]{xyz}}+\frac{\sqrt[3]{xyz}}{x+y+z}\geq\frac{10}{3},$$

$$let\ x+y+z=S,xyz=P$$

$$\frac{S}{\sqrt[3]{P}}+\frac{\sqrt[3]{P}}{S}\geq\frac{10}{3}\Rightarrow 3S^2+3\sqrt[3]{P^2}\geq 10S\sqrt[3]{P}\Rightarrow 3S^2-10S\sqrt[3]{P}+3\sqrt[3]{P^2}\geq 0$$

$\left(S-3\sqrt[3]{P}\right)\left(3S-3\sqrt[3]{P}\right)\geq 0$ *true because* $S\overset{AGM}{\geq}3\sqrt[3]{P}$ *and*

$$3S-\sqrt[3]{P}\geq 8\sqrt[3]{P}>0$$

Equality holds for $x=y=z$.*Therefore,*

$$\frac{6(x+y)}{\sqrt{xy}}+\frac{6(x+y+z)}{\sqrt[3]{xyz}}+\frac{6\sqrt{xy}}{x+y}+\frac{6\sqrt[3]{xyz}}{x+y+z}\geq 35$$

Equality holds for $x=y=z=1$.

A.100. *Solution by Tapas Das-India*

$$x^7+2x^6+5x^5+3x^4-16x^3-11x^2-20x-12=0$$

$$x^6(x+2)+5x^4(x+2)-7x^3(x+2)-2x^2(x+2)-7x(x+2)-6(x+2)=0$$

$$(x+2)(x^6+5x^4-7x^3-2x^2-7x-6)=0$$

$$(x+2)(x^6+x^4+4x^4+4x^2-7x^3-7x-6x^2-6)=0$$

$$(x+2)\left(x^4(x^2+1)+4x^2(x^2+1)-7x(x^2+1)-6(x^2+1)\right)=0$$

$$(x+2)(x^2+1)(x^4+4x^2-7x-6)=0$$

$$(x+2)(x^2+1)(x^4-x^3-x^2+x^3-x^2-x+6x^2-6x-6)=0$$

$$(x+2)(x^2+1)(x^2-x-1)(x^2+x+6)=0$$

$$x+2=0\Rightarrow x_1=-2$$

$$x^2+1=0\Rightarrow x^2=-1\Rightarrow x_{2,3}=\pm i$$

$$x^2-x-1=0\Rightarrow x_{4,5}=\frac{1\pm\sqrt{5}}{2}$$

$$x^2+x+6=0\Rightarrow x_{6,7}=\frac{-1\pm i\sqrt{23}}{2}$$

Complex roots are:

$$\left\{\pm i;\ \frac{-1\pm i\sqrt{23}}{2}\right\}$$

A.101. *Solution by Vivek Kumar-India*

$$\frac{(a+x)^2-(b+y)^2-(c+z)^2}{a+x}\ge\frac{a^2-b^2-c^2}{a}+\frac{x^2-y^2-z^2}{x}$$

$$a+x-\frac{(b+y)^2+(c+z)^2}{a+x}\ge a-\frac{b^2+c^2}{a}+x-\frac{y^2+z^2}{x}$$

$$\frac{b^2+c^2}{a}+\frac{y^2+z^2}{x}\ge\frac{(b+y)^2+(c+z)^2}{a+x}$$

$$\frac{b^2+c^2}{a}+\frac{y^2+z^2}{x}=\left(\frac{b^2}{a}+\frac{y^2}{x}\right)+\left(\frac{c^2}{a}+\frac{z^2}{x}\right)\ge$$

$$\ge\frac{(b+y)^2}{a+x}+\frac{(c+z)^2}{a+x}=\frac{(b+y)^2+(c+z)^2}{a+x}$$

A.102. *Solution by Ravi Prakash-New Delhi-India*

Let $b+ic=b_1$*, then* $X=\begin{pmatrix}a_1 & b_1\\ \bar{b}_1 & a_1\end{pmatrix}$*. Let* $Y=\begin{pmatrix}a_2 & b_2\\ \bar{b}_2 & a_2\end{pmatrix}$*, then*

$$XY=\begin{pmatrix} a_1a_2+b_1\bar{b}_2 & a_1b_2+b_1a_2 \\ a_2\bar{b}_1+a_1\bar{b}_2 & \bar{b}_1b_2+a_1a_2 \end{pmatrix}$$

$$Tr(XY)=2a_1a_2+b_1\bar{b}_2+\bar{b}_1b_2=2a_1a_2+2Re\left(b_1\bar{b}_2\right)$$

$$Tr(X^2)=2a_1^2+2Re\left(b_1\bar{b}_2\right)=2a_1^2+2|b_1|^2$$

$$Tr(XY)=2a_1^2+2|b_1|^2$$

$$Tr(XY)\le\frac{1}{2}[Tr(X^2)+Tr(Y^2)]$$

Similarly,

$$Tr(YZ)\le\frac{1}{2}[Tr(Y^2)+Tr(Z^2)]$$

$$Tr(ZX)\le\frac{1}{2}[Tr(Z^2)+Tr(X^2)]$$

Adding three above inequalities, we get:

$$Tr(XY)+Tr(YZ)+Tr(ZX)\le Tr(X^2)+Tr(Y^2)+Tr(Z^2)$$

A.103. *Solution by Ravi Prakash-New Delhi-India*

Let $b+ic=b_1$, *then* $X=\begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & a_1 \end{pmatrix}$. *Let* $Y=\begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & a_2 \end{pmatrix}$, *then*

$$XY=\begin{pmatrix} a_1a_2+b_1\bar{b}_2 & a_1b_2+b_1a_2 \\ a_2\bar{b}_1+a_1\bar{b}_2 & \bar{b}_1b_2+a_1a_2 \end{pmatrix}$$

$$Tr(XY)=2a_1a_2+b_1\bar{b}_2+\bar{b}_1b_2=2a_1a_2+2Re\left(b_1\bar{b}_2\right)$$

$$Tr(X^2)=2a_1^2+2Re\left(b_1\bar{b}_2\right)=2a_1^2+2|b_1|^2$$

Similarly,

$$Tr(Y^2)=2a_2^2+2|b_2|^2$$

$$Tr(X^2)Tr(Y^2)-Tr(XY)^2=$$

$$= 4(a_1^2 + |b_1|^2)(a_2^2 + |b_2|^2) - 4\left(a_1a_2 + Re(b_1\bar{b}_2)\right)^2 =$$

$$= 4\left[a_1^2a_2^2 + a_1^2|b_2|^2 + a_2^2|b_1|^2 + |b_1b_2|^2 - a_1^2a_2^2 - 2a_1a_2Re(b_1\bar{b}_2) - \left(Re(b_1\bar{b}_2)\right)^2\right] =$$

$$= 4\left[a_2^2|b_2|^2 + a_2^2|b_1|^2 - 2a_1a_2Re(b_1\bar{b}_2) + |b_1b_2|^2 - \left(Re(b_1\bar{b}_2)\right)^2\right] =$$

$$= 4\left[(a_1|b_2| - a_2|b_1|)^2 + 2a_1a_2\left(|b_1||b_2| - Re(b_1\bar{b}_2)\right) + \left(|b_1b_2| - Re(b_1\bar{b}_2)\right)\left(|b_1b_2| + Re(b_1\bar{b}_2)\right)\right]$$

As $\left|Re(b_1\bar{b}_2)\right| \le |b_1||b_2|$*, we get:*

$$Tr(X^2)Tr(Y^2) \ge \left(Tr(XY)\right)^2 \Rightarrow |Tr(XY)| \le \sqrt{Tr(X^2)Tr(Y^2)}$$

$$Tr(XY) \le \sqrt{Tr(X^2)Tr(Y^2)};\ \ (1)$$

Similarly,

$Tr(YZ) \le \sqrt{Tr(Y^2)Tr(Z^2)};\ \ (2)$ *and* $Tr(ZX) \le \sqrt{Tr(Z^2)Tr(X^2)};\ \ (3)$

By adding (1),(2) and (3), we get:

$$Tr(XY) + Tr(YZ) + Tr(ZX) \le \sqrt{Tr(X^2)Tr(Y^2)} + \sqrt{Tr(Y^2)Tr(Z^2)} + \sqrt{Tr(Z^2)Tr(X^2)}$$

A.104. *Solution by Ravi Prakash-New Delhi-India*

$$A + B = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}, A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1-i \\ -1+i & 2 \end{pmatrix},$$

$$B^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1+i \\ -1-i & 3 \end{pmatrix}$$

$$A^{-1} + B^{-1} = \frac{1}{4}\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}, \qquad (A^{-1} + B^{-1})^{-1} = \frac{4}{21}\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\frac{1}{2}(A+B)-2(A^{-1}+B^{-1})=\frac{1}{2}\begin{pmatrix}5 & 2\\ 2 & 5\end{pmatrix}-\frac{8}{21}\begin{pmatrix}5 & 2\\ 2 & 5\end{pmatrix}=\frac{5}{42}\begin{pmatrix}5 & 2\\ 2 & 5\end{pmatrix}$$

$$(x,y)\left(\frac{1}{2}(A+B)-2(A^{-1}+B^{-1})^{-1}\right)\begin{pmatrix}x\\ y\end{pmatrix}=$$

$$=\frac{5}{42}(5x^2+4xy+5y^2)=\frac{25}{42}\left[\left(x+\frac{2}{5}y\right)^2+\frac{21}{25}y^2\right]\geq 0$$

Equality holds when $x=y=0$.

A.105. *Solution by Soumava Chakraborty-Kolkata-India*

$$\sqrt{\frac{a^2+b^2}{2}}=Q,\frac{a+b}{2}=A,\sqrt{ab}=G\Rightarrow\frac{a^2+b^2}{a+b}=\frac{a^2+b^2}{2ab}.\frac{2ab}{a+b}=\frac{Q^2H}{G^2}$$

$$\therefore\left(\sqrt{\frac{a^2+b^2}{2}}\right)^5+\left(\frac{a^2+b^2}{a+b}-\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}\right)^5$$

$$\leq\left(\frac{a^2+b^2}{a+b}\right)^5+\left(\sqrt{ab}\right)^5$$

$$\Leftrightarrow Q^5-G^5\leq\left(\frac{Q^2H}{G^2}\right)^5-\left(\frac{Q^2H}{G^2}-(Q-G)\right)^5$$

$$\Leftrightarrow (Q-G)(Q^4+Q^3G+Q^2G^2+QG^3+G^4)$$

$$\leq\left(\frac{Q^2H}{G^2}-\left(\frac{Q^2H}{G^2}-(Q-G)\right)\right)(m^4+m^3n+m^2n^2+mn^3+n^4)$$

$$\left(assigning\ \frac{Q^2H}{G^2}=m,\frac{Q^2H}{G^2}-(Q-G)=n\right)$$

$$\Leftrightarrow (Q-G)(Q^4+Q^3G+Q^2G^2+QG^3+G^4)\overset{(*)}{\leqq}(Q-G)(m^4+m^3n+m^2n^2+mn^3+n^4)$$

$$\Leftrightarrow m^4+m^3n+m^2n^2+mn^3+n^4\overset{(*)}{\geqq}Q^4+Q^3G+Q^2G^2+QG^3+G^4\ (\because Q-G\geq 0)$$

$$Now, m=\frac{Q^2H}{G^2}=\frac{Q^2H}{AH}=\frac{Q^2}{A}=Q\left(\frac{Q}{A}\right)\geq Q\Rightarrow m\overbrace{\geq}^{(i)}Q \text{ and } n$$
$$=\frac{Q^2H}{G^2}-(Q-G)=m-Q+G\overbrace{\geq}^{via\,(i)}G\Rightarrow n\overbrace{\geq}^{(ii)}G$$
$$\therefore (i),(ii)\Rightarrow m^4\geq Q^4, m^3n\geq Q^3G, m^2n^2\geq Q^2G^2, mn^3\geq QG^3$$
$$\text{and } n^4\geq G^4\overbrace{\Rightarrow}^{summing\ up} m^4+m^3n+m^2n^2+mn^3+n^4$$
$$\geq Q^4+Q^3G+Q^2G^2+QG^3+G^4\Rightarrow(*)\text{ is true}$$
$$\therefore\left(\sqrt{\frac{a^2+b^2}{2}}\right)^5+\left(\frac{a^2+b^2}{a+b}-\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}\right)^5$$
$$\leq\left(\frac{a^2+b^2}{a+b}\right)^5+\left(\sqrt{ab}\right)^5\ (QED)$$

A.106. *Solution by George Florin Șerban-Romania*

$$abc\leq\left(\frac{a+b+c}{3}\right)^3$$

$$(a+b)(b+c)(c+a)\leq\left[\frac{2(a+b+c)}{3}\right]^3\Rightarrow$$

$$(a+b)^2(b+c)^2(c+a)^2\leq 64\left(\frac{a+b+c}{3}\right)^6$$

Therefore,

$$abc(a+b)^2(b+c)^2(c+a)^2\leq 64\left(\frac{a+b+c}{3}\right)^9$$

A.107. *Solution by Hikmat Mammadov-Azerbaijan*

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=0;(x,y,z\notin\{\pm1\})$$

$$\Rightarrow x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=0$$

$$x(1+y^2z^2-y^2-z^2)+y(1+z^2x^2-z^2-x^2)$$
$$+z(1+x^2y^2-x^2-y^2)=0$$

$$\Rightarrow (x+y+z)+xyz(xy+yz+zx)$$
$$-(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)=0$$

$$\Rightarrow (x+y+z)+(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2+3xyz)-$$

$$-(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2)=0$$

$$\Rightarrow (x+y+x)+3xyz=0 \Rightarrow xyz=0; (\because xyz=x+y+z)$$

$$x=0 \Rightarrow y=-z; y=0 \Rightarrow z=-x; z=0 \Rightarrow x=-y.$$

$$S=(0,a,-a),(a,0,-a),(a,-a,0)|a\in\mathbb{R}-\{\pm 1\}\}$$

A.108. *Solution by Bedri Hajrizi-Mitrovica-Kosovo*

$$(1)\ x\geq y>0 \Rightarrow 2(x-y)=(x+y)(1-1) \Rightarrow x=y \Rightarrow x=y=2$$

$$(2)\ x<0, y>0 \Rightarrow 2(y-x)=(-x+y)(-1-1) \Rightarrow$$

$x=y$ impossible.

$$(3)\ x\leq y<0 \Rightarrow 2(y-x)=-(x+y)(-1+1) \Rightarrow$$

$(x=y$ impossible.

$$(4)\ x>0, y<0 \Rightarrow 2(x-y)=(x-y)(1+1) \Rightarrow x+y=4$$

$$x>0, 4-x>0 \Rightarrow x<4$$

Therefore,

$$(x,y)\in\{(a,4-a)|0<a<4\}$$

A.109. *Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia*

$$(ab+bc+ca)(de+ef+fd)$$
$$=(abc)(def)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)\overset{AGM}{\geq}$$

$$\geq (abc)(def)\cdot\frac{9}{\sqrt[3]{(abc)(def)}}\overset{(*)}{\geq}$$

$$Let\ abc = x, def = y; (*)$$

$$\overset{(*)}{\geq} 3xy(x+y-5)+\frac{9xy}{\sqrt[3]{xy}} = 3xy\left(x+y-5+\frac{3}{\sqrt[3]{xy}}\right) =$$

$$= 3xy\left(x+y+\frac{1}{\sqrt[3]{xy}}+\frac{1}{\sqrt[3]{xy}}+\frac{1}{\sqrt[3]{xy}}-5\right)\overset{AGM}{\geq} 3xy\left(5\sqrt[5]{xy\left(\frac{1}{\sqrt[3]{xy}}\right)^3}-5\right) =$$

$$= 3xy(5-5) = 0$$

Equality holds for $a = b = c = d = e = f = 1$.

A.110. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$\frac{e^x}{\sqrt[3]{1+e^{-x}}}+\frac{e^{-x}}{\sqrt[3]{1+e^x}}\overset{AGM}{\geq} 2\sqrt{\frac{e^x}{\sqrt[3]{1+e^{-x}}}\cdot\frac{e^{-x}}{\sqrt[3]{1+e^x}}} = \frac{2}{\sqrt[3]{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}}$$

$$x \geq 0 \Rightarrow e^{\frac{x}{2}}+e^{-\frac{x}{2}} \geq 2 \Rightarrow e^{\frac{x}{2}}+e^{-\frac{x}{2}}-1 \geq 1$$

$$\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}-1\right)(e^x+e^{-x}) \geq (e^x+e^{-x}) \geq 2$$

$$\Rightarrow \left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)(e^x+e^{-x}) \geq e^x+e^{-x}+2$$

$$1+\frac{2}{e^x+e^{-x}} \leq e^{\frac{x}{2}}+e^{-\frac{x}{2}} \Rightarrow 1+\operatorname{sech} x \leq e^{\frac{x}{2}}+e^{-\frac{x}{2}} \Rightarrow$$

$$\frac{2}{\sqrt[3]{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}} \geq \frac{2}{\sqrt[3]{1+\operatorname{sech} x}}$$

Therefore,

$$\frac{e^x}{\sqrt[3]{1+e^{-x}}}+\frac{e^{-x}}{\sqrt[3]{1+e^x}} \geq \frac{2}{\sqrt[3]{1+\operatorname{sech} x}}$$

A.111. *Solution by Vivek Kumar-India*

$$\frac{ac^3}{(c^4+1)(c^2+c+1)} \overset{AGM}{\leq} \frac{ac^3}{2\sqrt{c^4\cdot 1}\cdot 3\sqrt[3]{c^2\cdot c\cdot 1}} = \frac{ac^3}{bc^2\cdot c} = \frac{a}{6}$$

Analogously,

$$\frac{ba^3}{(a^4+1)(a^2+a+1)} \leq \frac{b}{6} \text{ and } \frac{cb^3}{(b^4+1)(b^2+b+1)} \leq \frac{c}{6}$$

Therefore,

$$\frac{ac^3}{(c^4+1)(c^2+c+1)} + \frac{ba^3}{(a^4+1)(a^2+a+1)} + \frac{cb^3}{(b^4+1)(b^2+b+1)} \leq \frac{a+b+c}{6} = \frac{1}{2}$$

A.112. *Solution by Ravi Prakash-New Delhi-India*

Put $x = 5t_1, y = 6t_2$*. The equation becomes:*

$$\frac{25t_1^2}{5} + \frac{36t_2^2}{6} = \frac{1}{11}(5t_1+6t_2)^2 + \frac{5}{2}(t_1-t_2)^2$$

$$t_1^2\left(5-\frac{25}{11}-\frac{5}{2}\right) + t_2^2\left(6-\frac{36}{11}-\frac{5}{2}\right) = \left(\frac{60}{11}-5\right)t_1t_2$$

$$5t_1^2 + 5t_2^2 - 10t_1t_2 = 0 \Leftrightarrow (t_1-t_2)^2 = 0$$

$$So, S = \left\{\left(\alpha, \tfrac{6}{5}\alpha\right) | \alpha \in \mathbb{R}\right\}.$$

A.113. *Solution by Ravi Prakash-New Delhi-India*

Let $A=\frac{a+b}{2}, G=\sqrt{ab}, x=Gt_1, y=At_2$. *The inequality becomes:*

$$\frac{2(Gt_1)^2}{G}+\frac{4(At_2)^2}{2A}\geq\frac{4(Gt_1+At_2)^2}{2A+2G}+G\left(\frac{Gt_1}{G}-\frac{2At_2}{2A}\right)^2$$

$$2(Gt_1^2+At_2^2)-G(t_1-t_2)^2\geq\frac{2(Gt_1+At_2)^2}{A+G}$$

$$2(Gt_1^2+At_2^2)-G(t_1^2+t_2^2-2t_1t_2)\geq\frac{2(G^2t_1^2+A^2t_2^2-2t_1t_2)}{A+G}$$

$$(A+G)[Gt_1^2+(2A-G)t_2^2]+2(A+G)Gt_1t_2-2G^2t_1^2-2A^2t_2^2-4GAt_1t_2\geq 0$$

$$G(A-G)t_1^2+G(A-G)t_2^2+2G(G-A)t_1t_2\geq 0$$

$G(A-G)(t_1-t_2)^2\geq 0$, *which is true as* $G>0$ *and* $A\geq G$.

A.114. *Solution by Ravi Prakash-New Delhi-India*

$$Let: \Delta=\begin{vmatrix}1 & 3 & x & x\\ 5 & 9 & x & x\\ x & x & 1 & 3\\ x & x & 5 & 9\end{vmatrix}=x\Delta_1+\Delta_2, \qquad where$$

$$\Delta_1=\begin{vmatrix}1 & 3 & x & x\\ 1 & 9 & x & x\\ 1 & x & 1 & 3\\ 1 & x & 5 & 9\end{vmatrix} \; and\; \Delta_2=\begin{vmatrix}1-x & 3 & x & x\\ 5-x & 9 & x & x\\ 0 & x & 1 & 3\\ 0 & x & 5 & 9\end{vmatrix}$$

Using $r_2\to r_2-r_1; r_3=r_1; r_4-r_1$

$$\Delta_1 = \begin{vmatrix} 1 & 3 & x & x \\ 0 & 6 & 0 & 0 \\ 0 & x-3 & 1-x & 3-x \\ 0 & x-3 & 5-x & 9-x \end{vmatrix} = \begin{vmatrix} 6 & 0 & 0 \\ x-3 & 1-x & 3-x \\ x-3 & 5-x & 9-x \end{vmatrix} =$$

$$= 6\begin{vmatrix} 1-x & 3-x \\ 5-x & 9-x \end{vmatrix} = 6\begin{vmatrix} 1-x & 2 \\ 5-x & 4 \end{vmatrix} = 6\begin{vmatrix} 1-x & 2 \\ 4 & 2 \end{vmatrix} = 12\begin{vmatrix} 1-x & 1 \\ 4 & 1 \end{vmatrix}$$

$$= -12x - 36,$$

$$\Delta_2 = (1-x)\begin{vmatrix} 9 & x & x \\ x & 1 & 3 \\ x & 5 & 9 \end{vmatrix} - (5-x)\begin{vmatrix} 3 & x & x \\ x & 1 & 3 \\ x & 5 & 9 \end{vmatrix} =$$

$$= (1-x)\begin{vmatrix} 9 & x & x \\ x & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix} - (5-x)\begin{vmatrix} 3 & x & 0 \\ x & 1 & 2 \\ 0 & 4 & 2 \end{vmatrix} =$$

$$= (1-x)\begin{vmatrix} 9 & x & 0 \\ x & 1 & 2 \\ 0 & 4 & 2 \end{vmatrix} - (5-x)\begin{vmatrix} 3 & x & 0 \\ x & 1 & 2 \\ 0 & 4 & 2 \end{vmatrix} =$$

$$= (1-x)(18 - 72 - 2x^2) - (5-x)(6 - 24 - 2x^2) =$$

$$= (1-x)(-2x^2 - 54) - (5-x)(-18 - 2x^2) =$$

$$= 2[(x-1)(x^2+27) - (x-5)(x^2+9)] = 4(2x^2 + 9x + 9)$$

Thus,

$$\Delta = x(-12x - 36) + 4(2x^2 + 9x + 9) = -4x^2 + 36.$$

Therefore, $\Delta = 0 \Rightarrow x = \pm 3.$

A.115. *Solution by Ravi Prakash-New Delhi-India*

$$For\ x,y,z>0:\ \sqrt{\frac{x}{y}}+\sqrt[3]{\frac{y}{z}}=\frac{x^2}{4y^2}+\frac{y^2}{6z^2}+\frac{19}{12}=$$

$$=\left(\frac{x^4}{4y^4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{y^3}{6z^3}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right)\geq$$

$$\geq 4\cdot\left(\frac{x^2}{4y^2}\cdot\frac{1}{4^3}\right)^{\frac{1}{4}}+6\cdot\left(\frac{y^3}{6z^3}\cdot\left(\frac{1}{6}\right)^5\right)^{\frac{1}{6}}=\sqrt{\frac{x}{y}}+\sqrt[6]{\frac{y}{z}}$$

Equality holds for $\frac{x}{4y}=\frac{1}{4},\frac{y}{6z}=\frac{1}{6}\Rightarrow x=y=z.$

So, 2nd equation becomes

$$2x^4+18x+54=x^3+39x^2$$

$$2x^4-x^3-39x^2+18x+54=0\Rightarrow x\in\left\{-3\sqrt{2};-1;1.5;3\sqrt{2}\right\}$$

Thus,

$$(x,y,z)=\left\{(1.5;1.5;1.5),\left(3\sqrt{2},3\sqrt{2},3\sqrt{2}\right)\right\}$$

A.116. $x+y+z\overset{AGM}{\geq}3\sqrt[3]{xyz}$

$$\Rightarrow 2\sqrt[3]{xyz}+\frac{3xyz}{xy+yz+zx}\geq 3\sqrt[3]{xyz}\Rightarrow\frac{3xyz}{xy+yz+zx}\geq\sqrt[3]{xyz}$$

$$\Rightarrow M_h\geq M_g$$

But $M_h\leq M_g\Rightarrow M_g=M_h\Rightarrow x=y=z$. *Hence,*

$$x^5 - 560x^2 + x = 56x^4 + 10x^3 + 56$$

$$x^5 - 56x^4 - 10x^3 + 560x^2 + x - 56 = 0$$

$$x^4(x - 56) - 10x^2(x - 56) + (x - 56) = 0$$

$$(x - 56)(x^4 - 10x^2 + 1) = 0,$$

$$(I)\ x - 56 = 0 \Rightarrow x_1 = 56$$

$$(II)\ x^4 - 10x^2 + 1 = 0 \Leftrightarrow x^4 - 2x^2 + 1 - 8x^2 = 0 \Leftrightarrow$$

$$(x^2 - 1)^2 - \left(2\sqrt{2}x\right)^2 = 0 \Leftrightarrow \left(x^2 - 1 - 2\sqrt{2}x\right)\left(x^2 - 1 + 2\sqrt{2}x\right) = 0$$

$$\Leftrightarrow \left(x^2 - 2\sqrt{2}x + 2 - 3\right)\left(x^2 + 2\sqrt{2}x + 2 - 3\right) = 0$$

$$\left(\left(x - \sqrt{2}\right)^2 - \left(\sqrt{3}\right)^2\right)\left(\left(x + \sqrt{2}\right)^2 - \left(\sqrt{3}\right)^2\right) = 0 \Leftrightarrow$$

$$\left(x - \sqrt{2} - \sqrt{3}\right)\left(x - \sqrt{2} + \sqrt{3}\right)\left(x + \sqrt{2} - \sqrt{3}\right)\left(x + \sqrt{2} + \sqrt{3}\right) = 0$$

Hence, we have:

$$x_2 = \sqrt{2} + \sqrt{3}, x_3 = \sqrt{2} - \sqrt{3}, x_4 = \sqrt{3} - \sqrt{2}, x_5 = -\sqrt{3} - \sqrt{2}$$

Therefore,

$$S = \left\{\begin{matrix} (56,56,56); \left(\sqrt{2} + \sqrt{3}, \sqrt{2} + \sqrt{3}, \sqrt{2} + \sqrt{3}\right); \left(\sqrt{2} - \sqrt{3}, \sqrt{2} - \sqrt{3}, \sqrt{2} - \sqrt{3}\right); \\ \left(\sqrt{3} - \sqrt{2}, \sqrt{3} - \sqrt{2}, \sqrt{3} - \sqrt{2}\right); \left(-\sqrt{3} - \sqrt{2}, -\sqrt{3} - \sqrt{2}, -\sqrt{3} - \sqrt{2}\right) \end{matrix}\right\}$$

A.117. *Solution by Sanong Huayrerai-Nakon Pathom-Thailand*

$$x^8 + y^8 + z^8 + 15 \geq x^3 + y^3 + z^3 + 5\sqrt{3(xy + yz + zx)}$$

$$x^8 + y^8 + z^8 + 15 \geq x^3 + y^3 + z^3 + 5(x + y + z)$$

$$(x^8 + 1) + (y^8 + 1) + (z^8 + 1) + 12 \geq x^3 + y^3 + z^3 + 5(x + y + z)$$

$$2(x^4 + y^4 + z^4) + 12 \geq x^3 + y^3 + z^3 + 5(x + y + z)$$

$$(2x^4 - x^3 - 5x + 4) + (2y^4 - y^3 - 5y + 4) + (2z^4 - z^3 - 5z + 4) \geq 0$$

which is true, because

$$2x^4 - x^3 - 5x + 4 = (x - 1)(2x^3 + x^2 + x - 4) \geq 0$$

$$2y^4 - y^3 - 5y + 4 = (y - 1)(2y^3 + y^2 + y - 4) \geq 0$$

$$2z^4 - z^3 - 5z + 4 = (z - 1)(2z^3 + z^2 + z - 4) \geq 0$$

A.118. *Solution by Tapas Das-India*

Using AM-GM inequality, we have: $x^3 + y^3 + z^3 \geq 3xyz$

$$3\left(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}\right) \geq 3 \cdot 3\sqrt[3]{\sqrt[3]{xyz}} = 9\sqrt[9]{xyz}$$

Hence,

$$x^3 + y^3 + z^3 + 3\left(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}\right) \geq 3xyz + 9\sqrt[9]{xyz} = 12$$

Equality holds when $x = y = z$. *But* $xyz = 1 \Rightarrow x = y = z = 1$.

A.119. *Solution by Ravi Prakash-New Delhi-India*

$$log_x e \cdot (log\, x)^{-1} + log_{\frac{e}{x}} e \cdot \left(log\left(\frac{e}{x}\right)\right)^{-1} = 8$$

$$\frac{log_x e}{log\, x}+\frac{log_{\frac{e}{x}} e}{log\,\frac{e}{x}}=8,\qquad \frac{1}{log\, x\cdot log\, x}+\frac{1}{log\,\frac{e}{x}\cdot log\,\frac{e}{x}}=8$$

$$\frac{1}{log^2 x}+\frac{1}{log^2\left(\frac{e}{x}\right)}=8,\qquad \frac{1}{log^2 x}+\frac{1}{(log\, x-1)^2}=8;(1)$$

Put $log\, x=t+\frac{1}{2}$ *the equation becomes:*

$$\frac{1}{\left(t+\frac{1}{2}\right)^2}+\frac{1}{\left(\frac{1}{2}-t\right)^2}=8\Rightarrow 2\left(\frac{1}{4}+t^2\right)=8\left(\frac{1}{4}-t^2\right)^2$$

$$\frac{1}{2}+2t^2=8\left(\frac{1}{16}-\frac{1}{2}t^2+t^4\right)$$

$$\frac{1}{2}+2t^2=\frac{1}{2}-4t^2+8t^4\Leftrightarrow 8t^4=6t^2\Leftrightarrow t^2(4t^2-3)=0$$

$$t_1=0,t_{2,3}=\pm\frac{\sqrt{3}}{2},\qquad log\, x\in\left\{\frac{1}{2};\frac{1\pm\sqrt{3}}{2}\right\}$$

$$But\,\frac{1}{2}(1\pm\sqrt{3})=\sqrt{2}\left(\frac{1}{2\sqrt{2}}\pm\frac{\sqrt{3}}{2\sqrt{2}}\right)=\sqrt{2}\,sin\left(\frac{\pi}{4}\pm\frac{\pi}{3}\right)$$

Therefore,

$$x_1=e^{sin\frac{\pi}{6}},x_2=e^{\sqrt{2}\,sin\frac{7\pi}{12}},x_3=e^{-\sqrt{2}\,sin\frac{\pi}{12}}$$

A.120. *Solution by Ravi Prakash-New Delhi-India*

$\because x^2+xy+y^2=|x-y\omega|^2$, *where* $\omega\neq 1$, *is cube root of unit.*

$$\sum_{cyc}\sqrt{(a+3b)^2+(a+3b)(3a+b)+(3a+b)^2}$$

$$=\sum_{cyc}|a+3b-(3a+b)\omega|=$$

$$=\left|\sum_{cyc}(a+3b)-\omega\sum_{cyc}(3a+b)\right|=|4(a+b+c)-4\omega(a+b+c)|$$

$$=12|1-\omega|=12\left|\frac{3}{2}-\frac{\sqrt{3}}{2}i\right|=12\sqrt{3}\left|\frac{\sqrt{3}}{2}-\frac{i}{2}\right|=12\sqrt{3}.$$

A.121. *Solution by Christos Tsifakis-Greece*

$$x^{32}+x^{16}+y^2=2\sqrt{2}x^{12}y, \qquad y^2-2\sqrt{2}x^{12}y+x^{32}+x^{16}=0$$

$$\Delta_y=8x^{24}-4(x^{32}+x^{16})=-4(x^{16}-x^8)^2\le 0$$

If $\Delta<0$ *no solution in* $\mathbb{R}$

If $\Delta=0\Rightarrow x^8(x^8-1)=0\Rightarrow x\in\{-1,0,1\}$

$$(x,y)\in\{(0,0),(1,\sqrt{2}),(-1,\sqrt{2})\}$$

A.122. $x\in[a,b]\Rightarrow(x-a)(x-b)\le 0\Rightarrow x^2-(a+b)x+ab\le 0$

$$x^2+ab\le(a+b)x\Rightarrow x+\frac{ab}{x}\le a+b;(1)$$

Analogous: $z+\frac{ab}{z}\le a+b;(2)$

$$y\in[c,d]\Rightarrow(y-c)(y-d)\le 0\Rightarrow y^2-(c+d)y+cd\le 0$$

$$y^2 + cd \leq (c+d)y \Rightarrow y + \frac{cd}{y} \leq c + d; (3)$$

Analogous: $t + \frac{cd}{t} \leq c + d; (4)$

$$x + y + z + t + ab\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right) \overset{AGM}{\geq} 2\sqrt{(x+y+z+t)ab\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right)}; (5)$$

By adding (1),(2),(3) and (4), it follows that

$$x + y + z + t + ab\left(\frac{1}{x} + \frac{1}{z}\right) + cd\left(\frac{1}{y} + \frac{1}{t}\right) \leq 2(a+b+c+d); (6)$$

By (5) and (6):

$$2\sqrt{(x+y+z+t)ab\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right)} \leq 2(a+b+c+d)$$

$$ab(x+y+z+t)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right) \leq (a+b+c+d)^2$$

Equality holds for $x = t = z = t; ab = 2(a+b)$.

A.123. *Solution by Ravi Prakash-New Delhi-India*

$$2m_h + \sqrt{\frac{1}{2}\left((m_a - m_g)^2 + (m_g - m_h)^2 + (m_h - m_a)^2\right)} \leq m_g + m_a$$

$$m_a + m_g - 2m_h \geq \sqrt{\frac{1}{2}\left((m_a - m_g)^2 + (m_g - m_h)^2 + (m_h - m_a)^2\right)}$$

$$m_a^2+m_g^2+4m_h^2+2m_am_g-4m_am_h-4m_gm_h\geq$$

$$\geq m_a^2+m_g^2+m_h^2-m_am_g-m_am_h-m_gm_h$$

$$3m_h^2+3m_am_g-3m_am_h-3m_gm_h\geq 0$$

$$m_h^2-m_h(m_a+m_g)+m_am_g\geq 0,(m_a-m_h)(m_g-m_h)\geq 0\ \textit{true.}$$

Equality holds for $m_a=m_h=m_g\Leftrightarrow a=b$.

A.124. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$\Omega(a)=\prod_{k=1}^{n}\left(1+\frac{k}{an^2}\right)\overset{AGM}{\leq}\left(\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{an^2}\right)\right)^n=\left(\frac{n+\frac{n^2+n}{2an^2}}{n}\right)$$

$$=\left(1+\frac{n^2+n}{2an^3}\right)^n=e^{n\log\left(1+\frac{n^2+n}{2an^3}\right)}.$$

$$Let\ a_n=e^{n\log\left(1+\frac{n^2+n}{2an^3}\right)}\nearrow\ and\ \ a_n\geq 0\Rightarrow$$

$$e^{n\log\left(1+\frac{n^2+n}{2an^3}\right)}\leq\lim_{n\to\infty}e^{n\log\left(1+\frac{n^2+n}{2an^3}\right)}\leq e^{\frac{1}{2a}}$$

$\Omega(a)\leq e^{\frac{1}{2a}}$. *Analogous,* $\Omega(b)\leq e^{\frac{1}{2b}}$ *and* $\Omega(c)\leq e^{\frac{1}{2c}}$.

If $a\leq b\leq c\Rightarrow\Omega(a)\geq\Omega(b)\geq\Omega(c)$

$$a\Omega(a)+b\Omega(b)+c\Omega(c)\overset{Chebyshev}{\leq}\frac{a+b+c}{3}\big(\Omega(a)+\Omega(b)+\Omega(c)\big)$$

$$a\Omega(a)+b\Omega(b)+c\Omega(c)\leq e^{\frac{1}{2a}}+e^{\frac{1}{2b}}+e^{\frac{1}{2c}}$$

A.125. *Solution by Sanong Huayrerai-Nakon Pathom-Thailand*

$$\sum_{cyc}\left(\sqrt{a(a+2b)}+\sqrt{b(b+2a)}\right)=\frac{1}{\sqrt{3}}\sum_{cyc}\left(\sqrt{3a(a+2b)}+\sqrt{3b(b+2a)}\right)\le$$

$$\le\frac{1}{\sqrt{3}}\sum_{cyc}\left(\frac{3a+a+2b}{2}+\frac{3b+b+2a}{2}\right)=\frac{1}{\sqrt{3}}\cdot\frac{12(a+b+c)}{2}=$$

$$=\frac{1}{\sqrt{3}}\cdot 6(a+b+c)=\frac{6\cdot 3}{\sqrt{3}}=6\sqrt{3}$$

A.126. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$\sqrt{x^7\cdot 2^{7x-7}}(x+2^{x-1})=x^8+2^{8x-8}$$

$$(x+2^{x-1})^2x^7 2^{7x-7}=x^{16}+x^8 2^{8x-7}+2^{16x-16}$$

$$(x+2^{x-1})^2x^7 2^{7x}=x^{16}+x^8 2^{8x-7}+2^{16x-9}$$

$$(x^2+x2^x+2^{2x-2})x^7 2^{7x}=2^7x^{16}+x^8 2^{8x}+2^{16x-9}$$

$$x^9 2^{7x}+x^8 2^{8x}+x^7 2^{9x-2}=2^7x^{16}+x^8 2^{8x}+2^{16x-9}$$

Let $2^x=y$ *and* $2x=z\Rightarrow x^9 2^{7x}+x^7 2^{9x-2}=2^7x^{16}+2^{16x-9}$

$$2^9x^9y^7+2^7x^7y^9=2^{16}x^{16}+y^{16}\Rightarrow(2x)^9y^7+(2x)^7y^9$$
$$=(2x)^{16}+y^{16}$$

$$z^9y^7+z^7y^9=z^{16}+y^{16}\Rightarrow z^9(y^7-z^7)=y^9(y^7-z^7)$$

$$(y^7-z^7)(z^9-y^9)=0\Rightarrow y^7=z^7 \text{ or } y^9=z^9$$

$h(t)=t^7$ *and* $k(t)=t^9$ *are bijective on* $\mathbb{R}\Rightarrow 2^x=2x\Rightarrow x\in\{1,2\}$.

A.127. *Solution by Sanong Huayrerai-Nakon Pathom-Thailand*

$$\frac{8}{a+b}+\frac{24}{b+c}+\frac{12}{c+a}\le 8\left(\frac{1}{4a}+\frac{1}{4b}\right)+24\left(\frac{1}{4b}+\frac{1}{4c}\right)+12\left(\frac{1}{4c}+\frac{1}{4a}\right)$$
$$=\frac{2}{a}+\frac{2}{b}+\frac{6}{b}+\frac{6}{c}+\frac{3}{c}+\frac{3}{a}=\frac{5}{a}+\frac{8}{b}+\frac{9}{c}$$

SOLUTIONS GEOMETRY

G.001. *Solution by Ravi Prakash-India*

$$Let\ E=a_1b_1\sin^2 x+a_2b_2\cos^2 x-(a_1\sin^2 x+a_2\cos^2 x)(b_1\sin^2 x+b_2\cos^2 x)=a_1b_1\sin^2 x+a_2b_2\cos^2 x-$$

$$-(a_1b_1\sin^4 x+a_1b_2\sin^2 x\cos^2 x+a_2b_1\sin^2 x\cos^2 x+a_2b_2\cos^4 x)=$$
$$=(a_1b_1\sin^2 x-a_1b_1\sin^4 x)+a_2b_2\cos^2 x-a_2b_2\cos^4 x-(a_2b_1+a_1b_2)\sin^2 x\cos^2 x=$$

$$=a_1b_1\sin^2 x(1-\sin^2 x)+a_2b_2\cos^2 x(1-\cos^2 x)-(a_2b_1+a_1b_2)\sin^2 x\cos^2 x=$$

$$=(a_1b_1+a_2b_2-a_1b_2-a_2b_1)\sin^2 x\cos^2 x=$$

$$=(a_1-a_2)(b_1-b_2)\sin^2 x\cos^2 x=\frac{1}{4}(a_1-a_2)(b_1-b_2)\sin^2 2x$$

Therefore,

$$|E|\le\frac{1}{4}|a_1-a_2|\cdot|b_1-b_2|\le\frac{1}{4},where\ |a_1-a_2|\le 1,|b_1-b_2|\le 1$$

G.002. *Solution by Tapas Das-India*

$$\sum a^2cd\cot^{-1}(b)=$$

$$= a^2cd\,cot^{-1}(b) + b^2da\,cot^{-1}(c) + c^2ab\,cot^{-1}(d) + d^2bc\,cot^{-1}(a)$$

$$= abcd\left[\frac{a}{b}cot^{-1}(b) + \frac{b}{c}cot^{-1}(c) + \frac{c}{d}cot^{-1}(d) + \frac{d}{a}cot^{-1}(a)\right] \quad (1)$$

Let $f(x) = cot^{-1}x, x > 0$

$\therefore f'(x) = -\frac{1}{1+x^2} < 0 \therefore cot^{-1}x$ *is a decreasing function*

Again,

Let $f(t) = cot^{-1}(t), t > 0$

$$\therefore f'(t) = -\frac{1}{1+t^2}$$

$$f''(t) = \frac{2t}{(1+t^2)^2} > 0$$

$\therefore f$ *is convex; using Jensen's inequality*

$$\frac{a}{b}f(b) + \frac{b}{c}f(c) + \frac{c}{d}f(d) + \frac{d}{a}f(a) \geq$$

$$\geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)f\left(\frac{b\cdot\frac{a}{b}+c\cdot\frac{b}{c}+d\cdot\frac{c}{d}+a\cdot\frac{d}{a}}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right)$$

$$= \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)f\left(\frac{a+b+c+d}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right)$$

$$\overset{AM-GM}{\geq} 4\left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{a}\right)^{\frac{1}{4}} cot^{-1}\left(\frac{1}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right)$$

$$[\because a+b+c+d=1]$$

$$= 4\,cot^{-1}\left(\frac{1}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}\right) \geq 4\,cot^{-1}\left(\frac{1}{4}\right)$$

Since $cot^{-1}x$ *is decreasing function and* $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4$ *(AM-GM)*

$$\frac{a}{b}f(b)+\frac{b}{c}f(c)+\frac{c}{d}f(d)+\frac{d}{a}f(a)\geq 4\,cot^{-1}\left(\frac{1}{4}\right)$$

$$\Rightarrow \frac{a}{b}cot^{-1}(b)+\frac{b}{c}cot^{-1}(c)+\frac{c}{d}cot^{-1}(d)+\frac{d}{a}cot^{-1}(a)\geq 4\,cot^{-1}\left(\frac{1}{4}\right)\quad (2)$$

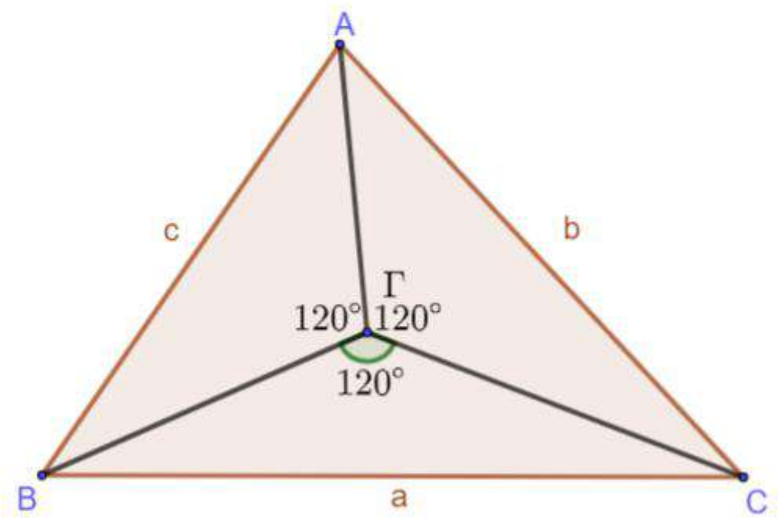
Now, from (1)

$$\sum a^2cd\,cot^{-1}(b)=$$

$$=abcd\left[\frac{a}{b}cot^{-1}(b)+\frac{b}{c}cot^{-1}c+\frac{c}{d}cot^{-1}d+\frac{d}{a}cot^{-1}(a)\right]\geq 4abcd\,cot^{-1}\frac{1}{4}$$

(using (2))

G.003. *Solution by Tapas Das-India*



Γ –Toricelli's point for triangle ABC, then $\angle A\Gamma B=\angle B\Gamma C=\angle C\Gamma A$

$$(\Gamma A^3+\Gamma B^3+\Gamma C^3)\left(\frac{1}{\Gamma A}+\frac{1}{\Gamma B}+\frac{1}{\Gamma C}\right)=$$

$$=\left(\left(\Gamma A^{\frac{3}{2}}\right)^2+\left(\Gamma B^{\frac{3}{2}}\right)^2+\left(\Gamma C^{\frac{3}{2}}\right)^2\right)\left(\frac{1}{\Gamma A}+\frac{1}{\Gamma B}+\frac{1}{\Gamma C}\right)\geq$$

$$\geq(\Gamma A+\Gamma B+\Gamma C)^2\geq 3(\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A)\geq$$

$$\geq 3\cdot 12r^2=36r^2$$

$$[AB\Gamma]=\frac{1}{2}\Gamma A\cdot\Gamma B\cdot sin\,120^\circ=\frac{\sqrt{3}}{4}\cdot\Gamma A\cdot\Gamma B$$

$$[BC\Gamma]=\frac{\sqrt{3}}{4}\cdot\Gamma B\cdot\Gamma C\ and\ [CA\Gamma]=\frac{\sqrt{3}}{4}\cdot\Gamma A\cdot\Gamma C$$

$$[ABC]=[\Gamma AB]+[\Gamma BC]+[\Gamma CA]=\frac{\sqrt{3}}{4}(\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A)$$

$$rs=\frac{\sqrt{3}}{4}(\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A)$$

$$\Gamma A\cdot\Gamma B+\Gamma B\cdot\Gamma C+\Gamma C\cdot\Gamma A=\frac{4rs}{\sqrt{3}}\overset{Mitrinovic}{\geq}\frac{4r\cdot 3\sqrt{3}r}{\sqrt{3}}=12r^2$$

G.004. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$We\ have:\ \frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}$$
$$=\left(a-\frac{a^2}{a+a'}\right)+\left(b-\frac{b^2}{b+b'}\right)+\left(c-\frac{c^2}{c+c'}\right)=$$

$$=2s-\left(\frac{a^2}{a+a'}+\frac{b^2}{b+b'}+\frac{c^2}{c+c'}\right)\overset{Bergström}{\leq}2s-\frac{(a+b+c)^2}{(a+b+c)+(a'+b'+c')}=$$

$$=2s-\frac{4s^2}{2s+2s'}=\frac{2s.2s'}{2(s+s')}\overset{Mitrinovic}{\leq}\frac{3\sqrt{3}R.3\sqrt{3}R'}{2(3\sqrt{3}r+3\sqrt{3}r')}=\frac{3\sqrt{3}RR'}{2(r+r')}.$$

$$Therefore,\ \frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\leq\frac{3\sqrt{3}RR'}{2(r+r')}.$$

G.005.

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}=\sum_{cyc}\frac{a(a+b)+c(a+b)}{(a+b+c)+a}$$
$$=\sum_{cyc}\frac{(a+b)(a+c)}{(a+b)+(a+c)}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{2}{\frac{1}{a+b}+\frac{1}{a+c}}\overset{AH-AM}{\leq}\frac{1}{2}\sum_{cyc}\frac{(a+b)+(a+c)}{2}=$$

$$=\frac{1}{4}\sum_{cyc}(2a+b+c)=\frac{1}{4}\cdot 4\sum_{cyc}a=2s\overset{Mitrinovic}{\leq}2\cdot\frac{3\sqrt{3}R}{2}=3\sqrt{3}R$$

Equality holds for $a=b=c$.

G.006. *Solution by Tapas Das-India*

$$\sin^4 x+\cos^2 x=\sin^4 x+1-\sin^2 x=1-\sin^2 x\cos^2 x$$

$$\cos^4 x+\sin^2 x=1-\cos^2 x+\cos^4 x=1-\sin^2 x\cos^2 x$$

$$\sqrt{\sin^4 x+\cos^2 x}+\sqrt{\sin^2 x+\cos^4 x}+\sqrt{1+\sin^2 x\cos^2 x}=3$$

$$\sqrt{\cos^4 x-\cos^2 x+1}+\sqrt{\cos^4 x-\cos^2 x+1}+\sqrt{1+\sin^2 x\cos^2 x}=3$$

$2\sqrt{\cos^4 x-\cos^2 x+1}+\sqrt{1+\sin^2 x\cos^2 x}=3$. Let $\sin^2 x\cos^2 x=p$, then:

$$2\sqrt{1-p}+\sqrt{1+p}=3,\qquad 4(1-p)=9+(1+p)-6\sqrt{1+p}$$

$$25p^2+24p=0\Leftrightarrow p=0\Rightarrow \sin 2x=0\Leftrightarrow x\in\left\{\frac{k\pi}{2}\middle|k\in\mathbb{Z}\right\}$$

G.007. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$From\ the\ identity\ \cot\omega=\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4F},$$

$$the\ desired\ inequality\ is\ succesively\ equivalent\ to:$$

$$a^2+b^2+c^2-4\sqrt{3}F\geq 2\sum_{cyc}(a^2-bc)+\frac{4abc}{s}\sum_{cyc}\left(\frac{s(s-a)}{bc}-\sqrt{\frac{s^2(s-b)(s-c)}{a^2bc}}\right)$$

$$\sum_{cyc}(ab+ca-a^2)-4\sqrt{3}F \geq 4\sum_{cyc}a(s-a)-4\sum_{cyc}\sqrt{bc(s-b)(s-c)}$$

$$2\sum_{cyc}a(s-a)\geq 2\sum_{cyc}\left(\sqrt{b(s-b)}-\sqrt{c(s-c)}\right)^2+4\sqrt{3}F$$

$$\sum_{cyc}\sqrt{a(s-a)}^2\geq\sum_{cyc}\left(\sqrt{b(s-b)}-\sqrt{c(s-c)}\right)^2+2\sqrt{3}F\ \ (1)$$

Now let's prove that $a'=\sqrt{a(s-a)},b'=\sqrt{b(s-b)},c'=\sqrt{c(s-c)}$ *can be the sides of a triangle :*

Let $x=(s-b)(s-c),\ \ y=(s-c)(s-a),\ \ z=(s-a)(s-b)$

Since : $a'^2=y+z$ *(and analogs) we have :* $a'^2+b'^2-c'^2=2z>0$ *then :* $a'+b'>c'$ *(and analogs)*

So a',b',c' *can be the sides of a triangle* Δ' *with area* F' *such that*

$$16F'^2=2\sum_{cyc}a'^2b'^2-\sum_{cyc}a'^4$$
$$=2\sum_{cyc}(y+z)(z+x)-\sum_{cyc}(y+z)^2=4\sum_{cyc}yz$$
$$=4(s-a)(s-b)(s-c)\sum_{cyc}(s-a)=4sr^2.s$$
$$=4F^2\ \text{ then :}$$

$$F'=\frac{F}{2}.$$

Applying now Hadwiger − Finsler inequality in Δ' *we get :*

$$\sum_{cyc}a'^2\geq\sum_{cyc}(b'-c')^2+4\sqrt{3}F'$$

$$\Leftrightarrow\sum_{cyc}\sqrt{a(s-a)}^2\geq\sum_{cyc}\left(\sqrt{b(s-b)}-\sqrt{c(s-c)}\right)^2+2\sqrt{3}F$$

which is (1) *and the proof is complete.*

G.008.

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1\Rightarrow\frac{a^2}{b+c}+\frac{ab}{c+a}+\frac{ca}{a+b}=a;(1)$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1\Rightarrow\frac{ab}{b+c}+\frac{b^2}{c+a}+\frac{bc}{a+b}=b;(2)$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1\Rightarrow\frac{ac}{b+c}+\frac{bc}{c+a}+\frac{c^2}{a+b}=c;(3)$$

By adding (1),(2) and (3), we get:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}+\frac{ab+bc}{c+a}+\frac{ca+bc}{a+b}+\frac{ac+ab}{b+c}=a+b+c$$

$$\sin x\cdot\sin y\cdot\sin z+\frac{b(a+c)}{a+c}+\frac{c(a+b)}{a+b}+\frac{a(b+c)}{b+c}=a+b+c$$

$$\sin x\cdot\sin y\cdot\sin z+a+b+c=a+b+c$$

$$\sin x\cdot\sin y\cdot\sin z=0$$

$$(x,y,z)\in\{(m\pi,n\pi,p\pi)|m,n,p\in\mathbb{Z}\}$$

G.009.

$$\begin{cases}\sin^3 x+\cos^3 y+z^3+3z=3z^2+2\\ \sin^2 x+\cos^2 y+z^2=2z+2\\ \sin x+\cos y+z=2\end{cases}. \text{ Denote: }\begin{cases}\sin x=u\\ \cos y=v\\ z-1=w\end{cases}$$

$$\begin{cases}S_1=u+v+w\\ S_2=uv+vw+wu\\ S_3=uvw\end{cases}$$

$$\begin{cases}u^3+v^3+z^3=1\\ u^2+v^2+z^2=1\\ u+v+w=1\end{cases}\Rightarrow\begin{cases}S_1^3-3S_3=S_1(S_1^2-3S_2)\\ S_1^2-2S_2=1\\ S_1=1\end{cases}\Rightarrow\begin{cases}S_1=1\\ S_2=0\Rightarrow uvw=0\\ S_3=0\end{cases}$$

$$\text{Hence, }\sin x\cdot\cos y\cdot(z-1)=0$$

$$\sin x=0\Rightarrow x\in\{k\pi|k\in\mathbb{Z}\}$$

$$\cos y=0\Rightarrow y\in\left\{\pm\frac{\pi}{2}+2k\pi\middle|k\in\mathbb{Z}\right\}$$

$$z-1=0 \Rightarrow z=1.$$

G.010. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$\sqrt{n(n+1)}.\cos\left(A-\frac{\pi}{7}\right)+\sqrt{n(n+2)}.\cos\left(B-\frac{\pi}{7}\right)+\sqrt{(n+1)(n+2)}.\cos\left(C-\frac{\pi}{7}\right)$$

$$\overset{(*)}{\lesssim} 3(n+1)\cos\frac{\pi}{21}$$

$We\ have$

$$:\ LHS_{(*)} \overset{CBS}{\lesssim} \sqrt{[n(n+1)+n(n+2)+(n+1)(n+2)]\left[\sum_{cyc}\cos^2\left(A-\frac{\pi}{7}\right)\right]}$$

$$With: n(n+1)+n(n+2)+(n+1)(n+2) < (n+1)[n+(n+2)]+[n(n+2)+1]=3(n+1)^2$$

$$\rightarrow It's\ suffices\ to\ prove: \sum_{cyc}\cos^2\left(A-\frac{\pi}{7}\right) \le 3\cos^2\left(\frac{\pi}{21}\right)$$

$$Let\ f(x)=\cos^2\left(x-\frac{\pi}{7}\right), x\in(0,\pi)\rightarrow f'(x) = -\sin\left(2x-\frac{2\pi}{7}\right)\ and\ f''(x)=-2\cos\left(2x-\frac{2\pi}{7}\right)$$

$$\rightarrow\ f-concave\ on\ \left]0,\frac{11\pi}{28}\right]\ and\ \left[\frac{25\pi}{28},\pi\right[, convex\ on\ \left[\frac{11\pi}{28},\frac{25\pi}{28}\right].$$

$$WLOG, we\ may\ assume\ that\ A\ge B\ge C.$$

■ $If\ A\ge \frac{25\pi}{28} \rightarrow B,C\in\left]0,\frac{11\pi}{28}\right]\ \left(\because \frac{25\pi}{28}+\frac{11\pi}{28}>\pi\right)$ $\rightarrow\ By\ Jensen's\ inequality, we\ have:$

$$\cos^2\left(B-\frac{\pi}{7}\right)+\cos^2\left(C-\frac{\pi}{7}\right)=f(B)+f(C)\le 2f\left(\frac{B+C}{2}\right) = 2f\left(\frac{\pi}{2}-\frac{A}{2}\right)=2\cos^2\left(\frac{5\pi}{14}-\frac{A}{2}\right)$$

$$Since\ \frac{25\pi}{28} \le A < \pi \ \rightarrow \ \frac{3\pi}{4} \le A - \frac{\pi}{7} < \frac{6\pi}{7} < \pi - \frac{\pi}{21}\ and\ \frac{\pi}{21} < \frac{5\pi}{56} \le \frac{A}{2} - \frac{5\pi}{14} < \frac{\pi}{7} \ \rightarrow \ -cos\frac{\pi}{21} < cos\left(A - \frac{\pi}{7}\right) < 0$$

$$And\ 0 < cos\left(\frac{A}{2} - \frac{5\pi}{14}\right) \le cos\frac{\pi}{21} \ \rightarrow \ cos^2\left(A - \frac{\pi}{7}\right), cos^2\left(\frac{A}{2} - \frac{5\pi}{14}\right) < cos^2\left(\frac{\pi}{21}\right) \ \rightarrow \sum_{cyc} cos^2\left(A - \frac{\pi}{7}\right) \le 3\,cos^2\left(\frac{\pi}{21}\right).$$

■ $If\ \frac{11\pi}{28} \le A \le \frac{25\pi}{28} \ \rightarrow \ \frac{\pi}{4} \le A - \frac{\pi}{7} \le \frac{3\pi}{4} \ \rightarrow \ cos^2\left(A - \frac{\pi}{7}\right) \le cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$

$$\rightarrow \sum_{cyc} cos^2\left(A - \frac{\pi}{7}\right) \overset{cos\,x\,\le\,1}{\lesssim} \frac{1}{2} + 1 + 1 = \frac{5}{2}\ and\ cos\frac{\pi}{21} > cos\frac{\pi}{12} = cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\rightarrow \ 3\,cos^2\left(\frac{\pi}{21}\right) > 3\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^2 = 3.\frac{2+\sqrt{3}}{4} \overset{\sqrt{3} > \frac{3}{2}}{\gtrsim} \frac{21}{8} > \frac{5}{2} \ge \sum_{cyc} cos^2\left(A - \frac{\pi}{7}\right).$$

■ $If\ A \le \frac{11\pi}{28} \ \rightarrow \ A, B, C \in \left]0, \frac{11\pi}{28}\right] \ \rightarrow \ By\ Jensen's\ inequality, we\ have:$

$$\sum_{cyc} cos^2\left(A - \frac{\pi}{7}\right) = f(A) + f(B) + f(C) \le 3f\left(\frac{\pi}{3}\right) = 3\,cos^2\left(\frac{\pi}{3} - \frac{\pi}{7}\right) = 3\,cos^2\left(\frac{4\pi}{21}\right) < 3\,cos^2\left(\frac{\pi}{21}\right).$$

$$\rightarrow \sum_{cyc} cos^2\left(A - \frac{\pi}{7}\right) \le 3\,cos^2\left(\frac{\pi}{21}\right), \forall \Delta ABC \ \rightarrow \ (*)\ is\ true.$$

$$Therefore,$$

$$\sqrt{n(n+1)}.\cos\left(A-\frac{\pi}{7}\right)+\sqrt{n(n+2)}.\cos\left(B-\frac{\pi}{7}\right)$$
$$+\sqrt{(n+1)(n+2)}.\cos\left(C-\frac{\pi}{7}\right)<3(n+1)\cos\frac{\pi}{21}$$

G.011. *Solution by Amir Sofi-Kosovo*

$$\frac{1}{1+\tan^4 x}+\frac{1}{10}=\frac{2}{1+3\tan^2 x}$$

$$10(1+3\tan^2 x)+(1+3\tan^2 x)(1+\tan^4 x)=20(1+\tan^4 x)$$

$$3\tan^6 x-19\tan^4 x+33\tan^2 x-9=0$$

$$(3\tan^6 x-18\tan^4 x+27\tan^2 x)-(\tan^4 x-6\tan^2 x+9)=0$$

$$3\tan^2 x\,(\tan^2 x-3)^2-(\tan^2 x-3)^2=0$$

$$(\tan^2 x-3)^2(3\tan^2 x-1)=0$$

$$\tan^2 x-3=0\Rightarrow \tan x=\pm\sqrt{3}\Rightarrow$$

$$x_1=\frac{\pi}{3}+n\pi,x_2=-\frac{\pi}{3}+m\pi;m,n\in\mathbb{Z}$$

$$3\tan^2 x-1=0\Rightarrow \tan x=\pm\frac{\sqrt{3}}{3}\Rightarrow$$

$$x_3=\frac{\pi}{6}+p\pi,x_4=-\frac{\pi}{6}+q\pi;p,q\in\mathbb{Z}$$

G.012. *Solution by George Florin Șerban-Romania*

$$m_a^2=\frac{2(b^2+c^2)-a^2}{4}\Rightarrow 4a^2=2(b^2+c^2)-a^2\Rightarrow 5a^2=2(b^2+c^2)$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{\frac{5a^2}{2}-a^2}{2bc}=\frac{5a^2-2a^2}{4bc}=\frac{3a^2}{4bc}\ and$$

$$\sin A=\frac{a}{2R},\qquad \tan A=\frac{\sin A}{\cos A}=\frac{a}{2R}\cdot\frac{4bc}{3a^2}=\frac{2abc}{3Ra^2}$$

$$a^2 \tan A = a^2 \cdot \frac{2abc}{3Ra^2} = \frac{3abc}{3R} = \frac{8rs}{3} \overset{Mitrinovic}{\geq} \frac{8r \cdot 3\sqrt{3}r}{3} = 8\sqrt{3}r^2 > 2\sqrt{3}r^2$$

G.013. *Solution by Adrian Popa-Romania*

$$\sum_{cyc}(b+c) \cdot \csc\frac{A}{2} = \sum_{cyc}\frac{b+c}{\sin\frac{A}{2}} = \sum_{cyc}\frac{b+c}{\sqrt{\frac{(s-b)(s-c)}{bc}}} \overset{AGM}{\geq}$$

$$\geq \sum_{cyc}\frac{2bc}{\sqrt{(s-b)(s-c)}} \overset{AGM}{\geq} \sum_{cyc}\frac{2bc}{\frac{s-b+s-c}{2}} = \sum_{cyc}\frac{4bc}{a} =$$

$$= 4abc\sum_{cyc}\frac{1}{a^2} \overset{Radon}{\geq} 16RF \cdot \frac{27}{(a+b+c)^2} = 16RF \cdot \frac{27}{4s^2} =$$

$$= \frac{4Rr \cdot 27}{s} \overset{Mitrinovic}{\geq} \frac{4Rr \cdot 27 \cdot 2}{3\sqrt{3}R} = \frac{72r}{\sqrt{3}} = \frac{27r\sqrt{3}}{3} = 24\sqrt{3}r.$$

Equality holds if and only if $a = b = c$.

G.014. *Solution by Ravi Prakash-New Delhi-India*

$$\frac{\sec^4 x + \csc^4 x}{2} \geq \left(\frac{\sec^2 x + \csc^2 x}{2}\right)^2$$

$$\Rightarrow \sec^4 x + \csc^4 x \geq \frac{1}{2}\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)^2 = \frac{1}{2}\left(\frac{4}{\sin^2 2x}\right)^2 \geq 2^3$$

Equality holds for $x = \frac{\pi}{4}$.

$$\frac{1}{2}(\sec^6 y + \csc^6 y) \geq \left(\frac{\sec^2 y + \csc^2 y}{2}\right)^3$$

$$\Rightarrow \sec^6 y + \csc^6 y \geq \frac{1}{4}\left(\frac{1}{\sin^2 y} + \frac{1}{\cos^2 y}\right)^3 = \frac{1}{4}\left(\frac{4}{\sin^2 2y}\right)^3 = 4^2 = 2^4$$

Equality holds for $y = \frac{\pi}{4}$.

Similarly, $sec^8 z + csc^8 z \geq 2^5$

Equality holds for $z = \frac{\pi}{4}$. *Now,*

$$16 = \sqrt[3]{(sec^4 x + csc^4 x)(sec^6 y + csc^6 y)(sec^8 z + csc^8 z)} \geq$$

$$\geq \sqrt[3]{2^3 \cdot 2^4 \cdot 2^5} = 16 \Leftrightarrow sec^4 x + csc^4 x = 2^3$$

$$sec^6 y + csc^6 y = 2^4$$

$$sec^8 z + csc^8 z = 2^5 \Leftrightarrow x = y = z = \frac{\pi}{4}.$$

G.015. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$We\ have:$$

$$(2a^3 + 3b^3 + 5c^3)(2a + 3b + 5c) \overset{CBS}{\geq} (2a^2 + 3b^2 + 5c^2)^2 \overset{CBS}{\geq} (2a^2 + 3b^2 + 5c^2).\frac{(2a + 3b + 5c)^2}{2 + 3 + 5}$$

$$Then: \frac{2a^3 + 3b^3 + 5c^3}{2a^2 + 3b^2 + 5c^2} \geq \frac{2a + 3b + 5c}{10}.$$

$$Similarly\ we\ have:$$

$$\frac{3a^3 + 5b^3 + 2c^3}{3a^2 + 5b^2 + 2c^2} \geq \frac{3a + 5b + 2c}{10} \quad \& \quad \frac{5a^3 + 2b^3 + 3c^3}{5a^2 + 2b^2 + 3c^2} \geq \frac{5a + 2b + 3c}{10}$$

$$Summing\ up\ these\ inequalities\ we\ get:$$

$$\frac{2a^3 + 3b^3 + 5c^3}{2a^2 + 3b^2 + 5c^2} + \frac{3a^3 + 5b^3 + 2c^3}{3a^2 + 5b^2 + 2c^2} + \frac{5a^3 + 2b^3 + 3c^3}{5a^2 + 2b^2 + 3c^2} \geq a + b + c = 2s \overset{Mitrinovic}{\geq} 2.3\sqrt{3}r = 6\sqrt{3}r.$$

G.016. *Solution 2 by Surjeet Singhania-India*

$$Let: f(x) = \frac{x^2}{1+x^2} tan^{-1} x, x \geq 0$$

$$f'(x) = \frac{x^2}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2} tan^{-1} x > 0, \forall x > 0 \Rightarrow f$$

$$\nearrow and\ since \frac{a+b}{2} \geq \sqrt{ab} \Rightarrow$$

$$f\left(\frac{a+b}{2}\right) \geq f(\sqrt{ab}) \Rightarrow \frac{(a+b)^2}{4\left(1+\frac{(a+b)^2}{4}\right)} tan^{-1}\left(\frac{a+b}{2}\right) \geq \frac{ab}{1+ab} tan^{-1}(\sqrt{ab})$$

Hence,

$$ab(4+(a+b)^2)\, tan^{-1}(\sqrt{ab}) \leq (1+ab)(a+b)^2\, tan^{-1}\left(\frac{a+b}{2}\right), \forall a, b \geq 0$$

G.017. *Solution by Adrian Popa-Romania*

$$\sqrt{\sum_{cyc} am_a^2} \cdot \sqrt[3]{\sum_{cyc} am_a^3} \cdot \sqrt[6]{\sum_{cyc} am_a^6} = \sqrt[6]{\left(\sum_{cyc} am_a^2\right)^3} \cdot \sqrt[6]{\left(\sum_{cyc} am_a^3\right)^2} \cdot \sqrt[6]{\sum_{cyc} am_a^6} =$$

$$= \sqrt[6]{(am_a^3 + bm_b^3 + cm_c^3)(am_a^6 + bm_b^6 + cm_c^6)} \overset{Holder}{\geq}$$

$$\geq \sqrt[6]{(am_a^3 + bm_b^3 + cm_c^3)^6} = am_a^3 + bm_b^3 + cm_c^3$$

$$\frac{(am_a + bm_b + cm_c)^3}{am_a^3 + bm_b^3 + cm_c^3} \leq 4s^2 = (a+b+c)^2$$

But: $(am_a^3 + bm_b^3 + cm_c^3)(a+b+c)^2 \overset{Holder}{\geq} (am_a + bm_b + cm_c)^3$

G.018. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Lemma: \ If\ x, y, \in (0,1)\ then:$$

$$\frac{1}{1+x} + \frac{1}{1+y} \leq \frac{2}{1+\sqrt{xy}}\ with\ equality\ holds\ iff\ x = y.$$

$$Proof: \ We\ have\ \frac{2}{1+\sqrt{xy}}-\left(\frac{1}{1+x}+\frac{1}{1+y}\right)$$
$$=\frac{2(1+x)(1+y)-\left(1+\sqrt{xy}\right)(2+x+y)}{\left(1+\sqrt{xy}\right)(1+x)(1+y)}=$$
$$=\frac{\left(x+y-2\sqrt{xy}\right)-\left(x\sqrt{xy}+y\sqrt{xy}-2xy\right)}{\left(1+\sqrt{xy}\right)(1+x)(1+y)}=\frac{\left(\sqrt{x}-\sqrt{y}\right)^2-\sqrt{xy}.\left(\sqrt{x}-\sqrt{y}\right)^2}{\left(1+\sqrt{xy}\right)(1+x)(1+y)}$$
$$=\frac{\left(1-\sqrt{xy}\right).\left(\sqrt{x}-\sqrt{y}\right)^2}{\left(1+\sqrt{xy}\right)(1+x)(1+y)}\geq 0$$

$So\ the\ proof\ of\ the\ lemma\ is\ completed.$

$Equality\ holds\ iff\ x=y. Now, we\ have:$

$$\sum\nolimits_{cyc}\frac{1}{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}=\sum\nolimits_{cyc}\frac{1}{1+\sin\alpha}$$
$$=\left(\frac{1}{1+\sin\alpha}+\frac{1}{1+\sin\beta}\right)+\left(\frac{1}{1+\sin\gamma}+\frac{1}{1+\sin\delta}\right)\leq$$
$$\overset{Lemma}{\leq}\ \frac{2}{1+\sqrt{\sin\alpha.\sin\beta}}+\frac{2}{1+\sqrt{\sin\gamma.\sin\delta}}\ \overset{Lemma}{\leq}$$
$$\leq\ 2.\frac{2}{1+\sqrt{\sqrt{\sin\alpha.\sin\beta}.\sqrt{\sin\gamma.\sin\delta}}}=\frac{4}{1+\sqrt[4]{\frac{9}{16}}}=\frac{8}{2+\sqrt{3}}=8\left(2-\sqrt{3}\right)$$

$$Therefore,\ 8\sqrt{3}+\sum\nolimits_{cyc}\frac{1}{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}\leq 16.$$

$$Equality\ holds\ iff\ \alpha=\beta=\gamma=\delta=\frac{\pi}{3}.$$

G.019. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$Lemma\ (Klamkin, 1975):\ If\ x,y,z \in R\ and\ M\ an\ arbitrary\ point\ then:$

$$(x+y+z)(x.MA^2+y.MB^2+z.MC^2)\geq yza^2+zxb^2+xyc^2.$$

$$Proof\ :For\ any\ point\ M\ we\ have:\ \left(x.\overrightarrow{MA}+y.\overrightarrow{MB}+z.\overrightarrow{MC}\right)^2\geq 0$$

$$Then\ \sum x^2.MA^2+\sum xy.\left(2\overrightarrow{MA}.\overrightarrow{MB}\right)\geq 0$$
$$\leftrightarrow \sum x^2.MA^2+\sum xy(MA^2+MB^2-AB^2)\geq 0$$

$$\leftrightarrow \sum x(x+y+z).MA^2\geq \sum yz.BC^2\leftrightarrow$$

$$(x+y+z)(x.MA^2+y.MB^2+z.MC^2)\geq yza^2+zxb^2+xyc^2$$

$$Using\ the\ lemma\ for\ M \equiv H, the\ orthocenter\ of\ \Delta ABC, and\ (x,y,z) = (a^2,b^2,c^2)\ we\ obtain:$$

$$(a^2+b^2+c^2)(a^2.HA^2+b^2.HB^2+c^2.HC^2)\geq 3a^2b^2c^2$$

$$Since\ HA=2R|\cos A|\ \ (And\ analogs), we\ have:$$

$$(a^2+b^2+c^2).4R^2(a^2\cos^2 A+b^2\cos^2 B+c^2\cos^2 C)\geq 48R^2F^2$$

$$Therefore,$$

$$(a^2+b^2+c^2)(a^2\cos^2 A+b^2\cos^2 B+c^2\cos^2 C)\geq 12F^2.$$

G.020. *Solution by Tapas Das-India*

$$\sin^2 x\,(2\sin^2 x\cdot\sin^2 2x+4\cos^4 x+1) = \cos^2 x\,(2\cos^2 x\cdot\sin^2 2x+4\sin^4 x+1)$$

$$\sin^2 x\,(2\sin^2 x\cdot 4\sin^2 x\cos^2 x+4\cos^4 x+1)=$$

$$=\cos^2 x\,(2\cos^2 x\cdot 4\sin^2 x\cos^2 x+4\sin^4 x+1)$$

$$(8\sin^6 x\cos^2 x-8\cos^6 x\sin^2 x)+(4\cos^4 x\sin^2 x-4\sin^4 x\cos^2 x)+$$

$$+(\sin^2 x-\cos^2 x)=0$$

$$8\sin^2 x\cos^2 x\,(\sin^2 x+\cos^2 x)(\sin^2 x-\cos^2 x)+$$

$$+4\sin^2 x\cos^2 x\,(\cos^2 x-\sin^2 x)+(\sin^2 x-\cos^2 x)=0$$

$$-8\sin^2 x\cos^2 x\cos 2x+4\sin^2 x\cos^2 x\cos 2x-\cos 2x=0$$

$$-\cos 2x\,(8\sin^2 x\cos^2 x - 4\sin^2 x\cos^2 x + 1) = 0$$

$$\cos 2x\,(4\sin^2 x\cos^2 x + 1) = 0$$

$$\cos 2x\,(1 + \sin^2 2x) = 0 \Rightarrow \cos 2x = 0; (\because 1 + \sin^2 2x \neq 0)$$

$$x = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$$

G.021. *Solution by Tapas Das-India*

$$a \le b \le c \Rightarrow \tan a \le \tan b \le \tan c \Rightarrow \frac{1}{\tan a} \ge \frac{1}{\tan b} \ge \frac{1}{\tan c}$$

$$(5+3+1)\left(\frac{1}{\tan a} + \frac{1}{\tan b} + \frac{1}{\tan c}\right) \overset{Chebyshev}{\le} 3\left(\frac{5}{\tan a} + \frac{3}{\tan b} + \frac{1}{\tan c}\right)$$

$$\frac{5}{\tan a} + \frac{3}{\tan b} + \frac{1}{\tan c} \ge \frac{9}{3}\left(\frac{1}{\tan a} + \frac{1}{\tan b} + \frac{1}{\tan c}\right) =$$

$$= 3\left(\frac{1}{\tan a} + \frac{1}{\tan b} + \frac{1}{\tan c}\right) \ge 3 \cdot \frac{(1+1+1)^2}{\tan a + \tan b + \tan c} = \frac{27}{\tan a + \tan b + \tan c}.$$ *Equality holds for* $a = b = c = \frac{\pi}{4}$.

G.022. *Proposed by George Titakis-Greece*

$$c + \frac{ad}{b\cot^2\frac{\pi}{20}} + \frac{bd}{c\cot^2\frac{3\pi}{20}} + \frac{d^2}{e\cot^2\frac{7\pi}{20}} + \frac{ed}{a\cot^2\frac{9\pi}{20}} \ge$$

$$\ge 5 \cdot \sqrt[5]{c \cdot \frac{ad}{b\cot^2\frac{\pi}{20}} \cdot \frac{bd}{c\cot^2\frac{3\pi}{20}} \cdot \frac{d^2}{e\cot^2\frac{7\pi}{20}} \cdot \frac{ed}{a\cot^2\frac{9\pi}{20}}} = 5d$$

Because: $\cot^2\frac{\pi}{20}\cot^2\frac{3\pi}{20}\cot^2\frac{7\pi}{20}\cot^2\frac{9\pi}{20} = 1$

This is true, due to the trigonometric identity:

$$\cot(a+b)\,(\cot a + \cot b) = \cot a \cdot \cot b - 1; (1)$$

If $a=\frac{\pi}{20}$ *and* $b=\frac{9\pi}{20}$*, then the left part of (1) is zero, due to* $cot\,\frac{\pi}{2}=0$.
So, from the right part we have that $cot\,\frac{\pi}{20}\,cot\,\frac{9\pi}{20}=1;\,(2)$

With the same way, if $a=\frac{7\pi}{20}$ *and* $b=\frac{3\pi}{20}$:

$$cot\,\frac{7\pi}{20}\,cot\,\frac{3\pi}{20}=1;\,(3)$$

From (2) and (3) it is obvious that: $cot^2\,\frac{\pi}{20}\,cot^2\,\frac{3\pi}{20}\,cot^2\,\frac{7\pi}{20}\,cot^2\,\frac{9\pi}{20}=1$

By AM-GM inequality we have that:

$$c+\frac{ad}{b\,cot^2\,\frac{\pi}{20}}+\frac{bd}{c\,cot^2\,\frac{3\pi}{20}}+\frac{d^2}{e\,cot^2\,\frac{7\pi}{20}}+\frac{ed}{a\,cot^2\,\frac{9\pi}{20}}\geq$$

$$\geq 5\cdot\sqrt[5]{c\cdot\frac{ad}{b\,cot^2\,\frac{\pi}{20}}\cdot\frac{bd}{c\,cot^2\,\frac{3\pi}{20}}\cdot\frac{d^2}{e\,cot^2\,\frac{7\pi}{20}}\cdot\frac{ed}{a\,cot^2\,\frac{9\pi}{20}}}=5d$$

G.023. *Solution by George Florin Şerban-Romania*

$$\frac{w_a^2+w_ah_a+h_a^2}{w_a+h_a}\leq\frac{3}{2}w_a\Leftrightarrow\frac{\left(\frac{w_a}{h_a}\right)^2+\frac{w_a}{h_a}+1}{\left(\frac{w_a}{h_a}\right)^2+\frac{w_a}{h_a}}\leq\frac{3}{2}$$

$$\frac{w_a}{h_a}=x\geq 1\text{ because }w_a\geq h_a$$

$$\frac{x^2+x+1}{x^2+x}\leq\frac{3}{2}\Leftrightarrow 2x^2+2x+2\leq 3x^2+3x$$

$$x^2+x-2\geq 0\Leftrightarrow(x-1)(x+2)\geq 0\text{ true.}$$

Equality holds if and only if $w_a=h_a$.

G.024. *Solution by Togrul Ehmedov-Azerbaijan*

$$(sin\,x+cos\,x)\,sin\,x\,cos\,x+1=\left(2+\sqrt{2}\right)sin\,x\,cos\,x\;\Big|\;sin\,x\,cos\,x\neq 0$$

$$\sin x+\cos x=y\Rightarrow \sin x\cos x=\frac{y^2-1}{2}$$

$$y\cdot\frac{y^2-1}{2}1=\left(2+\sqrt{2}\right)\cdot\frac{y^2-1}{2}$$

$$y^3-\left(2+\sqrt{2}\right)y^2-y+4+\sqrt{2}=0$$

$$\left(y-\sqrt{2}\right)\left(y^2-2y-2\sqrt{2}-1\right)=0$$

$$y-\sqrt{2}=0\Rightarrow y_1=\sqrt{2}$$

$$y^2-2y-2\sqrt{2}-1=0\Rightarrow y_{2,3}=1\pm\sqrt{2+2\sqrt{2}}$$

$$y_1=\sqrt{2}=\sin x+\cos x\Rightarrow \sqrt{2}\sin\left(\frac{\pi}{4}+x\right)=\sqrt{2}$$

$$\sin\left(\frac{\pi}{4}+x\right)=1\Rightarrow x_1=\frac{\pi}{6}+2k\pi,k\in\mathbb{Z}$$

$$y_2=1+\sqrt{2+2\sqrt{2}}=\sin x+\cos x\Rightarrow x\in\emptyset$$

$$y_3=1-\sqrt{2+2\sqrt{2}}=\sin x+\cos x\Rightarrow \sqrt{2}\sin\left(\frac{\pi}{4}+x\right)=1-\sqrt{2+2\sqrt{2}}$$

$$\sin\left(\frac{\pi}{4}+x\right)=\frac{1}{\sqrt{2}}-\sqrt{1+\sqrt{2}}\Rightarrow x_2=(-1)^k\sin^{-1}\left(\frac{1}{\sqrt{2}}-\sqrt{1+\sqrt{2}}\right)$$

G.025. *Solution by George Florin Șerban-Romania*

$$\frac{m_a^2+m_ah_a+h_a^2}{m_a+h_a}=\frac{(m_a+h_a)^2-m_ah_a}{m_a+h_a}=m_a+h_a-\frac{m_ah_a}{m_a+h_a}\overset{AGM}{\geq}$$

$$\geq m_a+h_a-\frac{m_a+h_a}{4}=\frac{3(m_a+h_a)}{4}\overset{m_a\geq h_a}{\geq}\frac{3}{2}h_a$$

$$\frac{m_a^2+m_ah_a+h_a^2}{m_a+h_a}\geq\frac{3}{2}h_a$$

G.026. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$The\ inequality\ in\ the\ statement\ is\ equivalent\ to:$$

$$\sum_{cyc}\sqrt{\frac{[2a(s-a)]^3}{b(s-b)+c(s-c)}}\geq 2\sum_{cyc}a(s-a)$$

$$By\ \text{Hölder}\ inequality, we\ have:$$

$$\left(\sum_{cyc}\sqrt{\frac{[2a(s-a)]^3}{b(s-b)+c(s-c)}}\right)^2\left(\sum_{cyc}[b(s-b)+c(s-c)]\right)\geq\left(\sum_{cyc}2a(s-a)\right)^3$$

$$Since\ \sum_{cyc}[b(s-b)+c(s-c)]=\sum_{cyc}2a(s-a)\ then\ \sum_{cyc}\sqrt{\frac{[2a(s-a)]^3}{b(s-b)+c(s-c)}}\geq 2\sum_{cyc}a(s-a)$$

$$Therefore,$$

$$\sum_{cyc}\frac{s-a}{bc}\sqrt{\frac{2a(s-a)}{b(s-b)+c(s-c)}}\geq\sum_{cyc}\frac{s-a}{bc}.$$

G.027. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Let\ AA', BB', CC'\ be\ the\ Lemoine\ cevians.$$

$$We\ know\ that:\ \frac{AB'}{B'C}=\frac{c^2}{a^2}\ and\ \frac{AC'}{C'B}=\frac{b^2}{a^2}.$$

$$From\ Van\ Aubel's\ theorem, we\ have:\ \frac{KA}{KA'}=\frac{AB'}{B'C}+\frac{AC'}{C'B}=\frac{b^2+c^2}{a^2}$$

$$Then:\ \frac{s_a}{KA}=1+\frac{KA'}{KA}=1+\frac{a^2}{b^2+c^2}=\frac{a^2+b^2+c^2}{b^2+c^2}$$

$$Thus,\ \ KA=\frac{b^2+c^2}{a^2+b^2+c^2}.s_a=\frac{2bc}{a^2+b^2+c^2}.m_a$$
$$=\frac{2bc}{a^2+b^2+c^2}.\frac{3}{2}GA$$
$$=\frac{3bc}{a^2+b^2+c^2}.GA\ \ (And\ analogs)$$

$$Therefore,\qquad a^2.KA.EA+b^2.KB.EB+c^2.KC.EC$$
$$=\frac{3abc}{a^2+b^2+c^2}(a.GA.EA+b.GB.EB+c.GC.EC)$$

$Lemma: If\ P\ and\ Q\ two\ arbitrary\ points\ in\ plane\ of\ \Delta ABC\ then$ $:\ \ a.PA.QA+b.PB.QB+c.PC.QC\geq abc.$

$Reference: D.S.Mitrinovic, J.E.Pecaric\ and\ V.Volenec,$

$Recent\ Advances\ in\ Geometric\ Inequalities,$

$Kluwer\ Academic\ Publishers, (1989), 298-299.$

$For\ P\equiv G\ and\ Q\equiv E\ we\ get:$

$$a.GA.EA+b.GB.EB+c.GC.EC\geq abc.$$

$Therefore,$

$$a^2.KA.EA+b^2.KB.EB+c^2.KC.EC=\frac{3abc}{a^2+b^2+c^2}.abc$$
$$=\frac{3a^2b^2c^2}{a^2+b^2+c^2}.$$

G.028. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$We\ have: abc=4Rrs=R.2r.2s\ \overset{\substack{Mitrinovic\\ Euler}}{\leq}\ R.R.3\sqrt{3}R$$
$$=\left(\sqrt{3}R\right)^3\ (1)$$

$$\to\prod_{cyc}\left(1+\frac{1}{\sqrt[3]{a}}\right)\overset{AM-GM}{\geq}\prod_{cyc}2\sqrt{1.\frac{1}{\sqrt[3]{a}}}=\frac{8}{\sqrt[6]{abc}}\overset{(1)}{\geq}\frac{8}{\sqrt{\sqrt{3}R}}.$$

$$Similarly, we\ have: \prod_{cyc}\left(1+\frac{1}{\sqrt[3]{a'}}\right) \geq \frac{8}{\sqrt{\sqrt{3}R'}}$$

$$\rightarrow \prod_{cyc}\left(1+\frac{1}{\sqrt[3]{a}}\right)\left(1+\frac{1}{\sqrt[3]{a'}}\right) \geq \frac{8}{\sqrt{\sqrt{3}R}}\cdot\frac{8}{\sqrt{\sqrt{3}R'}}$$

$$= \frac{64}{\sqrt{3}\sqrt{RR'}} \overset{AM-GM}{\geq} \frac{2.64\sqrt{3}}{3(R+R')}$$

$$Therefore, \quad \prod_{cyc}\left(1+\frac{1}{\sqrt[3]{a}}\right)\left(1+\frac{1}{\sqrt[3]{a'}}\right) \geq \frac{128\sqrt{3}}{3(R+R')}.$$

G.029. *Solution by Adrian Popa-Romania*

$$sin\left(4x-\frac{\pi}{3}\right) sin\left(6x-\frac{\pi}{3}\right) sin\left(\frac{8\pi}{3}-10x\right)+\frac{\sqrt{3}}{8}=0$$

$$\frac{1}{2}\left[cos(-2x)-cos\left(10x-\frac{2\pi}{3}\right)\right] sin\left(\frac{8\pi}{3}-10x\right)+\frac{\sqrt{3}}{8}=0$$

$$\frac{1}{2}\left(cos(2x)+sin\left(\frac{2\pi}{3}-10x\right)-cos\left(10x-\frac{2\pi}{3}\right) sin\left(\frac{2\pi}{3}-10x\right)\right)+\frac{\sqrt{3}}{8}=0$$

$$\frac{1}{4}\left(sin\left(\frac{2\pi}{3}-8x\right) sin\left(\frac{2\pi}{3}-12x\right)+sin\left(20x-\frac{4\pi}{3}\right)\right)=0$$

$$2\left(\frac{\sqrt{3}}{2}cos\,8x-\frac{1}{2}sin\,8x+\frac{\sqrt{3}}{2}cos\,12x-\frac{1}{2}sin\,12x-\frac{1}{2}sin\,20x+\frac{\sqrt{3}}{2}cos\,20x\right)+\sqrt{3}=0$$

$$\sqrt{3}(cos\,8x+cos\,12x+cos\,20x)-(sin\,8x+sin\,12x+sin\,20x)+\sqrt{3}=0$$

$$\sqrt{3}\,cos\,10x\,cos\,2x+\sqrt{3}\,cos^2 10x-sin\,10x\,cos\,2x-sin\,10x\,cos\,10x=0$$

$$2\,cos\,2x\left(sin\frac{\pi}{3}cos\,10x-cos\frac{\pi}{3}sin\,10x\right)+2\,cos\,10x\left(sin\frac{\pi}{3}cos\,10x-cos\frac{\pi}{3}sin\,10x\right)=0$$

$$2\cos 2x \sin\left(\frac{\pi}{3}-10x\right)+2\cos 10x \sin\left(\frac{\pi}{3}-10x\right)=0$$

$$\sin\left(\frac{\pi}{3}-10x\right)(2\cos 2x+2\cos 10x)=0$$

$$(I)\ \sin\left(\frac{\pi}{3}-10x\right)=0\Rightarrow x=\frac{\pi}{30}-\frac{k\pi}{10};k\in\mathbb{Z}$$

$$(II)\ 2\cos 2x+2\cos 10x=0\Rightarrow 2\cos 6x\cos 4x=0\Rightarrow x\in\left\{\frac{k\pi}{4}+\frac{\pi}{8};\frac{k\pi}{6}+\frac{\pi}{12}\middle|k\in\mathbb{Z}\right\}$$

Therefore,

$$S=\left\{\frac{\pi}{30}-\frac{k\pi}{10};\frac{k\pi}{4}+\frac{\pi}{8};\frac{k\pi}{6}+\frac{\pi}{12}\middle|k\in\mathbb{Z}\right\}$$

G.030. *Solution by George Florin Şerban-Romania*

$$\sum_{cyc}\sin^2 A=\sum_{cyc}\frac{a^2}{4R^2}=\frac{1}{4R}\sum_{cyc}a^2\overset{Leibniz}{\leq}\frac{9R^2}{4R^2}=\frac{9}{4}$$

$$\sin A+\frac{\sin B}{\sqrt{\varphi}}+\frac{\sin C}{\varphi}\overset{CBS}{\leq}\sqrt{\sum_{cyc}\sin^2 A\cdot\left(1+\frac{1}{\varphi}+\frac{1}{\varphi^2}\right)}\leq$$

$$\leq\sqrt{\frac{9}{4}\cdot\frac{\varphi^2+\varphi+1}{\varphi^2}}=\frac{3}{2\varphi}\sqrt{2\varphi^2}=\frac{3\sqrt{2}}{2}$$

$$\sin A+\frac{\sin B}{\sqrt{\varphi}}+\frac{\sin C}{\varphi}\leq\frac{3\sqrt{2}}{2}\overset{(1)}{<}\frac{1}{\varphi}+\frac{1+\sqrt{\varphi}+\varphi}{2\sqrt{\varphi}}$$

$$(1)\Leftrightarrow\frac{1}{\varphi}+\frac{\varphi^2}{2\sqrt{\varphi}}>\frac{3\sqrt{2}-1}{2}\Leftrightarrow\frac{1}{\varphi}+\frac{\varphi\sqrt{\varphi}}{2}>\frac{3\sqrt{2}-1}{2}$$

$$\frac{1}{\varphi}+\frac{\varphi\sqrt{\varphi}}{4}+\frac{\varphi\sqrt{\varphi}}{4}\overset{AGM}{\geq}3\sqrt[3]{\frac{1}{\varphi}\cdot\frac{\varphi\sqrt{\varphi}}{4}\cdot\frac{\varphi\sqrt{\varphi}}{4}}=\frac{3}{2}\sqrt[3]{\frac{\varphi^2}{2}}\overset{(1)}{>}\frac{3\sqrt{2}-1}{2}$$

$$\Leftrightarrow \frac{\varphi^2}{2} > \frac{\left(3\sqrt{2}-1\right)^3}{27} \Leftrightarrow \frac{1+\varphi}{2} > \frac{54\sqrt{2}-54+9\sqrt{2}-1}{27}$$

$$\Leftrightarrow \left(27\sqrt{5}\right)^2 > \left(252\sqrt{2}-301\right)^2 \textit{ true.}$$

Therefore,

$$\sin A + \frac{\sin B}{\sqrt{\varphi}} + \frac{\sin C}{\varphi} < \frac{1}{\varphi} + \frac{1+\sqrt{\varphi}+\varphi}{2\sqrt{\varphi}}$$

G.031. *Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{2\pi}{\sqrt{3}(cscA+cscB+cscC)} \overset{A-G}{\leq} \frac{2\pi}{3\sqrt{3}.\sqrt[3]{cscAcscBcscC}} \overset{?}{\leq} \sqrt[3]{\mu(A)\mu(B)\mu(C)}$$

$$\Leftrightarrow \sqrt[3]{\prod_{cyc}\left(\left(\mu(A)\right)cscA\right)} \overset{?}{\geq} \frac{2\pi}{3\sqrt{3}}$$

$$\Leftrightarrow ln\left(\sqrt[3]{\prod_{cyc}\left(\left(\mu(A)\right)cscA\right)}\right) \overset{?}{\geq} ln\left(\frac{2\pi}{3\sqrt{3}}\right)$$

$$\Leftrightarrow \frac{1}{3}.ln\prod_{cyc}\left(\left(\mu(A)\right)cscA\right) \overset{?}{\geq} ln\left(\frac{2\pi}{3\sqrt{3}}\right) \Leftrightarrow \sum_{cyc} ln\left(\left(\mu(A)\right)cscA\right) \underset{(*)}{\overset{?}{\geq}} 3ln\left(\frac{2\pi}{3\sqrt{3}}\right)$$

$$Let\ f(x) = ln(xcscx)\ \forall\ x \in (0,\pi) \therefore f''(x) \overset{(\bullet)}{=} \frac{\frac{x^2}{sin^2x}-1}{x^2}\ and\ g'(x)$$
$$= 1 - cosx \geq 0\ where\ g(x) = x - sinx\ \forall\ x \in [0,\pi)$$
$$\Rightarrow g(x)\ is \uparrow on\ [0,\pi) \Rightarrow g(x) \geq g(0) = 0$$
$$\Rightarrow \forall\ x \in [0,\pi), x - sinx \geq 0\ and\ " = "\ iff\ x = 0 \Rightarrow \forall\ x \in (0,\pi), x - sinx$$
$$> 0 \Rightarrow \frac{x}{sinx} > 1 \Rightarrow \frac{x^2}{sin^2x} > 1 \Rightarrow \frac{\frac{x^2}{sin^2x}-1}{x^2} > 0 \overset{via\ (\bullet)}{\Rightarrow} f''(x)$$
$$> 0 \Rightarrow f(x)\ is\ convex\ on\ (0,\pi)$$

$$\therefore \sum_{cyc} ln\left(\left(\mu(A)\right)cscA\right) \overset{Jensen}{\geq} 3ln\left(\left(\frac{\mu(A)+\mu(B)+\mu(C)}{3}\right)csc\left(\frac{A+B+C}{3}\right)\right)$$

$$= 3ln\left(\left(\frac{\pi}{3}\right)\left(\frac{2}{\sqrt{3}}\right)\right) = 3ln\left(\frac{2\pi}{3\sqrt{3}}\right) \Rightarrow (*)\ is\ true$$

$$\Rightarrow \frac{2\pi}{\sqrt{3}(cscA + cscB + cscC)} \leq \sqrt[3]{\mu(A)\mu(B)\mu(C)}\ (QED)$$

G.032. *Solution by Tapas Das-India*

$$(tan^2 x + tan^2 y + tan^2 z)(tan^2 u + tan^2 v + tan^2 w) \overset{CBS}{\geq}$$

$$\geq (tan\, x\, tan\, u + tan\, y\, tan\, v + tan\, z\, tan\, w)^2; \quad (1)$$

Now, we know that:

$$tan\, x = x + \frac{x^3}{x} + \frac{2x^5}{15} + \cdots$$

$$tan\, u = u + \frac{u^3}{3} + \frac{2u^5}{15} + \cdots$$

$$tan\, x\, tan\, u = \left(x + \frac{x^3}{x} + \frac{2x^5}{15} + \cdots\right)\left(u + \frac{u^3}{3} + \frac{2u^5}{15} + \cdots\right) =$$

$$= ux + \frac{ux(x^2+u^2)}{3} + \frac{2ux(u^4+x^4)}{15} + \cdots$$

Now,

$$tan(ux) = ux + \frac{(ux)^3}{3} + \frac{2(ux)^5}{15} + \cdots$$

$$tan\, x\, tan\, u \geq tan(ux)$$

$$tan\, y\, tan\, v \geq tan(yv)$$

$$tan\, z\, tan\, w \geq tan(zw)$$

From (1), we have:

If $x, y, z, u, v, w \in (-1,1)$ *then:*

$$(\tan^2 x + \tan^2 y + \tan^2 z)(\tan^2 u + \tan^2 v + \tan^2 w) \geq$$

$$\geq (\tan(xu) + \tan(yv) + \tan(zw))^2$$

G.033. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$We\ have:$$

$$s = \sum \frac{a^3}{bc + a^2} = \sum \left(a - \frac{abc}{bc + a^2}\right) \overset{AM-GM}{\geq} \sum \left(a - \frac{abc}{2\sqrt{bc.a^2}}\right) =$$

$$= \sum \left(a - \frac{\sqrt{bc}}{2}\right) \overset{AM-GM}{\geq} \sum \left(a - \frac{b + c}{2.2}\right) = \frac{1}{2} \sum a = s$$

$$With\ equality\ iff\ a = b = c \leftrightarrow \Delta ABC\ is\ equilateral \leftrightarrow 2s = 3\sqrt{3}R.$$

$$Therefore, \quad \sum \frac{a^3}{bc + a^2} = s \leftrightarrow 2s = 3\sqrt{3}R.$$

G.034. *Solution by Jose Ferreira Queiroz-Olinda-Brazil*

Using algebraic identity:

$$(x + y + z)^5 = x^5 + y^5 + z^5 + 5(x + y)(y + z)(z + x)(x^2 + y^2 + z^2 + xy + yz + zx); (1)$$

Now, replacing $x = s - a, y = s - b, z = s - c$ *we get:*

$$x + y + z = 3s - (a + b + c) = 3s - 2s = s; (2)$$

$$x + y = 2s - (a + b) = 2s - (2s - c) = c$$

$$x + z = b, y + z = a$$

Hence, $(x + y)(y + z)(z + x) = abc; (3)$

$$x^2+y^2+z^2+xy+yz+zx$$
$$=x^2+y^2+z^2+\frac{1}{2}[(a+y+z)^2-x^2-y^2-z^2]=$$
$$=\frac{1}{2}[(x+y+z)^2+x^2+y^2+z^2]$$
$$=\frac{1}{2}[s^2+(s-a)^2+(s-b)^2+(s-c)^2]=$$
$$=\frac{a^2+b^2+c^2}{4};(4)$$

Using (2),(3) and (4) in (1), we have:

$$s^5=(s-a)^5+(s-b)^5+(s-c)^5+5abc\cdot\frac{a^2+b^2+c^2}{2}$$
$$s^5=(s-a)^5+(s-b)^5+(s-c)^5+5\cdot 4Rrs\cdot\frac{a^2+b^2+c^2}{2}$$

Therefore,

$$(s-a)^5+(s-b)^5+(s-c)^5+10Rrs(a^2+b^2+c^2)=s^5$$

G.035. *Solution by Fayssal Abdelli-Bejaia-Algerie*

Let $a=\sin x, b=\cos y; a,b\in[-1,1]\Rightarrow\frac{1}{1+|a|}+\frac{1}{1+|b|}=1+\frac{1}{1+|a+b|}\Rightarrow$

$$\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}=\frac{|a+b|}{1+|a+b|}\Rightarrow\frac{|a|+2|ab|+|b|}{1+|a|+|b|+|ab|}=\frac{|a+b|}{1+|a+b|}$$
$$\Rightarrow|a|+2|ab|+|b|+|a|\cdot|a+b|+2|ab|\cdot|a+b|+|b|\cdot|a+b|-|a+b|-$$
$$-|a|\cdot|a+b|-|b|\cdot|a+b|-|ab|\cdot|a+b|=0$$
$$\Rightarrow|a|+|b|+|ab|\cdot|a+b|+2|ab|-|a+b|=0;(A)$$

1) $a=0$ *and* $b=0$ *true.*

2) $a>0$ *and* $b>0$, *from* $(A)\Rightarrow a+b+ab(a+b)+2ab-a-b=0$

$\Rightarrow ab(a+b+2)=0 \Rightarrow a+b+2=0 \Rightarrow a+b=-2 \Rightarrow a=-1$ *and* $b=-1$ *impossible, because* $a,b>0$.

3) $a<0$ *and* $b<0$*, from* $(A) \Rightarrow$

$$-a-b-ab(a+b)+2ab+a+b=0 \Rightarrow$$

$ab(2-a-b)=0 \Rightarrow a+b=2$ *impossible because* $a,b<0$.

4) $a<0$ *and* $b>0$*, from* $(A) \Rightarrow$

$$-a+b-ab|a+b|-2ab-|a+b|=0$$

4.1) If $a+b>0$*, from* $(A) \Rightarrow$

$$-a+b-ab(a+b)-2ab-a-b=0 \Rightarrow$$

$$-2a-a^2b-ab^2-2ab=0 \Rightarrow a(2+ab+b^2+2b)=0$$

impossible because $a<0$ *and* $2+ab+b^2+2b>0$.

4.2) If $a+b<0$*, from* $(A) \Rightarrow -a+b+ab(a+b)-2ab+a+b=0$

$\Rightarrow 2b+a^2b+ab^2-2ab=0 \Rightarrow b(2+a^2+ab-2a)=0$ *impossible.*

5) If $a>0$ *and* $b<0$*, from* $(A) \Rightarrow a-b-ab|a+b|-2ab-|a+b|=0$

5.1) If $a+b>0$*, from* $(A) \Rightarrow a-b-ab(a+b)-2ab-a-b=0$

$$-2b-a^2b-ab^2-2ab=0 \Rightarrow b(2+a^2+ab+2a)=0$$

impossible because $b<0$ *and* $2+a^2+ab+2a>0$.

5.2) If $a+b<0$*, from* $(A) \Rightarrow a-b+ab(a+b)-2ab+a+b=0 \Rightarrow$

$2a+a^2b+ab^2-2ab=0 \Rightarrow a(2+ab+b^2-2b)=0$ *impossible because* $a>0$ *and* $2+ab+b^2-2b>0$. *Finally,*

$$\frac{1}{1+|a|}+\frac{1}{1+|b|}=1+\frac{1}{1+|a+b|} \Rightarrow a=0, b=0 \Rightarrow \sin x=0, \cos y=0 \Rightarrow$$

$$(x,y)\in\left\{\left(k\pi,\frac{\pi}{2}+k\pi\right)\middle|k\in\mathbb{Z}\right\}$$

G.036. *Solution by George Florin Şerban-Romania*

First, we prove that:

$$sin^{-1}\left(\frac{4}{5}\right)+sin^{-1}\left(\frac{5}{13}\right)+sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$$

$$sin^{-1}\left(\frac{4}{5}\right)+sin^{-1}\left(\frac{5}{13}\right)=sin^{-1}\left(\frac{4}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{16}{25}}\right)=$$

$$=sin^{-1}\left(\frac{63}{65}\right)$$

$$sin^{-1}\left(\frac{63}{65}\right)+sin^{-1}\left(\frac{16}{65}\right)=sin^{-1}\left(\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2}+\frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right)=$$

$$=sin^{-1}1=\frac{\pi}{2}$$

Hence,

$$sin^{-1}\left(\frac{4}{5}\right)+sin^{-1}\left(\frac{5}{13}\right)+sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$$

Now, we have:

$$a\cdot\sqrt{sin^{-1}\left(\frac{4}{5}\right)}+b\cdot\sqrt{sin^{-1}\left(\frac{5}{13}\right)}+c\cdot\sqrt{sin^{-1}\left(\frac{16}{64}\right)}\overset{CBS}{\leq}$$

$$\leq\sqrt{a^2+b^2+c^2}\cdot\sqrt{sin^{-1}\left(\frac{4}{5}\right)+sin^{-1}\left(\frac{5}{13}\right)+sin^{-1}\left(\frac{16}{65}\right)}\overset{Leibniz}{\leq}$$

$$\leq\sqrt{9R^2}\cdot\sqrt{\frac{\pi}{2}}=\frac{3R\sqrt{2\pi}}{2}$$

Therefore,

$$a \cdot \sqrt{\sin^{-1}\left(\frac{4}{5}\right)} + b \cdot \sqrt{\sin^{-1}\left(\frac{5}{13}\right)} + c \cdot \sqrt{\sin^{-1}\left(\frac{16}{64}\right)} < \frac{3R\sqrt{2\pi}}{2}$$

$$a^2 + b^2 + c^2 \leq 9R^2 \ (Leibniz)$$

Equality holds if and only if $a = b = c$.

$$\frac{a}{\sqrt{\sin^{-1}\left(\frac{4}{5}\right)}} = \frac{b}{\sqrt{\sin^{-1}\left(\frac{5}{13}\right)}} = \frac{c}{\sqrt{\sin^{-1}\left(\frac{16}{64}\right)}} \ false\ because$$

$a = b = c$ *but* $\sin^{-1}\left(\frac{4}{5}\right) = \sin^{-1}\left(\frac{5}{13}\right)$. *So, inequality is strictly.*

G.037. *Solution by Ruxandra Daniela Tonilă-Romania*

Let be the function: $f: (0, \infty) \to \mathbb{R}, f(x) = x \cdot \log x, f'(x) = 1 + \log x$,

$f''(x) = \frac{1}{x} > 0 \Rightarrow f$ *−convexe function.*

$$f\left(\frac{\tan A + \tan B + \tan C}{3}\right) \leq \frac{f(\tan A) + f(\tan B) + f(\tan C)}{3} \Leftrightarrow$$

$$\frac{1}{3}(\tan A \cdot \log(\tan A) + \tan B \cdot \log(\tan B) + \tan C \cdot \log(\tan C) \geq$$

$$\geq \frac{1}{3}(\tan A + \tan B + \tan C) \cdot \log\left(\frac{\tan A + \tan B + \tan C}{3}\right) \Leftrightarrow$$

$$\log((\tan A)^{\tan A} \cdot (\tan B)^{\tan B} \cdot (\tan C)^{\tan C}) \geq$$

$$\geq (\tan A + \tan B + \tan C) \cdot \log\left(\frac{\tan A + \tan B + \tan C}{3}\right) \overset{AGM}{\geq}$$

$$\geq (\tan A + \tan B + \tan C) \cdot \log\left(\sqrt[3]{\tan A \cdot \tan B \cdot \tan C}\right) \Leftrightarrow$$

$$\log((\tan A)^{\tan A} \cdot (\tan B)^{\tan B} \cdot (\tan C)^{\tan C}) \geq \log(\tan A \cdot \tan B \cdot \tan C)^{\frac{\tan A + \tan B + \tan C}{3}} \Leftrightarrow$$

$$(\tan A)^{\tan A}\cdot(\tan B)^{\tan B}\cdot(\tan C)^{\tan C}\geq(\tan A\cdot\tan B\cdot\tan C)^{\frac{\tan A+\tan B+\tan C}{3}}\Leftrightarrow$$

$$(\tan A)^{3\tan A}\cdot(\tan B)^{3\tan B}\cdot(\tan C)^{3\tan C}\geq(\tan A\cdot\tan B\cdot\tan C)^{\tan A+\tan B+\tan C};(1)$$

$$\tan A+\tan B+\tan C=\frac{\sin(A+B)}{\cos A\cdot\cos B}+\tan C=\frac{\sin(\pi-C)}{\cos A\cdot\cos B}+\frac{\sin C}{\cos C}=$$

$$=\sin C\left(\frac{1}{\cos A\cdot\cos B}+\frac{1}{\cos C}\right)=\sin C\cdot\frac{\cos C+\cos A\cdot\cos B}{\cos A\cdot\cos B\cdot\cos C}=$$

$$=\tan C\cdot\frac{\cos\big(\pi-(A+B)\big)+\cos A\cdot\cos B}{\cos A\cdot\cos B}=\tan C\cdot\frac{\cos A\cdot\cos B-\cos(A+B)}{\cos A\cdot\cos B}=$$

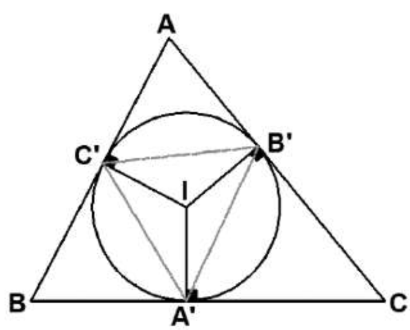
$$=\tan C\cdot\frac{\sin A\cdot\sin B}{\cos A\cdot\cos B}=\tan A\cdot\tan B\cdot\tan C;(2)$$

From (1),(2) it follows that:

$$(\tan A)^{3\tan A}\cdot(\tan B)^{3\tan B}\cdot(\tan C)^{3\tan C}\geq(\tan A\cdot\tan B\cdot\tan C)^{\tan A\cdot\tan B\cdot\tan C}$$

G.038. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$(*):xIA+yIB+zIC\geq 4\left(\frac{yzIA'}{y+z}+\frac{zxIB'}{z+x}+\frac{xyIC'}{x+y}\right)$$



$$We\ have:\quad IA'=IB'=IC'=r\ \ and$$

$$IA=\frac{r}{\sin\frac{A}{2}}, IB=\frac{r}{\sin\frac{B}{2}}, IC=\frac{r}{\sin\frac{C}{2}}.$$

$$Then\ (*)\ is\ equivalent\ to:$$

$$\frac{x}{\sin\frac{A}{2}}+\frac{y}{\sin\frac{B}{2}}+\frac{z}{\sin\frac{C}{2}}\ge 4\left(\frac{yz}{y+z}+\frac{zx}{z+x}+\frac{xy}{x+y}\right)$$

$$We\ have:$$

$$\frac{x}{\sin\frac{A}{2}}+\frac{y}{\sin\frac{B}{2}}+\frac{z}{\sin\frac{C}{2}}\overbrace{\ge}^{Bergström}\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}$$
$$\ge\frac{3\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}$$

$$Since\ x\to\sin x\ \ is\ a\ concave\ function\ on\ \left(0,\frac{\pi}{2}\right)\ then,$$

$$by\ Jensen's\ inequality, we\ have:$$

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le 3\sin\frac{A+B+C}{6}=3\sin\frac{\pi}{6}=\frac{3}{2}$$

$$Also, by\ GM-HM\ inequality, we\ have:$$

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\ge\frac{2yz}{y+z}+\frac{2zx}{z+x}+\frac{2xy}{x+y}$$

$$Therefore,$$

$$\frac{x}{\sin\frac{A}{2}}+\frac{y}{\sin\frac{B}{2}}+\frac{z}{\sin\frac{C}{2}}\ge 4\left(\frac{yz}{y+z}+\frac{zx}{z+x}+\frac{xy}{x+y}\right)\ \ and\ \ (*)\ is\ true.$$

G.039. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$By\ AM-GM\ inequality\ we\ have:$$

$$\frac{2bc}{(b+c)^2}+\frac{1}{2}\geq 2\sqrt{\frac{2bc}{(b+c)^2}.\frac{1}{2}}=\frac{2\sqrt{bc}}{b+c}=\frac{8abc}{2\sqrt{ab}.2\sqrt{ca}.(b+c)}$$
$$\geq\frac{8abc}{(a+b)(c+a)(b+c)},\ \ with:$$

$$(a+b)(c+a)(b+c)=(a+b+c)(ab+bc+ca)-abc$$
$$=2s(s^2+r^2+4Rr)-4sRr=2s(s^2+r^2+2Rr)$$

$$Therefore,$$

$$\frac{2bc}{(b+c)^2}+\frac{1}{2}\geq\frac{8.4sRr}{2s(s^2+r^2+2Rr)}=\frac{16Rr}{s^2+r^2+2Rr}.$$

G.040. *Solution by George Florin Șerban-Romania*

$$R^2R'F'=R^2R's'r'\overset{(1)}{\geq}8F(r')^3=8rs(r')^3\Leftrightarrow$$

$$R^2R's'\geq 8rs(r')^2\Leftrightarrow\frac{s'}{s}\geq\frac{8r(r')^2}{R^2R'};(2)$$

$$\frac{s'}{s}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}r'}{s}\geq\frac{3\sqrt{3}r'}{\frac{3\sqrt{3}R}{2}}=\frac{2r'}{R}\geq\frac{8r(r')^2}{R^2R'}$$

$RR'\geq 4rr'$ *which is true from* $R\geq 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

G.041. *Solution by George Florin Șerban-Romania*

$$\frac{m_a^3\cdot(a')^2}{a^2}+\frac{m_b^3\cdot(b')^2}{b^2}+\frac{m_c^3\cdot(c')^2}{c^2}=\sum_{cyc}\frac{m_a^3\cdot(a')^2}{a^2}=\sum_{cyc}\frac{m_a^3}{\left(\frac{a}{a'}\right)^2}\overset{Radon}{\geq}$$

$$\geq \frac{(\sum m_a)^3}{\left(\sum \frac{a}{a'}\right)^2} \geq \frac{(\sum m_a)^3}{\sum a^2 \cdot \sum \frac{1}{(a')^2}} \overset{Leibniz}{\geq} \frac{(\sum m_a)^3}{9R^2 \cdot \frac{1}{4(r')^2}}$$

$$= \frac{4(r')^2 (\sum m_a)^3}{9R^2} \overset{(1)}{\geq} \frac{32s^6(r')^2}{243R^5}$$

$$(1) \Leftrightarrow \left(\sum_{cyc} m_a\right)^3 \geq \frac{8s^6}{27R^3} \Rightarrow \sum_{cyc} m_a \geq \frac{2s^2}{3R}; (2)$$

$$\sum_{cyc} m_a \overset{Tereshin}{\geq} \sum_{cyc} \frac{b^2+c^2}{4R} = \frac{1}{2R}\sum_{cyc} a^2 = \frac{s^2-r^2-4Rr}{R} \geq \frac{2s^2}{3R}$$

$$s^2 \geq 16Rr - 5r^2 \geq 12Rr + 3r^2 \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow 4r(R-2r) \geq 0$$

true from

$R \geq 2r(Euler).$

$$\frac{m_a^3 \cdot (a')^2}{a^2} + \frac{m_b^3 \cdot (b')^2}{b^2} + \frac{m_c^3 \cdot (c')^2}{c^2} \geq \frac{32s^6(r')^2}{243R^5}$$

G.042. *Solution by Ertan Yildirim-Turkiye*

$$ab + bc + ca = s^2 + r^2 + 4Rr$$

$$\sum_{cyc} \sqrt{(a+b)(a+c)bc} \leq$$

$$\leq \sqrt{(a+b)c + (b+c)a + (c+a)b} \cdot \sqrt{(a+c)b + (b+a)c + (c+b)a} =$$

$$= \sqrt{2(ab+bc+ca)} \cdot \sqrt{2(ab+bc+ca)} = 2(ab+bc+ca) = 2(s^2 + R^2 + 4rR)$$

We must to prove that:

$$2s^2 + 2r^2 + 8Rr \leq 3s^2 - r^2 - 4Rr \Leftrightarrow 3r^2 + 12Rr \leq s^2$$

$$3r^2 + 12Rr \overset{Gerretsen}{\leq} 16Rr - 5r^2 \leq s^2 \Leftrightarrow R \geq 2r\ (Euler).$$

G.043. *Solution by Ravi Prakash-New Delhi-India*

$$\prod_{cyc}(2-\cos^2 x)+\prod_{cyc}\cos^2 x=8$$

$$8-4\sum_{cyc}\cos^2 x+2\sum_{cyc}\cos^2 x\cos^2 y\cos^2 z-\cos^2 x\cos^2 y\cos^2 z+$$

$$+\cos^2 x\cos^2 y\cos^2 z=8$$

$$\Rightarrow 2\sum_{cyc}\cos^2 x=\sum_{cyc}\cos^2 x\cos^2 y$$

$$\Rightarrow\sum_{cyc}\cos^2 x+\sum_{cyc}\cos^2 x\,(1-\cos^2 y)=0$$

$$\sum_{cyc}\cos^2 x+\sum_{cyc}\cos^2 x\sin^2 y=0$$

Equality holds for $\sin^2 x=\sin^2 y=\sin^2 z\Leftrightarrow$

$$\sin x=\sin y=\sin z\Leftrightarrow x=y=z=\frac{\pi}{2}.$$

Therefore, $S=\left\{\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)\right\}$

G.044. *Solution by Tapas Das-India*

$$2\cos^2 x\cos^2 y\cos^2 z=\frac{1}{4}(2\cos^2 x\cdot 2\cos^2 y\cdot 2\cos^2 z)=$$

$$=\frac{1}{4}(1+\cos 2x)(1+\cos 2y)(1+\cos 2z)=\frac{1}{4}(1+a)(1+b)(1+c)$$

$$a=\cos 2x\,,b=\cos 2y\,,c=\cos 2z\Rightarrow a,b,c\in(0,1)$$

$$0\le x\le\frac{\pi}{4}\Rightarrow 0\le 2x\le\frac{\pi}{2}$$

$$(a-1)(b-1)\ge 0\Rightarrow ab+1\ge a+b$$

$$2(ab+1)\geq a+b+ab+1=(a+1)(b+1);(1)$$

$$2(ab+1)\cdot 2(c+1)\geq 2(a+1)(b+1)(c+1)$$

$$4(ab+1)(c+1)\geq 2(a+1)(b+1)(c+1)$$

$$2(ab+1)(c+1)\geq (a+1)(b+1)(c+1)$$

$$(ab+1)(c+1)\geq \frac{1}{2}(a+1)(b+1)(c+1);(2)$$

From (1) and (2), we get :

$$2(abc+1)\geq \frac{1}{2}(a+1)(b+1)(c+1)$$

$$abc+1\geq \frac{1}{4}(a+1)(b+1)(c+1)$$

$$1+\cos 2x\cos 2y\cos 2z\geq \frac{1}{2}(1+\cos 2x)(1+\cos 2y)(1+\cos 2z)$$

$$1+\cos 2x\cos 2y\cos 2z\geq \frac{1}{4}\cdot 2\cos 2x\cos 2y\cos 2z=$$

$$=2\cos^2 x\cos^2 y\cos^2 z$$

$$8\cos^2 x\cos^2 y\cos^2 z=(1+\cos 2x)(1+\cos 2y)(1+\cos 2z)=$$

$$=1+a+b+c+ab+bc+ca\geq 1+abc=$$

$$=1+\cos 2x\cos 2y\cos 2z$$

G.045. *Solution by Fayssal Abdelli-Bejaia-Algerie*

$$\cos x\cdot\sqrt{\tan x}=\sin^3 x+\cos^3 x$$

$$\cos x\cdot\sqrt{\tan x}=(\sin x+\cos x)(1-\sin x\cos x)$$

$$\cos^2 x\cdot\tan x=(\sin x+\cos x)^2(1-\sin x\cos x)^2$$

$$\sin x\cdot\cos x=(1+2\sin x\cos x)(1+\sin^2 x\cos^2 x-2\sin x\cos x)$$

Let $y = \sin x \cos x\,; -\frac{1}{2} \le y \le \frac{1}{2}$ *because* $-\frac{1}{2} \le \frac{1}{2}\sin 2x \le \frac{1}{2}$

$$\Rightarrow y = (1+2y)(1-2y+y^2) \Rightarrow 2y^3 - 3y^2 - y + 1 = 0$$

$$\left(y-\frac{1}{2}\right)(y^2-y-1) = 0 \Rightarrow y = \frac{1}{2} \text{ or } y^2 - y - 1 = 0 \Rightarrow$$

$$y_1 = \frac{1-\sqrt{5}}{2} < -\frac{1}{2}$$

$y_2 = \frac{1+\sqrt{5}}{2} > 1$. *Thus,* $y = \frac{1}{2} \Rightarrow \sin x \cos x = \frac{1}{2} \Rightarrow \sin 2x = 1$

$$2x = \frac{\pi}{2} + 2k\pi \Rightarrow x = \frac{\pi}{4} + k\pi$$

$$S = \left\{\frac{\pi}{4} + k\pi \,\middle|\, x \in \mathbb{Z}\right\}$$

G.046. *Solution by Tapas Das-India*

$$\frac{2}{1+\tan x} + \frac{2}{1+\tan y} + \frac{1+\cot x \cdot \tan y}{1+\cot x} + \frac{1+\cot y \cdot \tan x}{1+\cot y} =$$

$$= \frac{2}{1+\tan x} + \frac{2}{1+\tan y} + \frac{1+\cot x \cdot \tan y}{1+\frac{1}{\tan x}} + \frac{1+\cot y \cdot \tan x}{1+\frac{1}{\tan y}} =$$

$$= 2\left(\frac{1}{1+\tan x} + \frac{1}{1+\tan y}\right) + \frac{\tan x + \tan y}{1+\tan x} + \frac{\tan y + \tan x}{1+\tan y} =$$

$$= \left(\frac{1}{1+\tan x} + \frac{1}{1+\tan y}\right)(2 + \tan x + \tan y) \ge$$

$$\ge 2\sqrt{\frac{1}{(1+\tan x)(1+\tan y)}} \cdot [(1+\tan x) + (1+\tan y)] \ge$$

$$\ge 2\sqrt{\frac{1}{(1+\tan x)(1+\tan y)}} \cdot 2\sqrt{(1+\tan x)(1+\tan y)} \ge 4$$

G.047. *Solution by Amir Sofi-Kosovo*

$$\sin z + \sqrt[3]{x^2y}\cdot\cos z \le \sqrt{1+\sqrt[3]{x^4y^2}}\cdot\sqrt{\sin^2 z+\cos^2 z} = \sqrt{1+\sqrt[3]{x^4y^2}}$$

$$\sin z + \sqrt[3]{xy^2}\cdot\cos z \le \sqrt{1+\sqrt[3]{x^2y^4}}\cdot\sqrt{\sin^2 z+\cos^2 z} = \sqrt{1+\sqrt[3]{x^2y^4}}$$

Hence, we get:

$$\left(\sin z+\sqrt[3]{x^2y}\cdot\cos z\right)\left(\sin z+\sqrt[3]{xy^2}\cdot\cos z\right) \le \sqrt{\left(1+\sqrt[3]{x^4y^2}\right)\left(1+\sqrt[3]{x^2y^4}\right)} \le$$

$$\le 1+\frac{\sqrt[3]{x^4y^2}+\sqrt[3]{x^2y^4}}{2} \le 1+\frac{x^2+y^2}{2} \le 1+\frac{(x+y)^2}{4}$$

Therefore,

$$4\left(\sin z+\sqrt[3]{x^2y}\cdot\cos z\right)\left(\sin z+\sqrt[3]{xy^2}\cdot\cos z\right) \le 4+(x+y)^2$$

G.048. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$We\ have:$$

$$\prod_{cyc}(1+\tan A\cot B) = 2+\sum_{cyc}\tan A\cot B+\sum_{cyc}\tan B\cot C\,.\tan C\cot A$$
$$= 2+\sum_{cyc}\frac{\tan A}{\tan B}+\sum_{cyc}\frac{\tan B}{\tan A}$$

$$By\ AM-GM\ inequality\ we\ have:$$

$$\sum_{cyc}\frac{\tan A}{\tan B} = \sum_{cyc}\frac{1}{3}\left(\frac{\tan A}{\tan B}+\frac{\tan A}{\tan B}+\frac{\tan B}{\tan C}\right) \ge \sum_{cyc}\sqrt[3]{\frac{\tan^2 A}{\tan B\,.\tan C}}$$
$$= \frac{\tan A+\tan B+\tan C}{\sqrt[3]{\tan A\,.\tan B\,.\tan C}}$$

$$And\ since\ \tan A+\tan B+\tan C = \tan A\,.\tan B\,.\tan C$$

$$then\ we\ get:$$

$$\sum_{cyc}\frac{\tan A}{\tan B}\geq\sqrt[3]{(\tan A.\tan B.\tan C)^2}.\ Similarly\ we\ have:$$

$$\sum_{cyc}\frac{\tan B}{\tan A}\geq\sqrt[3]{(\tan A.\tan B.\tan C)^2}$$

$$Then:$$

$$\prod_{cyc}(1+\tan A\cot B)\geq 2+2\prod_{cyc}\sqrt[3]{\tan^2 A}=2+2\prod_{cyc}\sqrt[3]{\left(\frac{4F}{b^2+c^2-a^2}\right)^2}$$

$$=2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}.$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral.$$

G.049. *Solution by Ertan Yildirim-Turkiye*

$$\sum m_a m_b\geq s^2;(1)$$

$$m_a\geq\sqrt{s(s-a)};(2)$$

$$\sum m_a\overset{(2)}{\geq}\sum\sqrt{s(s-a)}\overset{AGM}{\geq}3\sqrt[3]{\sqrt{s^2\cdot s(s-a)(s-b)(s-c)}}=$$

$$=3\sqrt[3]{sF}=3\sqrt[3]{s\cdot sr}=3\sqrt[3]{s^2r}\overset{Mitrinovic}{\geq}3\sqrt[3]{\left(3\sqrt{3}r\right)^2\cdot r}=$$

$$=3\sqrt[3]{27r^3}=9r$$

$$(m_am_b+m_bm_c+m_cm_a)(m_a+m_b+m_c)^2\geq s^2\cdot(9r)^2=81s^2r^2=81F^2$$

G.050. *Solution by George Florin Şerban-Romania*

Lemma. $AK=\frac{(b^2+c^2)s_a}{a^2+b^2+c^2}=\frac{(b^2+c^2)\cdot\frac{2bcm_a}{b^2+c^2}}{a^2+b^2+c^2}=\frac{2bcm_a}{a^2+b^2+c^2}$

Now, we have:

$$\sum_{cyc}\frac{m_a}{AK\cdot\sin A}=\sum_{cyc}\frac{m_a(a^2+b^2+c^2)}{2bcm_a\cdot\sin A}=\frac{1}{2}(a^2+b^2+c^2)\sum_{cyc}\frac{1}{bc\cdot\frac{a}{2R}}$$

$$=\frac{1}{2}(a^2+b^2+c^2)\cdot\sum_{cyc}\frac{2R}{abc}=R(a^2+b^2+c^2)\cdot\frac{3}{abc}=$$

$$=\frac{3R}{4RF}(a^2+b^2+c^2)=\frac{3}{4F}(a^2+b^2+c^2)\overset{Ionescu-Wetzenbock}{\geq}$$

$$\geq\frac{3}{4F}\cdot 4F\sqrt{3}=3\sqrt{3}$$

Therefore,

$$\frac{m_a}{AK\cdot\sin A}+\frac{m_b}{BK\cdot\sin B}+\frac{m_c}{CK\cdot\sin C}\geq 3\sqrt{3}$$

G.051. *Solution by George Florin Șerban-Romania*

Let $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\tan x$ *then*

$f'(x)=\frac{1}{\cos^2 x}, f''(x)=\frac{2\sin x}{\cos^3 x}>0\Rightarrow f$ *−convexe.*

Hence,

$$\tan\left(\frac{A+B+C}{3}\right)\leq\frac{1}{3}(\tan A+\tan B+\tan C)\Leftrightarrow$$

$$\sqrt{3}\leq\frac{1}{3}\tan A\tan B\tan C\Leftrightarrow\tan A\tan B\tan C\geq 3\sqrt{3}$$

Thus,

$$\sum_{cyc}\frac{\tan^4 A\cdot\tan^4 B}{x^3y^3}=\sum_{cyc}\frac{(\tan A\cdot\tan B)^4}{(xy)^3}\overset{Radon}{\geq}\frac{(\sum\tan A\cdot\tan B)^4}{(\sum xy)^3}\overset{AM-GM}{\geq}$$

$$\geq 3\left(\sqrt[3]{\tan A\tan B\tan C}\right)^8\geq 3\left(\sqrt[3]{3\sqrt{3}}\right)^8=243$$

Therefore,

$$\frac{\tan^4 A \cdot \tan^4 B}{x^3y^3} + \frac{\tan^4 B \cdot \tan^4 C}{y^3z^3} + \frac{\tan^4 C \cdot \tan^4 A}{z^3x^3} \geq 243$$

G.052. *Solution by Tapas Das-India*

$$\frac{1}{\frac{1}{\sin x + \sin y} + \frac{1}{\cos x + \cos y}} = \frac{1}{2} \cdot \frac{2}{\frac{1}{\sin x + \sin y} + \frac{1}{\cos x + \cos y}} \overset{HM-AM}{\leq}$$

$$\leq \frac{1}{2} \cdot \frac{\sin x + \sin y + \cos x + \cos y}{2} = \frac{1}{4}[(\sin x + \cos x) + (\sin y + \cos y)] \leq \frac{\sqrt{2}}{2}$$

$$\sin x + \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right) =$$

$$= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \leq \sqrt{2}(\text{ and analogs})$$

G.053. *Solution by Tapas Das-India*

$$Let\ f(x) = \frac{\sin^2 x}{\cos x} = \frac{1 - \cos^2 x}{\cos x} = \sec x - \cos x$$

$$f'(x) = \sec x \tan x + \sin x$$

$$f''(x) = \sec x + \tan^2 x + \sec^3 x + \cos x > 0 \Rightarrow f \text{ −convex function on } \left(0, \frac{\pi}{2}\right)$$

$$\frac{3\sin^2 A}{\cos A} + \frac{2\sin^2 B}{\cos B} + \frac{\sin^2 C}{\cos C} = 3f(A) + 2f(B) + f(C) \geq$$

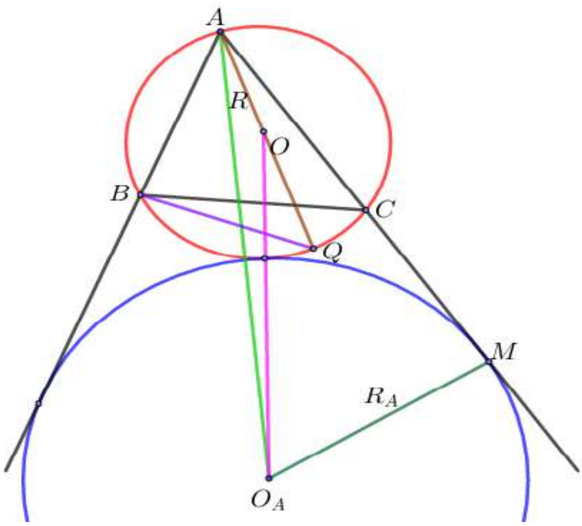
$$\geq (3 + 2 + 1)f\left(\frac{3A + 2B + C}{3 + 2 + 1}\right) = 6f\left(\frac{3A + 2B + C}{6}\right) =$$

$$= 6 \cdot \frac{\sin^2\left(\frac{3A + 2B + C}{6}\right)}{\cos\left(\frac{3A + 2B + C}{6}\right)} = 6 \cdot \frac{\sin^2\left(\frac{\pi + 2A + B}{6}\right)}{\cos\left(\frac{\pi + 2A + B}{6}\right)}$$

Therefore,

$$\left(\frac{3\,sin^2 A}{cos\,A}+\frac{2\,sin^2 B}{cos\,B}+\frac{sin^2 C}{cos\,C}\right)cos\left(\frac{\pi+2A+B}{6}\right)$$
$$\geq 6\,sin^2\left(\frac{\pi+2A+B}{6}\right)$$

G.054. *Solution by Adrian Popa-Romania*



O_A *−tangent of* $AB, AC \Rightarrow O, O_A$ *−tangent in* $T \Rightarrow O, T, O_A$ *−collinear and*

$$OO_A \equiv OT + TO_A = R + R_A$$

In $\Delta OO_AM, (\widehat{M} = 90^\circ)$: $sin\,\widehat{O_AOM} = \frac{O_AM}{AO_A} \Rightarrow sin\frac{A}{2} = \frac{R_A}{AO_A} \Rightarrow AO_A = \frac{R_A}{sin\frac{A}{2}}$

$\widehat{O_AAO} = \widehat{BAO} - \widehat{BAO_A}, AQ$ *−diameter, then* $\widehat{ABQ} = 90^\circ \Rightarrow$

$$\widehat{BAO} = 90^\circ - \widehat{ABQ} = 90^\circ - \frac{\widehat{AB}}{2} = 90^\circ - \hat{C} \Rightarrow \widehat{O_AAO} = 90^\circ - \hat{C} - \frac{\hat{A}}{2}$$
$$= \frac{\hat{B}-\hat{C}}{2}$$

From Law of cosines in ΔAO_AO: $OO_A^2 = AO_A^2 - 2OO_A \cdot AO \cdot cos\,\widehat{OAO_A}$

$$(R+R_A)^2=\frac{R_A^2}{sin^2\frac{A}{2}}+R^2-2\cdot\frac{R_A}{sin\frac{A}{2}}\cdot R\cdot cos\left(\frac{B-C}{2}\right)$$

$$R^2+2RR_A+R_A^2=\frac{R_A^2}{sin^2\frac{A}{2}}+R^2-\frac{2RR_A}{sin\frac{A}{2}}\cdot cos\left(\frac{B-C}{2}\right)$$

$$\frac{R_A^2}{sin^2\frac{A}{2}}-R_A^2=2RR_A+\frac{2RR_A}{sin\frac{A}{2}}\cdot cos\left(\frac{B-C}{2}\right)\Rightarrow$$

$$R_A-R_A\cdot sin^2\frac{A}{2}=2R\cdot sin^2\frac{A}{2}+2R\cdot sin\frac{A}{2}\cdot cos\left(\frac{B-C}{2}\right)\Rightarrow$$

$$R_A\cdot cos^2\frac{A}{2}=2R\cdot sin\frac{A}{2}\left(sin\frac{A}{2}+cos\left(\frac{B-C}{2}\right)\right)$$

$$sin\frac{A}{2}+cos\left(\frac{B-C}{2}\right)=sin\frac{\pi-(B+C)}{2}+cos\left(\frac{B-C}{2}\right)$$
$$=cos\left(\frac{B+C}{2}\right)+cos\left(\frac{B-C}{2}\right)=$$

$$=2\,cos\frac{\frac{B+C}{2}+\frac{B-C}{2}}{2}cos\frac{\frac{B+C}{2}-\frac{B-C}{2}}{2}=2\,cos\frac{B}{2}cos\frac{C}{2}$$

Hence,

$$R_A\,cos^2\frac{A}{2}=4R\cdot sin\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}\Rightarrow$$

$$R_A=\frac{4R\cdot sin\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}}{cos^2\frac{A}{2}}=\frac{r_a}{cos^2\frac{A}{2}}$$

Therefore, $R_A=\frac{r_a}{cos^2\frac{A}{2}},R_b=\frac{r_b}{cos^2\frac{B}{2}},R_c=\frac{r_c}{cos^2\frac{C}{2}}$.

$$\sum_{cyc}\frac{R_AR_B}{r_ar_b}=\sum_{cyc}\frac{1}{cos^2\frac{A}{2}cos^2\frac{B}{2}}=\frac{\sum cos^2\frac{A}{2}}{\prod cos^2\frac{A}{2}}=\frac{\frac{4R+r}{2R}}{\prod\frac{s(s-a)}{bc}}=\frac{\frac{4R+r}{2R}}{\frac{s^2F^2}{16R^2F^2}}=$$

$$=\frac{8R(4R+r)}{s^2}\geq\frac{32R^2+8Rr}{s^2}\overset{(*)}{\geq}\frac{64r^2}{3R^2}\Leftrightarrow$$

$$(*)\Leftrightarrow\frac{(32R^2+8Rr)\cdot 4}{27R^2}\geq\frac{64r^2}{3R^2}\Leftrightarrow 96R^2+24Rr\geq 432r^2$$

$$96R^2+24Rr\geq 96\cdot(2r)^2+24\cdot 2r\cdot r=324r^2+48r^2=432r^2$$

G.055. *Solution by Amrit Awasthi-India*

$$\sin 5x+10\sin x=5\sin 3x\Leftrightarrow$$

$$(\sin 5x+\sin x)+4\sin x=5(\sin 3x-\sin x)\Leftrightarrow$$

$$2\sin 3x\cos 2x+4\sin x=10\cos 2x\sin x\Leftrightarrow$$

$$2(3\sin x-4\sin^3 x)(1-2\sin^2 x)+10\sin x\,(1-2\cos 2x)=6\sin x;$$

$$\text{put }\sin x=t$$

$$\Leftrightarrow 2(3t-4t^3)(1-2t^2)+10t\cdot 2t^2=6t\Leftrightarrow 16t^5=0\Leftrightarrow\sin x=0$$

$$x=k\pi,k\in\mathbb{Z}.$$

G.056. *Solution by Amrit Awasthi-India*

$$r_a=\frac{F}{s-a},r_a=3\Rightarrow s-a=\frac{F}{3};(1)$$

$$r_b=\frac{F}{s-b},r_b=4\Rightarrow s-b=\frac{F}{4};(2)$$

$$r_c=\frac{F}{s-c},r_c=5\Rightarrow s-c=\frac{F}{5};(3)$$

Adding (1),(2),(3), we get: $3s-(a+b+c)=\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)F$

$$3s-2s=\frac{47F}{60}\Rightarrow s=\frac{47F}{60}$$

But $r=\frac{F}{s}\Rightarrow r=\frac{F}{47F}\cdot 60=\frac{60}{47}$

$$Now,\ F = \sqrt{r \cdot r_a r_b r_c} = \sqrt{3 \cdot 4 \cdot 5 \cdot \frac{60}{47}} = \frac{60}{\sqrt{47}}$$

G.057. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$We\ have:$$

$$4F\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}.\sum_{cyc} csc\, C.\sqrt{\frac{a+b}{c}} = 4F\sum_{cyc}\frac{2R}{c}.\sqrt{\frac{ab}{(b+c)(c+a)}}$$

$$= \sum_{cyc}\frac{2abc}{c}.\sqrt{\frac{ab}{(b+c)(c+a)}}$$

$$= \sum_{cyc} ab.2\sqrt{\frac{ab}{(b+c)(c+a)}} \overset{AM-GM}{\leq} \sum_{cyc} ab\left(\frac{a}{b+c}+\frac{b}{c+a}\right)$$

$$= \sum_{cyc}\frac{a^2b}{b+c}+\sum_{cyc}\frac{ab^2}{c+a} = \sum_{cyc}\frac{a^2b}{b+c}+\sum_{cyc}\frac{ca^2}{b+c} = \sum_{cyc} a^2\left(\frac{b}{b+c}+\frac{c}{b+c}\right) = \sum_{cyc} a^2$$

$$Therefore,$$

$$a^2+b^2+c^2 \geq 4F\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}.\sum_{cyc} csc\, C.\sqrt{\frac{a+b}{c}}$$

G.058. *Solution by Amrit Awasthi-India*

$$\begin{cases} 2\sin x + 2\sin y = 1 \\ 2\cos x + 2\cos y = \sqrt{3} \end{cases} \Rightarrow \begin{cases} 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}; (i) \\ 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{\sqrt{3}}{2}; (ii) \end{cases} \Rightarrow$$

$$\tan\left(\frac{x+y}{2}\right) = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} \Rightarrow x+y = \frac{\pi}{3}$$

Squaring and adding: $2 + 2\sin x \sin y + 2\cos x \cos y = \frac{1}{4}+\frac{3}{4}-1$

$$\cos(x-y) = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\frac{2\pi}{3} \Rightarrow x-y = \frac{2\pi}{3}; (iv)$$

$$(iii)+(iv)\Rightarrow 2x=\frac{2\pi}{3}\pm\pi\Rightarrow x=\frac{\pi}{2}$$

$$(iii)-(iv)\Rightarrow 2y=-\frac{\pi}{3}\Rightarrow y=-\frac{\pi}{6}$$

Therefore,

$$(x,y)\in\left\{\left(2n\pi+\frac{\pi}{2},2n\pi-\frac{\pi}{6}\right)\middle|n\in\mathbb{Z}\right\}$$

G.059. *Solution by Ravi Prakash-New Delhi-India*

$$a^3+\frac{1}{a^3}-\left(a+\frac{1}{a}\right)=\left(a+\frac{1}{a}\right)^3-4\left(a+\frac{1}{a}\right)=$$

$$=\left(a+\frac{1}{a}\right)\left[\left(a+\frac{1}{a}\right)^2-4\right]=\left(a+\frac{1}{a}\right)\left(a-\frac{1}{a}\right)^2\geq 0$$

Thus, for $x,y\in\left(0,\frac{\pi}{2}\right)$ *we have:*

$$tan^3 x+\frac{1}{tan^3 x}\geq tan\,x+\frac{1}{tan\,x};(1)$$

Similarly,

$$tan^3 y+\frac{1}{tan^3 y}\geq tan\,y+\frac{1}{tan\,y}$$

Hence,

$$tan^3 x+\frac{1}{tan^3 x}+tan^3 y+\frac{1}{tan^3 y}\geq tan\,x+\frac{1}{tan\,x}+tan\,y+\frac{1}{tan\,y}$$

Equality holds for $x=y=\frac{\pi}{4}$.

G.060. *Solution by Ravi Prakash-New Delhi-India*

$$Let\ f(t)=t^2+cos\,t-\frac{1}{12}t^4;\ 0\leq t\leq 1, then$$

$$f'(t) = 2t - \sin t - \frac{1}{3}t^3 \text{ and}$$

$$f''(t) = 2 - \cos t - t^2 = (1 - t^2) + (1 - \cos t) \geq 0, t \in (0,1)$$

$f'(x)$ *increasing on* $(0,1)$ *then* $f'(x) > f'(0) = 1 > 0; \forall t \in (0,1)$

Hence, $f(t)$ *increasing on* $(0,1)$. *If* $0 < a \leq b < \frac{\pi}{2}$, *we have:*

$$\sin b \geq \sin a \Rightarrow f(\sin b) \geq f(\sin a)$$

Therefore,

$$\sin^2 b - \sin^2 a + \cos(\sin b) - \cos(\sin a) \geq \frac{1}{12}(\sin^4 b - \sin^4 a)$$

G.061. *Solution by Ravi Prakash-New Delhi-India*

$$\vec{a} \cdot \vec{b} = 2\sqrt{x} + 3\sqrt{x-1}$$

$$|\vec{a}| \cdot |\vec{b}| \cos\theta = 2\sqrt{x} + 3\sqrt{x-1}$$

$$\vec{a} \cdot \vec{b} = \left(3\sqrt{x} - 2\sqrt{x-1}\right)' \cdot \vec{k} \Rightarrow |\vec{a}||\vec{b}| \sin\theta = 3\sqrt{x} - 2\sqrt{x-1}$$

$$\tan\theta = \frac{3\sqrt{x} - 2\sqrt{x-1}}{2\sqrt{x} + 3\sqrt{x-1}} = 13\sqrt{2} - 18 \Rightarrow$$

$$3\sqrt{x} - 2\sqrt{x-1} = \left(13\sqrt{2} - 18\right)\left(2\sqrt{x} + 3\sqrt{x-1}\right)$$

$$\left(26\sqrt{2} - 39\right)\sqrt{x} = \left(52 - 39\sqrt{2}\right)\sqrt{x-1}$$

$$\sqrt{\frac{x-1}{x}} = \frac{13\left(2\sqrt{2} - 3\right)}{13\left(4 - 3\sqrt{2}\right)} = \frac{1}{\sqrt{2}}$$

$$2(x-1) = x \Rightarrow x = 2.$$

G.062. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$From\ CBS, we\ have:$$

$$\left(x^2 \cot\frac{\pi}{19} + y^2 \cot\frac{2\pi}{19} + z^2 \cot\frac{4\pi}{19} + t^2 \tan\frac{8\pi}{19}\right)\left(\tan\frac{\pi}{19} + 2\tan\frac{2\pi}{19} + 4\tan\frac{4\pi}{19} + 8\cot\frac{8\pi}{19}\right) \ge \left(x + y\sqrt{2} + 2z + 2t\sqrt{2}\right)^2$$

$$We\ know\ that:\ \forall a \in \left(0, \frac{\pi}{2}\right);\ 2\cot 2a = \cot a - \tan a$$

$$\to\ 8\cot\frac{8\pi}{19} = 4\cot\frac{4\pi}{19} - 4\tan\frac{4\pi}{19} = 2\cot\frac{2\pi}{19} - 2\tan\frac{2\pi}{19} - 4\tan\frac{4\pi}{19} = \cot\frac{\pi}{19} - \tan\frac{\pi}{19} - 2\tan\frac{2\pi}{19} - 4\tan\frac{4\pi}{19}$$

$$\to\ \tan\frac{\pi}{19} + 2\tan\frac{2\pi}{19} + 4\tan\frac{4\pi}{19} + 8\cot\frac{8\pi}{19} = \cot\frac{\pi}{19}$$

$$Therefore,$$

$$x^2 \cot\frac{\pi}{19} + y^2 \cot\frac{2\pi}{19} + z^2 \cot\frac{4\pi}{19} + t^2 \tan\frac{8\pi}{19} \ge \left(x + y\sqrt{2} + 2z + 2t\sqrt{2}\right)^2 \tan\frac{\pi}{19}.$$

G.063. *Solution by Tapas Das-India*

$$\sqrt[3]{ab}\left(\sqrt[3]{a} + \sqrt[3]{b}\right) = \left(\sqrt[3]{a^2b} + \sqrt[3]{ab^2}\right) = \sqrt[3]{a \cdot a \cdot b} + \sqrt[3]{a \cdot b \cdot b} \overset{AM-GM}{\le}$$

$$\le \frac{a+a+b}{3} + \frac{a+b+b}{3} = \frac{3(a+b)}{3} = a + b$$

$$\sum_{cyc} \sqrt[3]{ab}\left(\sqrt[3]{a} + \sqrt[3]{b}\right) \le \sum_{cyc}(a+b) = 4s$$

$$\sum_{cyc} a\sqrt{bc} = \sum_{cyc} \sqrt{ab \cdot ac} \le \frac{1}{2}\sum_{cyc}(ab + ac) = ab + bc + ca$$

$$\left(\sum_{cyc} a\sqrt{bc}\right)\left(\sum_{cyc} \sqrt[3]{ab}\left(\sqrt[3]{a} + \sqrt[3]{b}\right)\right) \le 4s(ab + bc + ca) \le$$

$$\leq \frac{(a+b+c)^2}{3} \cdot 4s = \frac{4s^2 \cdot 4s}{3} \leq \frac{4}{3} \cdot \frac{27R^2}{4} \cdot 4 \cdot \frac{3\sqrt{3}}{2} R = 54\sqrt{3} \cdot R^3$$

G.064. *Solution by Ertan Yildirim-Izmir-Turkiye*

$$\text{Lemma 1. } \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\text{Lemma 2. } a^2 + b^2 + c^2 \geq 4\sqrt{3}F \ (Ionescu - Weitzenbock)$$

$$4ab\cos^2\frac{C}{2} = 4ab \cdot \frac{s(s-c)}{ab} = 4s(s-c) = 3c^2 \Rightarrow$$

$$4s^2 - 4sc - 3c^2 = 0 \Rightarrow (2s-3c)(2s+c) = 0 \Rightarrow 2s = 3c \Rightarrow$$

$$a + b + c = 3c \Rightarrow a + b = 2c$$

$$\Rightarrow a^2 + b^2 + 2ab = 4c^2$$

$$5c^2 = a^2 + b^2 + c^2 + 2ab \overset{Lemma\ 2}{\geq} 4\sqrt{3}F + 2ab$$

G.065. *Solution by Tapas Das-India*

$$\text{Let } f(x) = \tan^{-1}x - x; x > 0, \text{ then}$$

$$f'(x) = \frac{1}{1+x^2} - 1 = -\frac{x^2}{1+x^2} < 0$$

$$f \text{ -is decreasing function, then } f(x) < f(0), f(0) = 0$$

$$\tan^{-1}x \leq x; \forall x > 0$$

$$x \leq y \leq z < 1 \Rightarrow y - x > 0, z - x > 0, z - y > 0$$

$$\Rightarrow (y-x)\tan^{-1}x < (y-x)x$$

$$(z-x)\tan^{-1}y < (z-x)y$$

$$(z-y)\tan^{-1}z < (z-y)z$$

By adding, we get:

$$(y-z)\,tan^{-1}x+(z-x)\,tan^{-1}y+(z-x)\,tan^{-1}z<(y-x)x+(z-x)y+(z-y)z=z^2-x^2<0$$

$$and\ \frac{\pi}{2}-log\,2>0$$

Therefore,

$$(y-z)\,tan^{-1}x+(z-x)\,tan^{-1}y+(z-x)\,tan^{-1}z<\frac{\pi}{2}-log\,2$$

G.066. *Solution by Fayssal Abdelli-Bejaia-Algerie*

$$sin\,2x\cdot sin\,3x\cdot sin\,11x+sin^2 5x\cdot sin\,6x=sin\,x\cdot sin\,3x\cdot sin\,8x$$

$$sin\,3x\cdot[sin\,2x\cdot sin\,11x+2\,sin^2 5x\cdot cos\,3x]=sin\,x\cdot sin\,3x\cdot sin\,8x$$

$$sin\,3x=0\Rightarrow x=\frac{\pi}{3}k,k\in\mathbb{Z}$$

$$sin\,2x\cdot sin\,11x+2\,sin^2 5x\cdot cos\,3x=sin\,x\cdot sin\,8x\,;(A)$$

$$But:\ sin\,x\cdot sin\,11x=\frac{1}{2}cos\,9x-\frac{1}{2}cos\,13x\ and$$

$$sin\,x\cdot sin\,8x=\frac{1}{2}cos\,7x-\frac{1}{2}cos\,9x$$

$$(A)=\frac{1}{2}cos\,9x-\frac{1}{2}cos\,13x+2\,sin^2 5x\,cos\,3x-\frac{1}{2}cos\,7x-\frac{1}{2}cos\,9x$$

$$\Rightarrow cos\,3x\,[cos\,6x-2\,sin^2 3x+2\,sin^2 5x-cos\,7x\,cos\,3x+sin\,7x\,sin\,3x]=0$$

$$cos\,3x=0\Rightarrow x=\frac{\pi}{6}+\frac{k\pi}{3},k\in\mathbb{Z}.$$

$$cos\,6x-2\,sin^2 3x+2\,sin^2 5x-cos\,10x=0;(B)$$

$$(B)\Rightarrow 1-2\,sin^2 3x-2\,sin^2 3x+2\,sin^2 5x-1+2\,sin^2 5x=0$$

$$\Rightarrow sin^2 5x=sin^2 3x\Rightarrow sin\,5x=sin\,3x\ or\ sin\,5x=sin(-3x)\Rightarrow$$

$$\begin{cases} x = 2k\pi, & x = \frac{\pi}{8} + 2k\pi \\ x = \frac{k\pi}{4}, & x = \frac{\pi}{2} + k\pi \end{cases}$$

Therefore,

$$S = \left\{k\pi; \frac{\pi}{8} + \frac{k\pi}{4}; \frac{k\pi}{4}; \frac{\pi}{2} + k\pi; \frac{k\pi}{3}; \frac{\pi}{6} + \frac{k\pi}{3}\right\}$$

G.067. *Solution by Fayssal Abdeli-Bejaia-Algerie*

Let: $A = 1 + 2\cos x \cdot \cos 2x \cdot \cos 5x$; $B = \cos^2 x + \cos^2 2x + \cos^2 5x$

$$A = 1 + 2\cos x \cdot \cos 2x \cdot \cos 5x = 1 + (\cos 3x + \cos x)\cos 5x$$

$$= 1 + \frac{1}{2}\cos 8x + \frac{1}{2}\cos 2x + \frac{1}{2}\cos 4x + \frac{1}{2}\cos 6x =$$

$$= 1 + \frac{1}{2}(2\cos^2 4x - 1) + \frac{1}{2}(2\cos^2 x - 1) + \frac{1}{2}(2\cos^2 2x - 1) + \frac{1}{2}\cos 6x =$$

$$= \frac{1}{2}\cos 8x + \cos^2 x + \cos^2 2x + \frac{1}{2}\cos 6x = \frac{1}{2}\cos 8x + \frac{1}{2}\cos 6x = \cos^2 5x.$$

$$\frac{1}{2}\cos 8x + \frac{1}{2}\cos 6x = \frac{1}{2} + \frac{1}{2}\cos 10x$$

$$\cos 8x + \cos 6x - \cos 10x - 1 = 0$$

$$\cos 8x - 1 + 2\sin 8x \sin 2x = 0$$

$$2\cos^2 4x - 2 + 2\sin 8x \sin 2x = 0$$

$$-2\sin^2 4x + 4\sin 4x \cos 4x \sin 2x = 0$$

$$\sin 4x\,[-\sin 4x + 2\sin 2x \cos 4x] = 0, \qquad \sin 4x = 0 \Rightarrow x = \frac{k\pi}{4}$$

$$-\sin 4x + 2\sin 2x \cos 4x = 0; (B)$$

$$(B) \Rightarrow -2\sin 2x\cos 2x + 2\sin 2x\cos 4x = 0 \Rightarrow \sin 2x = 0 \Rightarrow x = k\pi$$

$$-\cos 2x = -\cos 4x \Rightarrow x = \frac{k\pi}{3}$$

Therefore,

$$S = \left\{\frac{k\pi}{4}; \frac{k\pi}{2}; k\pi; \frac{k\pi}{3}\right\}$$

SOLUTIONS
ANALYSIS

AN.001. *Solution by Adrian Popa-Romania*

$$b^3 - a^3 = x^3|_a^b = \int_a^b 3x^2\,dx$$

We must to prove:

$$3\int_a^b \sin x \cdot \sinh x\,dx \le 3\int_a^b x^2\,dx, \qquad \sin x \cdot \sinh x \le x^2, x \in [a,b]$$

Let $f(x) = \sin x \cdot \sinh x - x^2; f(0) = 0$*, then*

$$f'(x) = \cos x \cdot \sinh x + \sin x \cdot \cosh x - 2x; f'(0) = 0$$

$$f''(x) = 2\cos x \cdot \cosh x - 2; f''(0) = 0$$

$$f'''(x) = 2(-\sin x \cdot \cosh x + \cos x \cdot \sinh x); f'''(x) = 0$$

$$f^{(iv)}(x) = -2\sin x \cdot \sinh x \le 0; f^{(iv)}(0) = 0$$

x	0		$\frac{\pi}{2}$
$f^{(iv)}(x)$	0	$- \ - \ - \ - \ -$	
$f'''(x)$	0	$- \ - \ - \ - \ -$ ↘	
$f''(x)$	0	$- \ - \ - \ - \ -$ ↘	
$f'(x)$	0	$- \ - \ - \ - \ -$ ↘	
$f(x)$	0	$- \ - \ - \ - \ -$ ↘	

Thus, $f(x) \leq 0, \forall x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin x \cdot \sinh x \leq x^2, x \in [a,b]$

$$3\int_a^b \sin x \cdot \sinh x \, dx \leq 3\int_a^b x^2 \, dx$$

AN.002. *Solution by Ravi Prakash-New Delhi-India*

$$\frac{\tan\left(\frac{\pi - 2x}{4}\right)(1+\sin x)}{\sin x} = \frac{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\sin x} =$$

$$= \frac{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\sin x} = \frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\sin x} =$$

$$= \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin x} = \frac{\cos x}{\sin x}$$

Hence,

$$\int_a^b \frac{\tan\left(\frac{\pi-2x}{4}\right)(1+\sin x)}{\sin x}dx = \log(\sin x)|_a^b = \log\left(\frac{\sin b}{\sin a}\right)$$

$$= \log\left(1-\frac{\sin b-\sin a}{\sin a}\right) \le$$

$$\le \frac{\sin b-\sin a}{\sin a}; (\because \log(1+x) \le x, \forall x \ge 0)$$

AN.003. *Solution by Ravi Prakash-New Delhi-India*

$$S = \sum_{0\le i<j\le n} \binom{n}{i}\binom{n}{j} \cos\frac{2(j-i)\pi}{7} =$$

$$= \sum_{0\le i<j\le n} \binom{n}{i}\binom{n}{j} \cos\frac{2(j-i)\pi}{7} + \sum_{0\le i<j\le n} \binom{n}{i}\binom{n}{j} \cos\frac{2(i-j)\pi}{7} =$$

$$= Re\left[\sum_{i\ne j}\binom{n}{i}\binom{n}{j} e^{(i-j)\frac{\pi}{7}}\right] \Rightarrow \sum_{j=0}^{n}\binom{n}{j}^2 + S =$$

$$= Re\left\{\left[\binom{n}{0}+\binom{n}{1}e^{\frac{i\pi}{7}}+\binom{n}{2}e^{\frac{2\pi i}{7}}+\cdots+\binom{n}{n}e^{\frac{n\pi i}{7}}\right]\right.$$
$$\left.\cdot\left[\binom{n}{0}+\binom{n}{1}e^{-\frac{i\pi}{7}}+\binom{n}{2}e^{-\frac{2\pi i}{7}}+\cdots+\binom{n}{n}e^{-\frac{n\pi i}{7}}\right]\right\}$$

$$= Re\left[\left(1+e^{\frac{i\pi}{7}}\right)^n\left(1+e^{-\frac{i\pi}{7}}\right)^n\right] = Re\left[\left(1+2\cos\frac{\pi i}{7}+1\right)^n\right] =$$

$$= Re\left[2^n\left(1+\cos\frac{\pi}{7}\right)^n\right] = 2^n\left(1+\cos\frac{\pi}{7}\right)^n = 2^n\left(2\cos^2\frac{2\pi}{7}\right)^n$$
$$= 2^{2n}\cos^{2n}\left(\frac{2\pi}{7}\right)$$

$$\Rightarrow S = 2^{2n}\cos^{2n}\left(\frac{2\pi}{7}\right) - \sum_{j=0}^{n}\binom{n}{j}^2 = 2^{2n}\cos^{2n}\left(\frac{2\pi}{7}\right) - \binom{2n}{n}$$

$$\Rightarrow 2^{-2n} = cos^{2n}\left(\frac{2\pi}{7}\right) - \frac{1}{2^{2n}}\binom{2n}{n}$$

$$\Rightarrow cos^{2n}\left(\frac{2\pi}{7}\right) - 2^{1-2n}\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}cos\frac{(j-i)\pi}{7} = \frac{1}{2^{2n}}\binom{2n}{n}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\sqrt[n]{cos^{2n}\frac{\pi}{7} - 2^{1-2n}\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}cos\frac{2(j-i)\pi}{7}}$$

$$= \lim_{n\to\infty}\sqrt[n]{\frac{1}{2^{2n}}\binom{2n}{n}} =$$

$$\overset{C-D}{=} \frac{1}{4}\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 1$$

AN.004. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$\because tan^{-1}x \pm tan^{-1}y = tan^{-1}\left(\frac{x\pm y}{1\mp xy}\right) \Rightarrow$$

$$tan^{-1}\left(\frac{x}{sec\frac{\pi}{7} - x\,tan\frac{\pi}{7}}\right) - tan^{-1}\left(x\,sec\frac{\pi}{7} - tan\frac{\pi}{7}\right)$$

$$= tan^{-1}\left(\frac{x}{sec\frac{\pi}{7} - x\,tan\frac{\pi}{7}}\right) =$$

$$= \frac{\pi}{2} - tan^{-1}\left(\frac{sec\frac{\pi}{7} - x\,tan\frac{\pi}{7}}{x}\right)$$

$$tan^{-1}\left(x\,sec\frac{\pi}{7} - tan\frac{\pi}{7}\right) + tan^{-1}\left(\frac{sec\frac{\pi}{7} - x\,tan\frac{\pi}{7}}{x}\right) =$$

$$= tan^{-1}\left(\frac{\frac{sec\frac{\pi}{7}-x\,tan\frac{\pi}{7}}{x}+x\,sec\frac{\pi}{7}-tan\frac{\pi}{7}}{1-\left(x\,sec\frac{\pi}{7}-tan\frac{\pi}{7}\right)\cdot\frac{sec\frac{\pi}{7}-x\,tan\frac{\pi}{7}}{x}}\right)=$$

$$= tan^{-1}\left(\frac{sec\frac{\pi}{7}-x\,tan\frac{\pi}{7}+x^2\,sec\frac{\pi}{7}-x\,tan\frac{\pi}{7}}{x-x\,sec^2\frac{\pi}{7}+sec\frac{\pi}{7}tan\frac{\pi}{7}+x^2\,sec\frac{\pi}{7}tan\frac{\pi}{7}-x\,tan^2\frac{\pi}{7}}\right)=$$

$$= tan^{-1}\left(\frac{sec\frac{\pi}{7}-2x\,tan\frac{\pi}{7}+x^2\,sec\frac{\pi}{7}}{sec\frac{\pi}{7}tan\frac{\pi}{7}+x^2\,sec\frac{\pi}{7}tan\frac{\pi}{7}-2x\,tan^2\frac{\pi}{7}}\right)\overset{(*)}{=}$$

$$sec\frac{\pi}{7}tan\frac{\pi}{7}+x^2\,sec\frac{\pi}{7}tan\frac{\pi}{7}-2x\,tan^2\frac{\pi}{7}$$
$$= tan\frac{\pi}{7}\left(sec\frac{\pi}{7}-2x\,tan\frac{\pi}{7}+x^2\,sec\frac{\pi}{7}\right)$$

$$\overset{(*)}{=} tan^{-1}\left(\frac{1}{tan\frac{\pi}{7}}\right)=\frac{\pi}{2}-\frac{\pi}{7}$$

Therefore,

$$\Omega(a,b)=\frac{\pi}{7}(b-a).$$

AN.005. *Solution by Asmat Qatea-Afghanistan*

$$\int_a^b \frac{1}{\sqrt{2[x]+1}}\cdot\prod_{k=1}^{[x]} sin\left(\frac{k\pi}{2n+1}\right)dx \geq \frac{1}{2^a}-\frac{1}{2^b} \Leftrightarrow$$

$$\int_a^b \frac{1}{\sqrt{2[x]+1}}\cdot\prod_{k=1}^{[x]} sin\left(\frac{k\pi}{2n+1}\right)dx \overset{(?)}{\geq} log\,2\int_a^b 2^{-x}\,dx$$

$$\frac{1}{\sqrt{2[x]+1}}\cdot\prod_{k=1}^{[x]} sin\left(\frac{k\pi}{2n+1}\right) \overset{(?)}{\geq} 2^{-x}\,log\,2 \Rightarrow x\in[0,\infty)$$

Case 1. If $x \in [0,1)$ *then:*

$$\frac{1}{\sqrt{2\cdot 0+1}}\cdot\prod_{k=1}^{0} sin\left(\frac{k\pi}{2n+1}\right)\overset{(?)}{\geq} 2^{-x}\log 2 \Rightarrow 1 \geq 2^{-x}\log 2 \Rightarrow 2^{x} \geq \log 2 - true.$$

Case 2. If $x \in [n, n+1), n \in \mathbb{N}$ *then:*

$$\frac{1}{\sqrt{2\cdot n+1}}\cdot\prod_{k=1}^{n} sin\left(\frac{k\pi}{2n+1}\right)\overset{(?)}{\geq} 2^{-x}\log 2$$

$$\because \prod_{k=1}^{n} sin\left(\frac{k\pi}{2n+1}\right) = \frac{\sqrt{2n+1}}{2^{n}}$$

$$\frac{1}{2^{n}}\overset{(?)}{\geq} 2^{-x}\log 2 \Rightarrow 2^{x-n} \geq \log 2 - true, because\ x-n \geq 0$$

Therefore,

$$\int_{a}^{b}\frac{1}{\sqrt{2[x]+1}}\cdot\prod_{k=1}^{[x]} sin\left(\frac{k\pi}{2n+1}\right)dx \geq \frac{1}{2^{a}}-\frac{1}{2^{b}}$$

AN.006. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Let's\ prove\ that:$$

$$\left(\frac{1-xyz}{1+xyz}\right)^{3} \geq \left(\frac{1-x^{3}}{1+x^{3}}\right)\left(\frac{1-y^{3}}{1+y^{3}}\right)\left(\frac{1-z^{3}}{1+z^{3}}\right), \forall x,y,z \in (0,1).$$

$$Let\ x=e^{u},\ \ y=e^{v},\ \ z=e^{w}, where\ u,v,w \in (-\infty,0)\ and\ let$$

$$f(t)=\log\left(\frac{1-e^{3t}}{1+e^{3t}}\right), t \in (-\infty,0)$$

$$We\ have:\ \ f'(t)=-\frac{6e^{3t}}{1-e^{6t}}\ \ and$$

$$f''(t) = -\frac{18e^{3t}(1+e^{6t})}{(1-e^{6t})^2} \le 0 \ \ then\ f\ is\ concave\ on\ (-\infty, 0).$$

$$By\ Jensen's\ inequality, we\ have:$$

$$log\left(\prod_{cyc}\left(\frac{1-x^3}{1+x^3}\right)\right) = \sum_{cyc} log\left(\frac{1-x^3}{1+x^3}\right) = \sum_{cyc} f(u) \le 3f\left(\frac{u+v+w}{3}\right)$$
$$= log\left(\frac{1-e^{u+v+w}}{1+e^{u+v+w}}\right)^3 = log\left(\frac{1-xyz}{1+xyz}\right)^3$$

$$Then:\ \left(\frac{1-xyz}{1+xyz}\right)^3 \ge \left(\frac{1-x^3}{1+x^3}\right)\left(\frac{1-y^3}{1+y^3}\right)\left(\frac{1-z^3}{1+z^3}\right), \forall x,y,z \in (0,1).$$

$$Therefore,$$

$$\int_a^b\int_a^b\int_a^b \left(\frac{1-xyz}{1+xyz}\right)^3 dxdydz \ge \int_a^b\int_a^b\int_a^b \left(\frac{1-x^3}{1+x^3}\right)\left(\frac{1-y^3}{1+y^3}\right)\left(\frac{1-z^3}{1+z^3}\right)dxdydz$$
$$= \left(\int_a^b \frac{1-x^3}{1+x^3}dx\right)^3$$

AN.007. *Solution by Adrian Popa-Romania*

$$\frac{x+y}{2} \overset{AHQ}{\le} \sqrt{\frac{x^2+y^2}{2}} \Leftrightarrow \left(\frac{x+y}{2}\right)^2 \le \frac{x^2+y^2}{2}$$

$$-\left(\frac{x+y}{2}\right)^2 \ge -\frac{x^2+y^2}{2} \Leftrightarrow 1-\left(\frac{x+y}{2}\right)^2 \ge 1-\frac{x^2+y^2}{2}$$

$$1-\left(\frac{x+y}{2}\right)^2 = \frac{(1-x^2)+(1-y)^2}{2}$$

$$I \ge \int_0^1\int_0^1 \sqrt{\frac{(1-x^2)+(1-y)^2}{2}}dxdy$$

$$\begin{cases} x = \sin t \\ y = \sin q \end{cases} \Rightarrow I \geq \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-\sin^2 t)+(1-\sin^2 q)}{2}} \cos t \cos q \, dtdq$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos^2 t + \cos^2 q}{2}} \cos t \cos q \, dtdq$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\cos^2 t \cos q + \cos t \cos^2 q) dtdq =$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos^2 t \cos q \, dtdq = \int_0^{\frac{\pi}{2}} \cos^2 t \, dt \int_0^{\frac{\pi}{2}} \cos q \, dq$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} dt \sin q |_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Therefore,

$$\int_0^1 \int_0^1 \sqrt{1 - \left(\frac{x+y}{2}\right)^2} \, dxdy > \frac{\pi}{4}$$

AN.008. *Solution by Adrian Popa-Romania*

$$\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin(-2x)\cos(-4x)\cos(-8x)}{\sin^5\left(\frac{\pi}{4}+x\right) + \cos^5\left(\frac{\pi}{4}+x\right)} dx =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{-\sin(2x)\cos(4x)\cos(8x)}{\cos^5\left(\frac{\pi}{2} - \left(\frac{\pi}{4}+x\right)\right) + \sin^5\left(\frac{\pi}{2} - \left(\frac{\pi}{4}+x\right)\right)} dx =$$

$$= -\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x \cos 4x \cos 8x}{\cos^5\left(\frac{\pi}{4}-x\right) + \sin^5\left(\frac{\pi}{4}-x\right)} dx = -\Omega \Rightarrow 2\Omega = 0 \Rightarrow$$

$$\Rightarrow \Omega = 0$$

AN.009. *Solution by Pham Duc Nam-Vietnam*

$$\int_0^1 \frac{tx^t}{(1+x^t)^2}dt$$
$$= \frac{1}{\ln x}\int_0^1 \frac{t}{(1+x^t)^2}d(x^t+1), \begin{cases} u = t \\ dv = \frac{1}{(1+x^t)^2}d(x^t+1) \end{cases} \Rightarrow \begin{cases} du = dt \\ v = -\frac{1}{1+x^t} \end{cases}$$
$$\Rightarrow \int_0^1 \frac{tx^t}{(1+x^t)^2}dt = -\frac{t}{\ln x\,(1+x^t)}\Big|_0^1 + \frac{1}{\ln x}\int_0^1 \frac{1}{1+x^t}dt$$
$$= -\frac{1}{\ln x\,(1+x)} + \frac{1}{\ln^2 x}\int_0^1 \frac{1}{x^t(x^t+1)}d(x^t)$$
$$= -\frac{1}{\ln x\,(1+x)} + \frac{1}{\ln^2 x}\int_0^1 \frac{1}{x^t}d(x^t) - \frac{1}{\ln^2 x}\int_0^1 \frac{1}{x^t+1}d(x^t)$$
$$= -\frac{1}{\ln x\,(1+x)} + \frac{1}{\ln^2 x}\left(\ln\frac{x^t}{x^t+1}\right)\Big|_0^1$$
$$= -\frac{1}{\ln x\,(1+x)} + \frac{1}{\ln^2 x}\left(\ln\frac{x}{x+1} + \ln 2\right)$$
$$* \Rightarrow \Omega = \lim_{x\to\infty}\left(\ln^2 x \,.\int_0^1 \frac{tx^t}{(1+x^t)^2}dt\right)$$
$$= \lim_{x\to\infty}\left(\ln^2 x\left(-\frac{1}{\ln x\,(1+x)} + \frac{1}{\ln^2 x}\left(\ln\frac{x}{x+1} + \ln 2\right)\right)\right)$$
$$= \underbrace{\lim_{x\to\infty}\left(-\frac{\ln x}{1+x}\right)}_{=0} + \lim_{x\to\infty}\left(\ln\frac{x}{x+1} + \ln 2\right) = \ln 2$$

AN.010. *Solution by Tapas Das-India*

$$u = 5(1 + e^x\sin x) + 3e^x\cos x, \qquad du = e^x(2\sin x + 8\cos x)dx$$
$$\frac{1}{2}du = e^x(\sin x + 4\cos x)dx$$

$$\Omega=\int\frac{sinx+4cosx}{5(e^{-x}+sinx)+3cosx}dx=\int\frac{e^x(sinx+4cosx)}{5(1+e^xsinx)+3e^xcosx}dx=$$

$$=\int\frac{1}{2u}du=\frac{1}{2}log|u|+C=\frac{1}{2}log|5(1+e^xsinx)+3e^xcosx|+C$$

AN.011. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Lemma:\ \ If\ x,y,z>0\ then:\ \frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq\frac{3(x^2+y^2+z^2)}{x+y+z}\quad(1)$$

$$Proof:\ We\ have:\ \ (1)\ \Leftrightarrow\ \sum_{cyc}\left(\frac{x^2}{y}-2x+y\right)$$

$$\geq\frac{3(x^2+y^2+z^2)}{x+y+z}-(x+y+z)$$

$$\Leftrightarrow\sum_{cyc}\frac{(x-y)^2}{y}\geq\frac{2(x^2+y^2+z^2)-2(xy+yz+zx)}{x+y+z}=\sum_{cyc}\frac{(x-y)^2}{x+y+z}$$

$$Which\ is\ true\ because\ x+y+z>y\ \ (and\ analogs).$$

$$Now\ we\ have:\ \int_a^b\int_a^b\int_a^b(x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)dxdydz\geq$$

$$\overset{Lemma}{\geq}\int_a^b\int_a^b\int_a^b3(x^2+y^2+z^2)dxdydz=3\int_a^bdy\int_a^bdz\int_a^b3x^2dx=$$

$$=3(b-a)^2(b^3-a^3)=3(b-a)^3(a^2+ab+b^2),\ \ as\ desired.$$

Equality holds for $a=b$.

AN.012. *Solution by Togrul Ehmedov-Azerbaijan*

$$\because\ sin\left(x+\frac{\pi}{3}\right)sin\left(\frac{\pi}{3}-x\right)=\frac{1}{4}sin\,3x$$

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{sin\,x\cdot sin\left(x+\frac{\pi}{3}\right)\cdot sin\left(x+\frac{2\pi}{3}\right)}{sin\,3x+cos\,3x}dx=$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\sin 3x}{\sin 3x+\cos 3x}dx=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\sin 3\left(\frac{\pi}{6}-x\right)}{\sin 3\left(\frac{\pi}{6}-x\right)+\cos 3\left(\frac{\pi}{6}-x\right)}dx=$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\cos 3x}{\sin 3x+\cos 3x}dx$$

$$2\Omega=\frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{\sin 3x+\cos 3x}{\sin 3x+\cos 3x}dx=\frac{1}{4}\int_0^{\frac{\pi}{6}}dx=\frac{\pi}{24}$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx=\frac{\pi}{48}$$

AN.013. *Solution by Serlea Kabay-Liberia*

$$\Omega(a,b)=\int_a^b\int_a^b\int_a^b \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)dxdydz=$$

$$=\int_a^b\int_a^b\int_a^b(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z)\,dxdydz\overset{(*)}{=}$$

$$\left(\because\int\tan^{-1}x\,dx=x\tan^{-1}x-\log(\sqrt{1+x^2})\right)$$

$$\overset{(*)}{=}\int_a^b\int_a^b\int_a^b\tan^{-1}x\,dxdydz+\int_a^b\int_a^b\int_a^b\tan^{-1}y\,dxdydz$$

$$+\int_a^b\int_a^b\int_a^b\tan^{-1}z\,dxdydz=$$

$$=(b-a)^2\left(\int_a^b\tan^{-1}x\,dx+\int_a^b\tan^{-1}y\,dy+\int_a^b\tan^{-1}z\,dz\right)=$$

$$=3(b-a)^2\left(b\tan^{-1}b-a\tan^{-1}a+\frac{1}{2}\left(\frac{a^2+1}{b^2+1}\right)\right)$$

AN.014. *Solution by Ravi Prakash-New Delhi-India*

$$\text{For } x \geq 0 \Rightarrow x^{20}(x^{10}-2)^2 \geq 0 \Rightarrow x^{20}(x^{20}-4x^{10}+4) \geq 0$$

$$x^{40}-4x^{30}+4x^{20} \geq 0 \Rightarrow x^{40}+4x^{20}+4 \geq 4(1+x^{30})$$

$$(x^{20}+2)^2 \geq 4(1+x^{30}) \Rightarrow \frac{1}{\sqrt{1+x^{30}}} \geq \frac{2}{x^{20}+2}$$

$$\frac{x^{19}}{\sqrt{1+x^{30}}} \geq \frac{2x^{19}}{x^{20}+2}$$

$$\int_a^b \frac{x^{19}}{\sqrt{1+x^{30}}}dx \geq 2\int_a^b \frac{x^{19}}{x^{20}+2}dx = \frac{1}{10}log(x^{20}+2)\Big|_a^b = log\sqrt[10]{\frac{2+b^{20}}{2+a^{20}}}$$

AN.015. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$We\ know\ that:\ \sin x < x < \tan x,\ \ \forall x \in \left(0,\frac{\pi}{2}\right)\ and\ \ e^t-1 \geq t,\ \ \forall t \in R.$$

$$Then:\ e^{\tan x - x}-1 \geq \tan x - x\ \ or\ \ \frac{\tan x - x}{e^{\tan x}-e^x} \leq e^{-x},\ \ \forall x \in \left(0,\frac{\pi}{2}\right).$$

$$Thus,\ \ \frac{(e^x-e^{\sin x})(\tan x - x)}{e^{\tan x}-e^x} \leq (e^x-e^{\sin x})e^{-x} = 1-e^{\sin x - x} \leq 1,\ \ \forall x \in \left(0,\frac{\pi}{2}\right).$$

$$Therefore,\ \ \int_a^b \frac{(e^x-e^{\sin x})(\tan x - x)}{e^{\tan x}-e^x}dx \leq \int_a^b dx = b-a \leq (b-a)\left(1+\sin\frac{a+b}{2}\right).$$

$$Equality\ holds\ for\ a=b.$$

AN.016. *Solution by Rana Ranino-Setif-Algerie*

$$\Omega(a,b) = \int_a^b \tan^{-1}\left(\frac{30x^3-10x}{31x^2-1}\right)dx = \int_a^b \tan^{-1}\left(\frac{10x-30x^3}{1-31x^2}\right)dx$$

$$tan^{-1}\left(\frac{10x-30x^3}{1-31x^2}\right)=tan^{-1}\left(\frac{5x+5x(1-6x^2)}{1-31x^2}\right)=tan^{-1}\left(\frac{\frac{5x}{1-6x^2}+5x}{1-\frac{25x^2}{1-6x^2}}\right)$$

$$For\ x\geq\frac{1}{\sqrt{31}}\Rightarrow\frac{25x^2}{1-6x^2}\geq 1\Rightarrow tan^{-1}\left(\frac{10x-30x^3}{1-31x^2}\right)=$$

$$=tan^{-1}\left(\frac{5x}{1-6x^2}\right)+tan^{-1}(5x)-\pi$$

$$tan^{-1}\left(\frac{5x}{1-6x^2}\right)=tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right)=tan^{-1}(2)+tan^{-1}(3x)$$

$$tan^{-1}\left(\frac{10x-30x^3}{1-31x^2}\right)=tan^{-1}(2x)+tan^{-1}(3x)+tan^{-1}(5x)-\pi$$

$$\Omega(a,b)=\int_a^b(-\pi+tan^{-}(2x)+tan^{-1}(3x)+tan^{-1}(5x))\,dx=$$

$$=\Big[-\pi x+x\,tan^{-1}(2x)+x\,tan^{-1}(3x)+x\,tan^{-1}(5x)$$
$$-\frac{1}{4}log(4x^2+1)-\frac{1}{6}(9x^2+1)-\frac{1}{10}(25x^2+1)\Big]_a^b=$$

$$=\left[x\,tan^{-1}\left(\frac{30x^3-10x}{31x^2-1}\right)-\frac{1}{4}log(4x^2+1)-\frac{1}{6}(9x^2+1)-\frac{1}{10}(25x^2+1)\right]_a^b$$

Therefore,

$$\Omega(a,b)=b\,tan^{-1}\left(\frac{30b^3-10b}{31b^2-1}\right)-a\,tan^{-1}\left(\frac{30a^3-10a}{31a^2-1}\right)-\frac{1}{4}log\left(\frac{4b^2+1}{4a^2+1}\right)-$$
$$-\frac{1}{6}log\left(\frac{9b^2+1}{9a^2+1}\right)-\frac{1}{10}log\left(\frac{25b^2+1}{25a^2+1}\right)$$

AN.017. *Solution by Asmat Qatea-Afghanistan*

$$\Omega(a,b)=\int_a^b \frac{(1+tan^2 x)^2}{cos^2 x-3\,sin^2 x}dx$$
$$=\int_a^b \frac{(1+tan^2 x)^2}{cos^2 x\,(1+3\,tan^2 x)}dx \overset{tan\,x=t}{\Rightarrow}$$

$$\Omega=\int \frac{(1+t^2)^2}{1-3t^2}dt=\int \frac{1+2t^2+t^4}{1-3t^2}dt$$
$$=\int\left(-\frac{1}{3}t^2-\frac{7}{9}+\frac{\frac{16}{9}}{1-3t^2}\right)dt=$$

$$=-\frac{1}{9}t^3-\frac{7}{9}t+\frac{16}{27}\int\frac{1}{\frac{1}{3}-t^2}dt$$

$$=-\frac{1}{9}t^3-\frac{7}{9}t+\frac{16}{27}\cdot\frac{\sqrt{3}}{2}log\left(\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right)+C=$$

$$=-\frac{1}{9}t^3-\frac{7}{9}t+\frac{16\sqrt{3}}{27}\cdot\frac{1}{2}log\left(\frac{1+\sqrt{3}t}{1-\sqrt{3}t}\right)+C=$$

$$=-\frac{1}{9}t^3-\frac{7}{9}t+\frac{16\sqrt{3}}{27}tanh^{-1}\left(\sqrt{3}t\right)+C$$

$$\Omega(a,b)=\left[-\frac{1}{9}tan^3 x-\frac{7}{9}tan\,x+\frac{16\sqrt{3}}{27}tanh^{-1}\left(\sqrt{3}\,tan\,x\right)\right]_a^b$$

$$\Omega(a,b)=-\frac{1}{9}tan^3 b-\frac{7}{9}tan\,b+\frac{16\sqrt{3}}{27}tanh^{-1}\left(\sqrt{3}\,tan\,b\right)+\frac{1}{9}tan^3 a+\frac{7}{9}tan\,a-$$

$$-\frac{16\sqrt{3}}{27}tanh^{-1}(\sqrt{3}\,tan\,a)$$

AN.018. *Solution by Tapas Das-India*

$$We\ know\ that: 0 < \sin x < 1; (\forall)x \in \left(0, \frac{\pi}{2}\right), then$$

$$\sin^3 x > \sin^5 x\ and$$

$$1 - \sin^2 x < 1 - \sin^5 x\ , also\ we\ have:\ \sin x < x; (\forall)x \in \left(0, \frac{\pi}{2}\right)$$

$$Hence: \frac{(1 - \sin^5 x)^7}{(1 - \sin^3 x)^8} > \frac{(1 - \sin^5 x)^7}{(1 - \sin^5 x)^8} = \frac{1}{1 - \sin^5 x} \geq 1$$

$$Therefore, \int_a^b \frac{(1 - \sin^5 x)^7}{(1 - \sin^3 x)^8} dx \geq \int_a^b dx = b - a$$

AN.019. *Solution by Ruxandra Daniela Tonilă-Romania*

If $a = b$ is nothing to prove.

If $a \neq b$ using Cauchy's inequality, we have:

$$\int_a^b \left(e^{x^2}\right)^2 dx \cdot \int_a^b dx \geq \left(\int_a^b e^{x^2} dx\right)^2$$

$$\int_a^b e^{x^2} dx \geq \frac{1}{b - a}\left(\int_a^b e^{x^2} dx\right)^2 \cdot \int_a^b e^{-x^2} dx$$

$$\int_a^b e^{2x^2} dx \cdot \int_a^b e^{-x^2} dx \geq \frac{1}{b - a}\int_a^b e^{x^2} dx \cdot \int_a^b e^{x^2} dx \cdot \int_a^b e^{-x^2} dx$$

$$\int_a^b e^{2x^2} dx \cdot \int_a^b e^{-x^2} dx \geq \frac{1}{b - a}\int_a^b e^{x^2} dx \cdot \int_a^b \left(\sqrt{e^{x^2}}\right)^2 dx \cdot \int_a^b \left(\frac{1}{\sqrt{e^{x^2}}}\right)^2 dx \overset{CBS}{\geq}$$

$$\geq \frac{1}{b - a}\int_a^b e^{x^2} dx \cdot \left(\int_a^b \left(\sqrt{e^{x^2}} \cdot \frac{1}{\sqrt{e^{x^2}}}\right) dx\right)^2$$

Hence,

$$\int_a^b e^{2x^2}\,dx \cdot \int_a^b e^{-x^2}\,dx \geq \frac{1}{b-a}\int_a^b e^{x^2}\,dx \cdot \left(\int_a^b dx\right)^2$$

Therefore,

$$\left(\int_a^b e^{2x^2}\,dx\right)\left(\int_a^b e^{-x^2}\,dx\right) \geq (b-a)\int_a^b e^{x^2}\,dx$$

AN.020. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$By\ Schur's\ inequality, we\ have:$$

$$u^3+v^3+w^3+3uvw \geq uv(u+v)+vw(v+w)+wu(w+u), \forall u,v,w>0$$

$$Also, by\ AM-GM\ inequality, we\ have:\ u+v \geq 2\sqrt{uv}\ \ (And\ analogs)$$

$$Then:\ \ u^3+v^3+w^3+3uvw \geq 2\left(\sqrt{(uv)^3}+\sqrt{(vw)^3}+\sqrt{(wu)^3}\right)$$

$$Taking\ \ u=\sqrt[3]{f(x)},\ \ v=\sqrt[3]{f(y)},\ \ w=\sqrt[3]{f(z)}, (x,y,z\in[a,b]), we\ obtain:$$

$$f(x)+f(y)+f(z)+3\sqrt[3]{f(x).f(y).f(z)} \geq 2\left(\sqrt{f(x).f(y)}+\sqrt{f(y).f(z)}+\sqrt{f(z).f(x)}\right), \forall x,y,z\in[a,b]$$

$$Integrating\ the\ both\ sides, we\ obtain:$$

$$\int_a^b\int_a^b\int_a^b \left(f(x)+f(y)+f(z)+3\sqrt[3]{f(x).f(y).f(z)}\right)dxdydz \geq 2\int_a^b\int_a^b\int_a^b \left(\sqrt{f(x).f(y)}+\sqrt{f(y).f(z)}+\sqrt{f(z).f(x)}\right)dxdydz$$

$$Therefore,$$

$$3(b-a)^2\int_a^b f(x)dx+3\left(\int_a^b \sqrt[3]{f(x)}dx\right)^3 \geq 2.3(b-a)\left(\int_a^b \sqrt{f(x)}dx\right)^2$$
$$=6(b-a)\left(\int_a^b \sqrt{f(x)}dx\right)^2$$

AN.021. *Solution by Hikmat Mammadov-Azerbaijan*

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \text{ and } \Gamma(\alpha)=\int_0^\infty S^{\alpha-1}e^{-s}\,ds$$

$$\Gamma(\lambda x+(1-\lambda)y)=\int_0^\infty S^{\lambda x+(1-\lambda)y-1}e^{-S}\,ds=$$

$$=\int_0^\infty S^{\lambda(x-1)+(1-\lambda)(y-1)}e^{-\lambda s}e^{-(1-\lambda)s}\,ds=$$

$$=\int_0^\infty (S^{x-1}e^{-S})^{\lambda}(S^{y-1}e^{-S})^{1-\lambda}\,ds \overset{Holder}{\leq}$$

$$\leq\left(\int_0^\infty \left((S^{x-1}e^{-S})^{\lambda}\right)^{\frac{1}{\lambda}}ds\right)^{\lambda}\left(\int_0^\infty \left((S^{y-1}e^{-S})^{1-\lambda}\right)^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda}=$$

$$=\left(\int_0^\infty S^{x-1}e^{-S}\,ds\right)^{\lambda}\left(\int_0^\infty S^{y-1}e^{-S}\,ds\right)^{1-\lambda}=\Gamma(x)^{\lambda}\Gamma(y)^{1-\lambda}$$

$\Gamma(\cdot)$ *—is log-convex, then:*

$$\Gamma\left(\frac{1}{2}\cdot 2x+\frac{1}{2}\cdot 2y\right)\leq \Gamma(2x)^{\frac{1}{2}}\Gamma(y)^{\frac{1}{2}}$$

$$\Gamma(x+y)\leq\sqrt{\Gamma(2x)\Gamma(2y)}$$

Hence:

$$\frac{1}{\Gamma(x+y)}\geq\frac{1}{\sqrt{\Gamma(2x)\Gamma(2y)}}\Rightarrow\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\geq\frac{\Gamma(x)\Gamma(y)}{\sqrt{\Gamma(2x)\Gamma(2y)}}$$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \geq \sqrt{\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+x)}} \cdot \sqrt{\frac{\Gamma(x)\Gamma(y)}{\Gamma(y+y)}}$$

$$B(x,y) \geq \sqrt{B(x,x)} \cdot \sqrt{B(y,y)}$$

$$B(y,z) \geq \sqrt{B(y,y)} \cdot \sqrt{B(z,z)}$$

$$B(z,x) \geq \sqrt{B(z,z)} \cdot \sqrt{B(x,x)}$$

Therefore,

$$B(x,y) \cdot B(y,z) \cdot B(z,x) \geq B(x,x) \cdot B(y,y) \cdot B(z,z)$$

AN.022. *Solution by Max Wong , Timmy Wong-Hong Kong*

Using M.V.T. $\exists c \in (a,b)$ *such that:*

$$cot\,B - cot\,a + \int_a^b csc^2 x\, e^{sin^2 x}\, dx = \left(e^{sin^2 c} - 1\right) \int_a^b \frac{dx}{sin^2 x}$$

$$\leq (e-1) \int_a^b \frac{dx}{sin^2 x} = (e-1)(cot\,b - cot\,a)$$

Also, $\forall 0 < a < b < \frac{\pi}{2}, \exists d \in (a,b)$ *such that*

$$cot\,b - cot\,a \overset{MVT}{=} -csc^2 d \cdot (b-a)$$

When $a = b, cot\,b - cot\,a = b - a = 0.$ *Hence,*

$$cot\,b - cot\,a + \int_a^b csc^2 x\, e^{sin^2 x}\, dx \leq (e-1)(b-a); \forall 0 < a \leq b < \frac{\pi}{2}$$

AN.023. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$tan\left(\frac{\pi}{4} - x - y\right) = \frac{1 - tan(x+y)}{1 + tan(x+y)} \Rightarrow 1 + tan\left(\frac{\pi}{4} - x - y\right) = \frac{2}{1 + tan(x+y)}$$

$$(1 + tan(x+y))\left(tan(\frac{\pi}{4} - (x+y)\right) = 1 - tan(x+y)$$

$$= [1 + tan(x+y)]\left[1 + tan\,x\,tan\,y\,tan\left(\frac{\pi}{4} - (x+y)\right)\right] =$$

$$= 1 + tan\,x\,tan\,y + tan(x+y) - tan\,x\,tan\,y\,tan(x+y) =$$

$$= 1 + tan\,x\,tan\,y + tan\,x + tan\,y$$

Hence,

$$\frac{(1+tan\,x)(1+tan\,y)\left(1+tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+tan\,x\cdot tan\,y\cdot tan\left(\frac{\pi}{4}-x-y\right)} = \frac{2(1+tan\,x)(1+tan\,y)}{1+tan\,x\,tan\,y\,tan\left(\frac{\pi}{4}-(x+y)\right)}$$

$$= \frac{2(1+tan\,x)(1+tan\,y)}{1+tan\,x\,tan\,y+tan\,x\,tan\,y} = \frac{2(1+tan\,x)(1+tan\,y)}{(1+tan\,y)(1+tan\,x)} = 2$$

Therefore,

$$\Omega(a,b) = \int_a^b\int_a^b \frac{(1+tan\,x)(1+tan\,y)\left(1+tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+tan\,x\cdot tan\,y\cdot tan\left(\frac{\pi}{4}-x-y\right)}dx\,dy$$
$$= 2(b-a)^2$$

AN.024. *Solution by Khaled Abd Imouti-Damascus-Syria*

We want to prove that:

$$e^{t^3} + t \le 1 + t\cdot e^{t^2} \le 1 + e; \forall t \in [0,1]$$

$$e^{t^3} - t\cdot e^{t^2} - 1 \le 0 \Leftrightarrow e^{t^3} - 1 \le t\cdot e^{t^2} - t \Leftrightarrow$$

$$\frac{e^{t^3}-1}{t^3} \le \frac{e^{t^2}-1}{t^2}; (*)$$

$$Let\ f(t) = \frac{e^x - 1}{x}, then\ f'(x) = \frac{x\cdot e^x - e^x + 1}{x^2} > 0$$

$$Let\ g(x) = x\cdot e^x - e^x + 1, then\ g'(x) = x\cdot e^x$$

$$g'(x) = 0 \Leftrightarrow x = 0$$

x	$-\infty$ $\quad$ 0 $\quad$ $+\infty$
$g'(x)$	$- - - - - - - - - -0 + + + + + + + + +$
$g(x)$	$1 \searrow \searrow \searrow 0 \nearrow \nearrow \nearrow +\infty$

Thus, $g(x) \geq 0$ *and then* f *strictly increasing on* $\mathbb{R}$.

From $t^3 < t^2$ *imply* $f(t^3) < f(t^2) \Rightarrow (*)$ *is true.*

Let $h(t) = t \cdot e^{t^2}$ *then* $h'(t) = (1 + 2t^2)e^{t^2} > 0$*, therefore for* $t \in [0, e]$

$h(t) \leq h(e)$*. So,* $e^{t^3} + t \leq 1 + t \cdot e^{t^2} \leq 1 + e; \forall t \in [0,1]$

By integrating, it follows that:

$$2\int_a^b e^{x^3}\,dx + (b-a)^2 \leq 2(b-a) + e^{b^2} - e^{a^2}$$

AN.025. *Solution by Amrit Awasthi-India*

$$\Omega(a,b) = \int_a^b tan^{-1}\left(\frac{4x - 4x^3}{x^4 - 6x^2 + 1}\right)dx \overset{x=tan\,t}{=}$$

$$= \int_{tan^{-1} a}^{tan^{-1} b} sec^2 t \cdot tan^{-1}\left(\frac{4\,tan\,t - 4\,tan^3 t}{tan^4 t - 6\,tan^2 t + 1}\right)dt =$$

$$= \int_{tan^{-1} a}^{tan^{-1} b} sec^2 t \cdot tan^{-1}(tan\,4t)\,dt = 4\int_{tan^{-1} a}^{tan^{-1} b} t \cdot sec^2 t\,dt =$$

$$= [4t \cdot tan\,t + 4\,log|cos\,t|]_{tan^{-1} a}^{tan^{-1} b} =$$

$$= 4b \cdot tan^{-1} b - 4a \cdot tan^{-1} a + 4\,log\frac{cos(tan^{-1} b)}{cos(tan^{-1} a)} =$$

$$= 4\left[b\,tan^{-1} b - a\,tan^{-1} a + \frac{1}{2}log\left(\frac{a^2+1}{b^2+1}\right)\right]$$

AN.026. *Solution by Hikmat Mammadov-Azerbaijan*

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n\cdot 27^n}\sum_{i=1}^{n}\sum_{j=1}^{n}3^{i+j}\binom{3n-i-j}{n}\binom{2n-i-j}{n-i}\right)$$

$$2n-i-j \geq 2n-1;\ \ n+1 \geq i+j$$

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i+j\leq n+1}}^{n}\frac{1}{3^{3n-(i+j)}}\cdot\frac{(3n-i-j)!}{(2n-i-j)!\,n!}\cdot\frac{(2n-i-j)!}{(n-1)!\,(n+1-i-j)}\cdot\frac{(2n-1)!}{(2n-1)!}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i+j\leq n+1}}^{n}\frac{1}{3^{3n-(i+j)}}\binom{2n-1}{n}\binom{3n-i-j}{2n-1}$$

$$= \lim_{n\to\infty}\frac{1}{n}\binom{2n-1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i+j\leq n+1}}^{n}\frac{1}{3^{3n-(i+j)}}\binom{3n-i-j}{2n-1}$$

$$= \lim_{n\to\infty}\frac{1}{n}\binom{2n-1}{n}\sum_{v=2}^{n+1}(v-1)\cdot\frac{1}{3^{3n-v}}\binom{3n-v}{2n-1}$$

$$\overset{k=v-1}{=} \lim_{n\to\infty}\frac{1}{n}\binom{2n-1}{n}\sum_{k=1}^{n}\frac{k}{3^{3n-k-1}}\binom{3n-k-1}{2n-1}$$

$$\overset{\substack{appendix\\ bellow}}{=} \lim_{n\to\infty}\frac{1}{2n}\binom{2n-1}{n}\frac{1}{3^{3n-2}}(3n-1)\binom{3n-2}{2n-1}$$

$$\Gamma(x) = \sqrt{2}\pi e^{x\log x - x - \frac{1}{2}\log x + O\left(\frac{1}{x^2}\right)}\left(1+O\left(\frac{1}{x}\right)\right)$$

$$v\in\mathbb{Z}_{\geq 1} \Rightarrow \Gamma(z) = (v-1)!$$

$$\Omega=\frac{1}{2}\lim_{n\to\infty}\frac{1}{3^{3n-2}}\cdot\frac{1}{n^2}\cdot\frac{\Gamma(3n)}{\Gamma(n)^3}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1}{3^{3n-2}}\cdot\frac{1}{n^2}\cdot\frac{\sqrt{2}\pi e^{3n\log(3n)-3n-\frac{1}{2}\log(3n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n}\right)\right)}{2\pi\sqrt{2}\pi e^{3n\log(3n)-3n-\frac{1}{2}\log(3n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n}\right)\right)}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1}{3^{3n-2}}\cdot\frac{1}{n^2}\cdot\frac{1}{2\pi}\cdot 3^{3n}\cdot\frac{n}{\sqrt{3}}=0$$

Appendix:

$$\sum_{k=1}^{n}\frac{k}{3^{3n-k-1}}\binom{3n-1-k}{2n-1}\stackrel{?}{=}\frac{1}{2}\cdot\frac{1}{3^{3n-2}}\binom{3n-2}{2n-1}$$

$$\sum_{k=1}^{n}\binom{3n-1-k}{2n-1}z^k\frac{1}{z\binom{3n-2}{2n-1}}\stackrel{m=k-1}{=}\sum_{m=0}^{n-1}\frac{\binom{3n-m-2}{2n-1}}{\binom{3n-2}{2n-1}}$$

$$=\sum_{m=0}^{n-1}z^m\frac{\frac{(3n-m-2)!}{(n-m-1)!}}{\frac{(3n-2)!}{(n-1)!}}=$$

$$=\sum_{m=0}^{n-1}z^m\frac{(n-1)!}{(n-1-m)!}\cdot\frac{1}{\frac{(3n-2)!}{(3n-2-m)!}}=$$

$$=\sum_{m=0}^{n}\frac{z^m}{m!}\cdot\frac{m!\big(-(n-1)\big)\big(-(n-2)\big)\dots\big(-(n-1-(m-1)\big)}{\big(-(3n-2)\big)\big(-(3n-3)\big)\dots\big(-(3n-2-(m-1)\big)}=$$

$$=\ {}_2F_1(1;-(n-1);-(3n-2);z)$$

$$\frac{d}{dz}\left(\sum_{k=1}^{n}\binom{3n-1-k}{2n-1}z^k\right)=\sum_{k=1}^{n}k\binom{3n-1-k}{2n-1}=\binom{3n-2}{2n-1}\frac{d}{dz}=$$

$$=\ {}_2F_1(1;-(n-1);-(3n-2);z)=$$

$$= \binom{3n-2}{2n-1}\Big({}_2F_1(1;-(n-1);-(3n-2);z) + z\frac{n-1}{3n-2}\, {}_2F_1(2;-(n-1);-(3n-2);z)\Big)$$

$$= \binom{3n-2}{2n-1}(1+\frac{n-1}{3n-2}x+\frac{(n-1)(n-2)}{(3n-2)(3n-3)}z^2 + \frac{(n-1)(n-2)(n-3)}{(3n-2)(3n-3)(3n-4)}z^3+\cdots$$

$$+\cdots+\frac{n-1}{3n-2}z+2\frac{(n-1)(n-2)}{(3n-2)(3n-3)}z+3\ldots)$$

$$= \binom{3n-2}{2n-1}\, {}_2F_1(1;-(n-1);-(3n-2);z)$$

$$\sum_{k=1}^{n} k\binom{3n-1-k}{2n-1}z^{k-1} = \binom{3n-2}{2n-1}\, {}_2F_1(1;-(n-1);-(3n-2);z)$$

$$z=3 \Rightarrow \sum_{k=1}^{n} k\binom{3n-1-k}{2n-1}3^{k-1} = \binom{3n-2}{2n-1}\, {}_2F_1(1;-(n-1);-(3n-2);3)$$

$$\sum_{k=1}^{n} \frac{k}{3^{3n-k-1}}\binom{3n-1-k}{2n-1} = \frac{1}{3^{3n-2}}\binom{3n-2}{2n-1}\, {}_2F_1(1;-(n-1);-(3n-2);3) =$$

$$= \frac{1}{3^{3n-2}}\binom{3n-2}{2n-1}\frac{3n-1}{2}$$

AN.027. *Solution by Remus Florin Stanca-Romania*

$$Let\ a_n = \tfrac{1}{n^2}\left(1+2\textstyle\sum_{k=1}^{n}\tfrac{1}{2k+5}\right)^n.$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}\left(1+2\sum_{k=1}^{n+1}\frac{1}{2k+5}\right)^n}{\frac{1}{n^2}\left(1+2\sum_{k=1}^{n}\frac{1}{2k+5}\right)^n}=$$

$$=\lim_{n\to\infty}\left(1+2\sum_{k=1}^{n+1}\frac{1}{2k+5}\right)\left(\frac{1+2\sum_{k=1}^{n+1}\frac{1}{2k+5}}{1+2\sum_{k=1}^{n}\frac{1}{2k+5}}\right)^n$$

$$2k+5\le 2k+6\Rightarrow\frac{2}{2k+5}>\frac{1}{2k+3}\Rightarrow\sum_{k=1}^{n+1}\frac{1}{2k+5}$$

$$\ge\sum_{k=1}^{n+1}\frac{1}{k+3};\left(\lim_{k\to\infty}\frac{1}{k+3}=\infty\right)$$

$$\Rightarrow\lim_{n\to\infty}\sum_{k=1}^{n+1}\frac{2}{2k+5}=\infty\Rightarrow 1+2\sum_{k=1}^{n+1}\frac{2}{2k+5}=\infty;(1)$$

$$let\ u(n)=\frac{1+2\sum_{k=1}^{n+1}\frac{1}{2k+5}}{1+2\sum_{k=1}^{n}\frac{1}{2k+5}}$$

$$\lim_{n\to\infty}\left(\frac{1+2\sum_{k=1}^{n+1}\frac{1}{2k+5}}{1+2\sum_{k=1}^{n}\frac{1}{2k+5}}\right)^n=\lim_{n\to\infty}\left(1+(u(n)-1)\right)^{\frac{1}{u(n)-1}\cdot n(u(n)-1)}=$$

$$=exp\left\{\lim_{n\to\infty}\frac{2n}{2n+7}\cdot\frac{1}{1+2\sum_{k=1}^{n}\frac{1}{2k+5}}\right\}=e^0=1;(2)$$

From (1),(2) it follows that:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^2}\left(1+2\sum_{k=1}^{n}\frac{1}{2k+5}\right)^n$$

AN.028. *Solution by Adrian Popa-Romania*

$$\Omega(a,b)=\int_a^b \frac{3+\cos 4x}{1-\cos 4x}dx=-\int_a^b \frac{-3-\cos 4x}{1-\cos 4x}dx$$
$$=-\int_a^b \frac{1-\cos 4x-4}{1-\cos 4x}dx=$$

$$=-\int_a^b dx+\int_a^b \frac{4}{1-\cos 4x}dx=(a-b)+\int_a^b \frac{2}{\sin^2 2x}dx=$$

$$=a-b-\cot 2x|_a^b=a-b+\cot 2a-\cot 2b$$

AN.029. *Solution by Remus Florin Stanca-Romania*

$$\Omega=\lim_{n\to\infty}\frac{1}{n(H_n-1)}\left(\log(n!)-\sum_{k=2}^{n}\frac{\Gamma'(k)}{\Gamma(k)}\right)$$
$$=\lim_{n\to\infty}\frac{\log(n!)-\sum_{k=2}^{n}\psi(k)}{n(H_n-1)}\overset{C-S}{=}$$

$$=\lim_{n\to\infty}\frac{\log(n+1)-\psi(n+1)}{(n+1)\left(H_n+\frac{1}{n+1}\right)-nH_n+n}=$$

$$=\lim_{n\to\infty}\frac{\log(n+1)-\psi(n+1)}{nH_n+H_n+1-n-1-nH_n+n}=\lim_{n\to\infty}\frac{\log(n+1)-\psi(n+1)}{H_n}\overset{C-S}{=}$$

$$=\lim_{n\to\infty}\frac{\log\left(\frac{n+2}{n+1}\right)-\psi(n+2)+\psi(n+1)}{\frac{1}{n+1}}=\lim_{n\to\infty}\frac{\log\left(\frac{n+2}{n+1}\right)-\frac{1}{n+1}}{\frac{1}{n+1}}=$$

$$=\lim_{n\to\infty}(n+1)\log\left(\frac{n+2}{n+1}\right)-1=\lim_{n\to\infty}\log\left(1+\frac{1}{n+1}\right)^{n+1}-1$$
$$=\log e-1=0$$

AN.030. *Solution by Surjeet Singhania-India*

As we known logarithmic function is increasing $\log:\mathbb{R}_+^*\to\mathbb{R}$*,*

$$0<1+\sin^n x<1+\sin^n 1\,;\forall x\in[0,1]$$

Also, $x+1<1+x+x^n<1+x+1=2+x$, *then*

$$\frac{1+\sin^n x}{1+x+x^n}<\frac{1+\sin^n x}{1+x}$$

$$Let\ \left|\int_0^1 \log\left(\frac{(1+\sin^n x)(1+x)}{1+x+x^n}\right)dx\right|$$
$$\le\left|\int_0^1 \log\left(\frac{(1+\sin^n x)(1+x)}{1+x}\right)dx\right|=$$

$$=\log(1+\sin^n 1)<\sin^n 1<1;\forall n\ge 1$$

Hence, we have:

$$\lim_{n\to\infty}\int_0^1 \log\left(\frac{(1+\sin^n x)(1+x)}{1+x+x^n}\right)dx=0$$

$$\int_0^1 \log\left(\frac{1+\sin^n x}{1+x+x^n}\right)dx\to\int_0^1 \log(1+x)\,dx$$

Therefore,

$$\Omega=-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^n\,dx=-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}=-\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}-\frac{(-1)^{n+1}}{n+1}\right)=$$

$$=-(\log 2+\log 2-1)=1-\log 4$$

AN.031. *Solution by Mohammed Diai-Morocco*

$$\int_0^a (2x-a)\log(1+x+x^2)\,dx\ge 0;(*)$$

$$\int_0^a (2x-a)\log(1+x+x^2)\,dx=$$

$$=\int_0^a (2x+1)\log(1+x+x^2)\,dx-\int_0^a (a+1)\log(1+x+x^2)\,dx$$

$$\int_0^a (2x+1)\,log(1+x+x^2)\,dx$$
$$= [(1+x+x^2)\,log(1+x+x^2) - (x+x^2)]|_0^a =$$
$$= (1+a+a^2)\,log(1+a+a^2)$$

Therefore,

$$(*) \Leftrightarrow \int_0^a log(1+x+x^2)\,dx \le \frac{(1+a+a^2)\,log(1+a+a^2)}{1+a}$$

The function $x \to log(1+x+x^2)$ *is increasing in* $[0,a]$*, so:*

$$\int_0^a log(1+x+x^2)\,dx \le \int_0^a log(1+a+a^2)\,dx = a\,log(1+a+a^2)$$
$$\le \frac{(1+a+a^2)\,log(1+a+a^2)}{1+a}$$

AN.032. *Solution by Tapas Das-India*

We know that:

$$tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$x \cdot tan^{-1} x = x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \frac{x^{10}}{9} - \cdots$$

Now,

$$tan^{-1}(x^2) = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \frac{(x^2)^9}{9} - \cdots =$$
$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \frac{x^{18}}{9} - \cdots$$

Hence,

$$tan^{-1}(x^2) - x \cdot tan^{-1} x$$
$$= \frac{1}{3}(x^4 - x^6) + \frac{1}{5}(x^{10} + x^6) + \frac{1}{7}(x^8 - x^{14}) + \cdots$$

For $x \in (0,1)$: $x^4 > x^6 > x^8 > x^{10} > x^{14} \dots$

Clearly, $tan^{-1}(x^2) - x \cdot tan^{-1} x \geq 0 \Leftrightarrow tan^{-1}(x^2) \geq x \cdot tan^{-1} x$, *thus*

$$\frac{tan^{-1}(x^2)}{x \cdot tan^{-1} x} \geq 1$$

Therefore,

$$\int_a^b \frac{sin\, x \cdot tan^{-1}(x^2)}{x \cdot tan^{-1} x} dx \geq \int_a^b sin\, x\, dx$$

$$\int_a^b \frac{sin\, x \cdot tan^{-1}(x^2)}{x \cdot tan^{-1} x} dx \geq cos\, a - cos\, b$$

AN.033. *Solution by Naren Bhandari-Bajura-Nepal*

Due to Lambert continued fraction (particular case of Gauss continued fraction)

$$tan\, x = \frac{x}{1 + \mathbb{K}_{n=1}^{\infty} \frac{-x^2}{2n+1}}$$

Now we replace x *by* ix *giving us*

$$tanh\, x = \frac{x}{1 + \mathbb{K}_{n=1}^{\infty} \frac{x^2}{2n+1}}$$

So, we need to integrate $I + \int_a^b tanh\, x\, dx$ *which is easy to see*

$$I = \int_a^b \frac{d}{dx} log(cosh\, x)\, dx = log\left(\frac{cosh\, b}{cosh\, a}\right)$$

AN.034. *Solution by Adrian Popa-Romania*

$$log\left(\frac{(a-\sqrt{2})(b+\sqrt{2})}{(b-\sqrt{2})(a+\sqrt{2})}\right)$$
$$= log(\sqrt{2}-a) - log(\sqrt{2}-b) + log(\sqrt{2}+b)$$
$$- log(\sqrt{2}+a)$$

$$= \log x|_{\sqrt{2}-b}^{\sqrt{2}-a} + \log x|_{\sqrt{2}+a}^{\sqrt{2}+b} = \int_{\sqrt{2}-b}^{\sqrt{2}-a} \frac{1}{x} dx + \int_{\sqrt{2}+a}^{\sqrt{2}+b} \frac{1}{x} dx =$$

$$= \int_a^b \frac{1}{\sqrt{2}-t} dt + \int_a^b \frac{1}{\sqrt{2}+t} dt = \int_a^b \frac{2\sqrt{2}}{2-x^2} dx = 2\sqrt{2} \int_a^b \frac{dx}{2-x^2}$$

We need to prove:

$$exp\left(2\sqrt{2}\int_a^b x^x \, dx\right) \geq 2\sqrt{2}\int_a^b \frac{dx}{2-x^2}$$

Let $f:(0,1) \to \mathbb{R}, f(x) = x^x = e^{x \log x}$, *then* $f'(x) = x^x(1 + \log x)$

$$f'(x) = 0 \Leftrightarrow x = \frac{1}{e}$$

x	$0 \qquad \frac{1}{e} \qquad 1$
$f'(x)$	$- - - - - - - - 0 + + + + + + + + +$
$f(x)$	$\searrow \qquad 0 \qquad \nearrow$

$$g:(0,1) \to \mathbb{R}, g(x) = \frac{1}{2-x^2}, g'(x) = \frac{2x}{(2-x^2)^2} > 0; \forall x \in (0,1)$$

Hence, $f(x) \geq g(x); \forall x \in (0,1)$.

AN.035. *Solution by Ravi Prakash-New Delhi-India*

$$Let\ m \in \mathbb{N} - \{0\}, f(x) = (x+1)^x(m+1-x)^{m-x}, 0 \leq x \leq \left[\frac{m}{2}\right]$$

$$\log f(x) = x \log(x+1) + (m-x)\log(m+1-x)$$

$$\frac{f'(x)}{f(x)} = \log(x+1) - \log(m+1-x) - \frac{m-x}{m+1-x} =$$

$$= \log\left(\frac{x+1}{m+1-x}\right) - \frac{m-x}{m+1-x} < 0, \forall 0 < x < \left[\frac{m}{2}\right]$$

Thus, f decreases on $\left[0,\left[\frac{m}{2}\right]\right] \Rightarrow f(x) \geq f\left(\left[\frac{m}{2}\right]\right), \forall x \in \left[0,\left[\frac{m}{2}\right]\right]$.

Hence,

$$\frac{1}{(k+1)^k(2n+1-k)^{2n-k}} \leq \frac{1}{(n+1)^n(n+1)^n} = \frac{1}{(n+1)^{2n}}$$

and

$$\frac{1}{(k+1)^k(2n+2-k)^{2n+1-k}} \leq \frac{1}{(n+1)^{2n+1}}$$

$$(2n-1)!\sum_{k=0}^{2n}\frac{1}{(k+1)^k(2n-k+1)^{2n-k}} < \frac{(2n-1)!\,(2n+1)}{(n+1)^{2n}} = \frac{(2n+1)!}{2n(n+1)^{2n}}$$

Let $b_n = \frac{(2n+1)!}{2n(n+1)^{2n}}$ *and*

$$(2n+1-1)!\sum_{k=0}^{2n+1}\frac{1}{(k+1)^k(2n+1-k+1)^{2n+1-k}} < \frac{(2n)!\,(2n+2)}{(n+1)^{2n+1}} = \frac{(2n+2)!}{(2n+1)(n+1)^{2n+1}}$$

Let $c_n = \frac{(2n+2)!}{(2n+1)(n+1)^{2n+1}}$. *We prove that:* $b_n, c_n \to \infty$ *for* $n \to \infty$.

$$\frac{b_n}{b_{n+1}} = \left(1+\frac{1}{n}\right)\cdot\frac{(n+2)^2}{(2n+2)(2n+3)}\left[\left(1+\frac{1}{n+1}\right)^{n+1}\right]^2 \to \frac{e^2}{4}$$

As $\frac{e^2}{4} > 1, b_n \to 0$ *as* $n \to \infty$. *Similarly,* $c_n \to 0$ *as* $n \to \infty$.

Now,

$$0 < \sum_{k=0}^{n}\frac{1}{(k+1)^k(n-k+1)^{n-k}} < b_n, c_n$$

As $b_n \to 0, c_n \to 0$ *as* $n \to \infty$. *Therefore,*

$$\Omega = \lim_{n\to\infty}(n-1)!\sum_{k=0}^{n}\frac{1}{(k+1)^k(n-k+1)^{n-k}} = 0$$

AN.036. *Solution by Amrit Awasthi-India*

$$\psi(z) = \frac{1}{\Gamma(z)}\int_0^\infty x^{z-1}e^{-x}\log x\,dx$$

$$\Omega(n) = \int_0^\infty x^{n+1}e^{-x}\log x\,dx - n\int_0^\infty x^{n-1}e^{-x}\log x\,dx =$$

$$= \Gamma(n+1)\psi(n+1) - n\Gamma(n)\psi(n) = n!\,[\psi(n+!) - \psi(n)] =$$

$$= n!\left(\frac{1}{n}\right) = (n-1)!$$

Therefore,

$$\Omega = \sum_{n=1}^{\infty}\frac{1}{\Omega(n)} = \sum_{n-1}^{\infty}\frac{1}{(n-1)!} = e$$

AN.037. *Solution by Ravi Prakash-New Delhi-India*

$$\Omega_n(x) = \int\frac{dx}{x(1+x^n)} = \int\frac{x^{n-1}}{x^n(1+x^n)}dx = \int\left(\frac{1}{x^n} - \frac{1}{1+x^n}\right)x^{n-1}\,dx$$

$$= \frac{1}{n}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \frac{1}{n}\log\left(\frac{t}{t+1}\right) + C = \frac{1}{n}\log\left(\frac{x^n}{1+x^n}\right) + C$$

$$\Omega_n(1) = \frac{1}{n}\log\left(\frac{1}{2}\right) = -\frac{1}{n}\log 2 + C \Rightarrow C = \frac{n+1}{n}\log 2$$

Thus,

$$n\Omega_n(x) = (n+1)\log 2 + \log\left(\frac{x^n}{1+x^n}\right) = \log 2 + \log\left(\frac{2^n x^n}{1+x^n}\right)$$

If $0 < x < \frac{1}{2}, 0 < 2x < 1 \Rightarrow (2x)^n \to 0$

$$\lim_{n\to\infty} n\Omega_n(x) = -\infty \ if \ 0 < x < \frac{1}{2}.$$

For $x = \frac{1}{2}$ *we have:* $n\Omega_n(x) = \log 2 + \log\left(\frac{1}{1+\left(\frac{1}{2}\right)^n}\right) \to \log 2$ *as* $n \to \infty$.

For $\frac{1}{2} < x < 1, (2x)^n \to \infty, x^n \to \infty$ *ad* $n\Omega_n(x) \to \infty$ *as* $n \to \infty$.

For $x \ge 1, n\Omega_n(x) = \log 2 - \log\left(\frac{1}{2^n} + \frac{1}{2^n x^n}\right) \to \infty$ *as* $n \to \infty$.

AN.038. *Solution by Sergio Esteban-Buenos Aires-Argentina*

$$S_1 = \sum_{k=1}^{n} k(n-k+1) = \sum_{k=1}^{n}(nk - k^2 + k) = n\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= n \cdot \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

$$S_2 = \sum_{k=1}^{n} k(n+1)^2 = \sum_{k=1}^{n} k[(n+1)^2 + k^2 - 2(n+1)k] =$$

$$= \sum_{k=1}^{n} k(n+1)^2 + \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} 2(n+1)k^2 =$$

$$= \frac{n(n+1)}{2} \cdot (n+1)^2 + \left(\frac{n(n+1)}{2}\right)^2 - 2(n+1) \cdot \frac{n(n+1)(2n+1)}{6}$$

$$\Omega = \lim_{n\to\infty} \frac{nS_1}{S_2} = 2$$

AN.039. *Solution by Ravi Prakash-New Delhi-India*

Let $f(x) = \log(1+x) + x + \log x - \sqrt{x}; x > 0$, *then*

$$f'(x) = \frac{1}{1+x} + 1 + \frac{1}{x} - \frac{1}{2\sqrt{x}}; x > 0$$

If $0 < x \leq 1$, *then* $2\sqrt{x} \geq \sqrt{x} \geq x$ *and* $\frac{1}{2\sqrt{x}} \leq \frac{1}{x} \Rightarrow \frac{1}{x} - \frac{1}{2\sqrt{x}} \geq 0 \Rightarrow$
$f'(x) > 0; \forall x \in (0,1]$

If $x > 1$, *then* $2\sqrt{x} > \sqrt{x} > 1 \Rightarrow \frac{1}{2\sqrt{x}} < 1 \Rightarrow 1 - \frac{1}{2\sqrt{x}} > 0 \Rightarrow$
$f'(x).0; \forall x > 1$

Thus, $f'(x) > 0; \forall x > 0 \Rightarrow f(x)$ *−strictly increasing on* $(0, \infty)$.

As $b \geq a > 1, log\ b \geq log\ a \Rightarrow f(log\ b) \geq f(log\ a)$

$$log(1 + log\ b) + log\ b + log(log\ b) - \sqrt{log\ b} \geq$$

$$\geq log(1 + log\ a) + log\ a + log(log\ a) - \sqrt{log\ a}$$

Hence,

$$\sqrt{log\ b} - \sqrt{log\ a} \leq$$

$$\leq log(1 + log\ b) + log\ b + log(log\ b) - log(1 + log\ a) - log\ a - log(log\ a)$$

$$2\int_a^b \frac{dx}{x\sqrt{log\ x}} \leq log\left[\frac{(1 + log\ b)\ log\ b^b}{(1 + log\ a)\ log\ a^a}\right]$$

Therefore,

$$exp\left(2\int_a^b \frac{dx}{x\sqrt{log\ x}}\right) \leq \frac{(1 + log\ b)\ log\ b^b}{(1 + log\ a)\ log\ a^a}$$

AN.040. *Solution by Adrian Popa-Romania*

$$\Omega = n\int_a^b \sqrt[n]{\frac{f^{(n)}(x)}{f(x)}}dx = n\int_a^b \sqrt[n]{\frac{f^{(n)}(x)}{f^{(n-1)}(x)} \cdot \frac{f^{(n-1)}(x)}{f^{(n-2)}(x)} \cdot \ldots \cdot \frac{f^{(1)}(x)}{f^{(0)}(x)}}dx \overset{AGM}{\leq}$$

$$\leq \int_a^b \left(\frac{f^{(n)}(x)}{f^{(n-1)}(x)} + \frac{f^{(n-1)}(x)}{f^{(n-2)}(x)} + \cdots + \frac{f^{(1)}(x)}{f^{(0)}(x)}\right)dx =$$

$$= \sum_{k=1}^{n-1}\int_a^b \frac{f^{(k+1)}(x)}{f^{(k)}(x)}dx;$$

$$Let\ f^{(k)}(x)=t\Rightarrow f^{(k+1)}(x)dx=dt,\ then$$

$$\Omega\le\sum_{k=0}^{n-1}log\ t\Big|_{f^{(k)}(a)}^{f^{(k)}(b)}=\sum_{k=0}^{n-1}log\left(\frac{f^{(k)}(a)}{f^{(k)}(b)}\right)<\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{f^{(k)}(a)}$$

AN.041. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$Let\ k=n-i+1\ and\ p=n-j+1,\ then$$

$$\Omega_1(n)=\sum_{k=1}^{n}\sum_{p=1}^{n}\left|(p-k)\left(\frac{1}{n+k}-\frac{1}{n+p}\right)\right|=\sum_{k=1}^{n}\sum_{p=1}^{n}\frac{(p-k)^2}{(n+p)(n+k)}=$$

$$=\sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{(p-k)^2}{n+p}\Rightarrow$$

$$\Omega_1(n)\ge\sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{(k-1)^2}{2n}\ge\sum_{k=1}^{n}\frac{(k-1)^2}{2(n+k)}$$

Because:

$$\sum_{p=1}^{n}\frac{(p-k)^2}{n+p}(p-k)^2\ge(k-1)^2;p\ge1\ and\ \frac{1}{n+p}\ge\frac{1}{2n};\forall p\le n$$

$$\sum_{p=1}^{n}\frac{(k-1)^2}{2n}=n\frac{(k-1)^2}{2n}=\frac{(k-1)^2}{2}$$

$$\frac{1}{n+k}\ge\frac{1}{2n}\Rightarrow\sum_{k=1}^{n}\frac{(k-1)^2}{2(n+k)}\ge\sum_{k=1}^{n}\frac{(k-1)^2}{4n}$$

$$\Omega_1(n)\ge\sum_{k=1}^{n}\frac{(k-1)^2}{4n}=\sum_{k=0}^{n-1}\frac{k^2}{4n}=\frac{1}{12}n^2-\frac{1}{8}n+\frac{1}{24}$$

$$\Omega_2(n)=\sum_{k=1}^{n}\sum_{p=1}^{n}\left|\left(\frac{1}{n+k}-\frac{1}{n+p}\right)\right|=\sum_{k=1}^{n}\sum_{p=1}^{n}\frac{|p-k|}{(n+p)(n+k)}$$

$$\Rightarrow\Omega_2(n)=\sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{|p-k|}{n+p}\le\sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{|n-k|}{n+1}$$

$$|p-k|\le|n-k|\le n-1\ \textit{and}\ \frac{1}{n+p}\le\frac{1}{n+1}$$

$$\Omega_2(n)\le\sum_{k=1}^{n}\frac{1}{n+k}\sum_{k=1}^{n}\frac{n-k}{n+1}\le\sum_{k=1}^{n}\frac{1}{n+k}\sum_{k=1}^{n}\frac{n-1}{n+1}$$
$$\le\frac{n(n-1)}{n+1}\sum_{k=1}^{n}\frac{1}{n+k}\le$$

$$\le\frac{n(n-1)}{n+1}\cdot\frac{n}{n+1}\le\frac{n^3}{(n+1)^2}\le n$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{\Omega_1(n)}{\Omega_2(n)}\ge\lim_{n\to\infty}\frac{\frac{n^2}{12}-\frac{n}{8}+\frac{1}{24}}{n}=+\infty$$

AN.042. *Solution by Ravi Prakash-New Delhi-India*

$$\sqrt{\frac{rr_a}{r_br_c}}=\sqrt{\frac{\frac{F^2}{s(s-a)}}{\frac{F^2}{(s-b)(s-c)}}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\tan\frac{A}{2}$$

$$\tan^{-1}\sqrt{\frac{rr_a}{r_br_c}}+\tan^{-1}\sqrt{\frac{rr_b}{r_cr_a}}+\tan^{-1}\sqrt{\frac{rr_c}{r_ar_b}}=\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\sin^2x+\cos x+2}{\sin x+\cos x+7}dx;\ (1)$$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\cos^2 x+\sin x+2}{\cos x+\sin x+7}dx\ ;(2)$$

By adding (1) and (2), we get:

$$2\Omega=\int_0^{\frac{\pi}{2}}\frac{3+\cos x+\sin x+4}{\cos x+\sin x+7}dx=\frac{\pi}{2}$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\sin^2 x+\cos x+2}{\sin x+\cos x+7}dx=\frac{\pi}{4}$$

AN.043. *Solution by Ravi Prakash-New Delhi-India*

Let $f(t)=\frac{t\cdot\log t}{t^4+x^2}, 0<t<x$ *−continuous.*

If $0<x<1(t)<0\Rightarrow\int_1^x f(t)\,dt<0$

If $x>1, f(t)>0\Rightarrow\int_1^x f(t)\,dt>0$

For $x=1\Rightarrow\int_1^x f(t)\,dt=0\Rightarrow x=1$ *is only possible solution.*

AN.044. *Solution by Ravi Prakash-New Delhi-India*

$$\Omega=\int_1^{21}\frac{dx}{e^{\left[2x+\frac{1}{4}\right]}}\overset{\left(t=2x+\frac{1}{4}\right)}{=}\frac{1}{2}\int_{2+\frac{1}{4}}^{42+\frac{1}{4}}\frac{dt}{e^{[t]}}=$$

$$=\frac{1}{2}\left[\int_{2+\frac{1}{4}}^{3}\frac{dt}{e^2}+\sum_{k=3}^{41}\int_k^{k+1}\frac{dt}{e^k}+\int_{42}^{42+\frac{1}{4}}\frac{dt}{e^{42}}\right]=$$

$$=\frac{1}{2}\left[\frac{1}{e^2}\cdot\frac{3}{4}+\sum_{k=3}^{41}\frac{1}{e^k}+\frac{1}{e^{42}}\cdot\frac{1}{4}\right]=\frac{1}{2}\left[\frac{3}{4}\cdot\frac{1}{e^2}+\frac{\frac{1}{e^3}\left(1-\left(\frac{1}{e}\right)^{39}\right)}{1-\frac{1}{e}}+\frac{1}{4e^{42}}\right]=$$

$$= \frac{3}{8e^2} + \frac{1}{8e^{42}} + \frac{1}{2e^2(e-1)}\left[1 - \frac{1}{e^{39}}\right]$$

AN.045. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Firstly, let's\ prove\ that:$$

$$f'(x+y+t) + f'(t) \geq f'(x+t) + f'(y+t), \forall x, y, t \geq 0 \quad (1)$$

$$(1)\ is\ symmetrical\ on\ variables\ x, y, WLOG,$$

$$we\ may\ assume\ that\ x \geq y.$$

$$Since\ f'\ is\ convex\ then\ f''\ is\ increasing\ on\ [0, \infty)$$

$$Applying\ Mean\ Value\ Theorem, we\ obtain:$$

$$f'(u) - f'(v) \geq (u-v)f''(v), \forall u, v \geq 0\ (\because f''\ is\ increasing)$$

$$Then:$$

$$f'(x+y+t) - f'(x+t) \geq yf''(x+t) \quad and$$

$$f'(t) - f'(y+t) \geq -yf''(y+t)$$

$$Therefore,$$

$$[f'(x+y+t) - f'(x+t)] + [f'(t) - f'(y+t)] \geq y[f''(x+t) - f''(y+t)] \geq 0$$

$$because\ x \geq y \geq 0\ and\ f''\ is\ increasing\ on\ [0, \infty). \quad Hence, (1)\ is\ true.$$

$$Integrating\ the\ inequality\ (1)\ on\ [0, z], we\ obtain$$

$$\int_0^z \left(f'(x+y+t) + f'(t)\right)dt \geq \int_0^z \left(f'(x+t) + f'(y+t)\right)dt$$

$$Or, \quad [f(x+y+z) + f(z)] - [f(x+y) + f(0)] \geq [f(z+x) + f(y+z)] - [f(x) + f(y)]$$

$$Then: \quad f(x)+f(y)+f(z)+f(x+y+z) \geq f(x+y)+f(y+z)+f(z+x)$$

$$By\ AM-GM\ inequality, we\ have:$$

$$x+y \geq 2\sqrt{xy}, y+z \geq 2\sqrt{yz},$$

$$z+x \geq 2\sqrt{zx}\ and\ since\ f\ is\ increasing\ then$$

$$f(x)+f(y)+f(z)+f(x+y+z) \geq f(2\sqrt{xy})+f(2\sqrt{yz})+f(2\sqrt{zx}), \forall x,y,z \geq 0.$$

AN.046. *Solution by Pham Duc Nam-Vietnam*

$$\int e^{5x}(tan^4 x + tan^5 x + tan^6 x)dx = \int e^{5x} tan^4 x\,(1 + tan\,x + tan^2 x)dx \overset{t=tan\,x}{=}$$

$$= \int e^{5\,tan^{-1} t} t^4 \left(\frac{t}{t^2+1}+1\right) dt$$

$$= \int e^{5\,tan^{-1} t} t^4\, dt + \int \frac{t^5 e^{5\,tan^{-1} t}}{t^2+1} dt \overset{IBP}{=}$$

$$\Rightarrow \int e^{5\,tan^{-1} t} t^4\, dt + \int \frac{t^5 e^{5\,tan^{-1} t}}{t^2+1} dt = \frac{1}{5} t^5 e^{5\,tan^{-1} t} - \int \frac{t^5 e^{5\,tan^{-1} t}}{t^2+1} dt + \int \frac{t^5 e^{5\,tan^{-1} t}}{t^2+1} dt$$

$$= \frac{1}{5} t^5 e^{5\,tan^{-1} t} + C = \frac{1}{5} tan^5(x e^{5x}) + C$$

$$\int_0^n e^{5x}(tan^4 x + tan^5 x + tan^6 x)dx = \frac{1}{5} tan^5(n e^{5n})$$

$$\Omega = \lim_{n\to\infty} e^{5n+1} tan^5 n \left(\int_0^n e^{5x}(tan^4 x + tan^5 x + tan^6 x)dx\right)^{-1} =$$

$$= 5\lim_{n\to\infty}\frac{e^{5n+1}\tan^5 n}{\tan^5(ne^{5n})} = 5e$$

AN.047. *Solution by Ruxandra Daniela Tonilă-Romania*

$$2\int_a^b\int_a^b\int_a^b\left(\frac{y+x}{y+z}+\frac{y+z}{y+x}\right)dxdydz + 2(b-a)^3 =$$

$$= 2(b-a)^3 + \int_a^b\int_a^b\int_a^b\left((y+x)\frac{1}{y+z}+(y+z)\frac{1}{y+x}\right)dxdydz \overset{AHM}{\leq}$$

$$\leq 2(b-a)^3 + \int_a^b\int_a^b\int_a^b\left((y+x)\cdot\frac{\frac{1}{y}+\frac{1}{z}}{2}+(y+z)\cdot\frac{\frac{1}{y}+\frac{1}{x}}{2}\right)dxdydz$$

$$= 2(b-a)^3 + \frac{1}{2}\int_a^b\int_a^b\int_a^b\left(2+\frac{y}{z}+\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{z}{y}+\frac{z}{x}\right)dxdydz =$$

$$= 2(b-a)^3\int_a^b\int_a^b\int_a^b dxdydz$$
$$+\frac{1}{2}\int_a^b\int_a^b\int_a^b\left(x\left(\frac{1}{y}+\frac{1}{z}\right)+(y+z)\frac{1}{x}+\frac{y}{z}+\frac{z}{y}\right)dxdydz$$

$$= 3(b-a)^3 + \frac{1}{2}\int_a^b\int_a^b\left(\frac{b^2-a^2}{2}\left(\frac{1}{y}+\frac{1}{z}\right)+(y+z)\log\frac{b}{a}\right.$$
$$\left.+(b-a)\left(\frac{y}{z}+\frac{z}{y}\right)\right)dydz =$$

$$= 3(b-a)^3 + \frac{1}{2}\int_a^b\left(\frac{b^2-a^2}{2}\log\frac{b}{a}+\frac{b-a}{z}\cdot\frac{b^2-a^2}{2}+\frac{b^2-a^2}{2}\log\frac{b}{a}\right)dz +$$

$$+\int_a^b\left((b-a)z\log\frac{b}{a}+\frac{b-a}{z}\cdot\frac{b^2-a^2}{2}+(b-a)z\log\frac{b}{a}\right)dz =$$

$$= 3(b-a)^3 + \frac{3}{2}(b+a)(b-a)^2 \log\frac{b}{a}$$

Therefore, we have to prove:

$$3(b-a)^3 + \frac{3}{2}(b+a)(b-a)^2 \log\frac{b}{a} \le 3(b+a)(b-a)^2 \log\left(\frac{b}{a}\right) \Leftrightarrow$$

$$3(b-a)^3 \le \frac{3}{2}(b+a)(b-a)^2 \log\frac{b}{a} \Leftrightarrow$$

$$b-a \le \frac{b+a}{2}\log\frac{b}{a} \Leftrightarrow \frac{2}{b+a} \le \frac{\log\frac{b}{a}}{b-a} \Leftrightarrow$$

$$\frac{b+a}{2} \ge \frac{b-a}{\log b - \log a}; \forall a,b > 0$$

AN.048. *Solution by Syed Shahabudeen-Kerala-India*

$$\Omega = \lim_{n\to\infty}\frac{H_n}{n(H_{2n-1}-2H_{n-1})} = \lim_{n\to\infty}\frac{H_n}{n\left(H_{2n-1}-2\left(H_n-\frac{1}{n}\right)\right)} =$$

$$= \lim_{n\to\infty}\frac{H_n}{nH_{2n-1}-2nH_n-1} = \lim_{n\to\infty}\frac{1}{n\frac{H_{2n-1}}{H_n}-2n-\frac{1}{H_n}}$$

$$Here: \lim_{n\to\infty} n\left(\frac{H_{2n-1}}{H_n}-2\right) = \lim_{n\to\infty}\frac{\frac{H_{2n-1}}{H_n}-2}{\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{H_{2n-1}}{H_n} \overset{Cesaro-S}{=} \lim_{n\to\infty}\frac{H_{2n+1}-H_{2n-1}}{H_{n+1}-H_n} = \lim_{n\to\infty}\frac{\frac{1}{2n+1}+\frac{1}{2n}}{\frac{1}{n}} = 1$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{H_{2n-1}}{H_n}-2\right) = -\infty$$

$$\lim_{n\to\infty}\frac{1}{H_n}=0$$

Therefore,

$$\lim_{n\to\infty}\left(n\frac{H_{2n-1}}{H_n}-2n-\frac{1}{H_n}\right)=0\Rightarrow \Omega=\lim_{n\to\infty}\frac{1}{n\frac{H_{2n-1}}{H_n}-2n-\frac{1}{H_n}}=0$$

AN.049. *Solution by Kamel Gandoulli Rezgui-Tunisia*

Let $f(t)=\sqrt{t^4+t^2+1}>0; a,b>0$ and let $c=\frac{\sqrt{a^2+ab+b^2}}{\sqrt{3}}\leq\frac{\sqrt{3}}{\sqrt{3}}b$.
From M.V.T., we get:

$$\exists\alpha\in[a,b]: f(\alpha)=\frac{1}{b-a}\int_a^b f(t)\,dt$$

$$f\nearrow\Rightarrow \alpha\geq\frac{a+b}{2};\frac{\alpha}{\frac{a+b}{2}}\neq 0$$

If $a\to 0, c\to\frac{b}{\sqrt{3}}$ and $\frac{a+b}{2}\to\frac{b}{2}$ then, $\frac{\alpha}{\frac{a+b}{2}}=\frac{2}{\sqrt{3}}=c$ impossible!

$$c<\alpha\Rightarrow f(\alpha)\geq f(c)\Rightarrow$$

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f\left(\frac{\sqrt{a^2+ab+b^2}}{\sqrt{3}}\right)=\frac{\sqrt{(a^2+ab+b^2)^2+3(a^2+ab+b^2)+9}}{3}$$

Therefore,

$$3\int_a^b\sqrt{x^4+x^2+1}\,dx\geq(b-a)\sqrt{(a^2+ab+b^2)^2+3(a^2+ab+b^2)+9}$$

AN.050. *Solution by Rana Ranino-Setif-Algerie*

$$\Omega(n)=\int\frac{x^{2n-1}(1-x^2)}{e^{nx^2}}dx\overset{t=x^2}{=}\frac{1}{2}\int t^{n-1}e^{-nt}\,dt-\frac{1}{2}\int t^ne^{-nt}\,dt$$

$$\frac{1}{2}\int t^{n-1}e^{-nt}\,dt\overset{IBP}{=}\frac{1}{2n}t^ne^{-nt}+\frac{1}{2}\int t^ne^{-nt}\,dt$$

$$\Omega(n)=\frac{1}{2n}t^ne^{-nt}+C=\frac{x^{2n}}{2ne^{nx^2}}+C$$

AN.051.

Denote $f(x)=y; y\in\left[-\frac{5}{2},\frac{5}{2}\right]$ *then* $2y\in[-5,5]$ *and*

$-2y\in[-5,5], 15-2y\in[10,20]$, *hence* $15-2y>0$.

We will start from $(5-6y)^2\geq 0; \forall y\in\left[-\frac{5}{2},\frac{5}{2}\right]\Leftrightarrow$

$$25+36y^2-60y\geq 0\Leftrightarrow 200-32y^2\leq 225-60y+4y^2$$

$$200-32y^2\leq(15-2y)^2\Leftrightarrow\sqrt{200-32y^2}\leq 15-2y$$

$$\sqrt{200-32y^2}+2y\leq 15\Leftrightarrow\sqrt{200-32f^2(x)}+2f(x)\leq 15$$

$$2\sqrt{50-8f^2(x)}+2f(x)\leq 15\Leftrightarrow\sqrt{50-8f^2(x)}+f(x)\leq\frac{15}{2}$$

Therefore,

$$\int_{-\frac{5}{2}}^{\frac{5}{2}}\sqrt{50-8f^2(x)}\,dx+\int_{-\frac{5}{2}}^{\frac{5}{2}}f(x)\,dx\leq\int_{-\frac{5}{2}}^{\frac{5}{2}}\frac{15}{2}dx=\frac{75}{2}$$

Equality holds for $f(x)=\frac{5}{6}; \forall x\in\left[-\frac{5}{2},\frac{5}{2}\right]$.

AN.052. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$Let\ x\in[0,a], since\ a\leq\frac{\pi}{12}$$
$$\to\ \cos x, \cos(2x), \cos(4x), \cos(6x), \sin x\geq 0$$

$$We\ have:\ \cos(2x)=2\cos^2 x-1\overset{AM-GM}{\leq}(\cos^4 x+1)-1=\cos^4 x$$
$$\to\ \cos(2x)\leq\cos^4 x\ \ (1)$$

$$\to\ \cos(4x)\overset{(1)}{\leq}\cos^4(2x)\overset{(1)}{\leq}(\cos^4 x)^4=\cos^{16}x\ \to\ \cos(4x)$$
$$\leq\cos^{16}x\ \ (2)$$

$$cos(3x) = cos\,x\,.(4\,cos^2 x - 3) \overset{AM-GM}{\lesssim}$$

$$cos\,x\,.[(cos^8 x + 1 + 1 + 1) - 3] = cos^9 x \;\rightarrow\; cos(3x) \leq cos^9 x \;\;(i)$$

$$\rightarrow\; cos(6x) \overset{(1)}{\lesssim} cos^4(3x) \overset{(i)}{\lesssim} (cos^9 x)^4 = cos^{36} x \;\rightarrow$$

$$cos(6x) \leq cos^{36} x \;\;(3)$$

$$(1),(2),(3) \rightarrow cos(6x)\,.\,cos^6(4x)\,.\,cos^{15}(2x) \leq cos^{36} x\,.(cos^{16} x)^6.(cos^4 x)^{15} = cos^{192} x$$

$$\rightarrow \int_0^a sin\,x\,.\,cos(6x)\,.\,cos^6(4x)\,.\,cos^{15}(2x)\,dx \leq \int_0^a sin\,x\,.\,cos^{192} x\,dx$$

$$= \left[\frac{-1}{193} cos^{193} x\right]_0^a = \frac{1}{193}(1 - cos^{193} a).$$

AN.053. *Solution by Adrian Popa-Romania*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)} \Rightarrow \Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right) = \frac{\pi}{sin\left(\frac{\pi x}{2}\right)}$$

$$\Gamma\left(\frac{x-1}{2}\right)\Gamma\left(1-\frac{x+1}{2}\right) = \frac{\pi}{sin\left(\frac{\pi(x+1)}{2}\right)} = \frac{\pi}{sin\left(\frac{\pi}{2}+\frac{\pi x}{2}\right)} = \frac{\pi}{cos\left(\frac{\pi x}{2}\right)}$$

So, we have:

$$\Omega(x) = \frac{\pi}{sin\left(\frac{\pi x}{2}\right)} \cdot \frac{\pi}{cos\left(\frac{\pi x}{2}\right)} \cdot 2\,sin\left(\frac{\pi x}{2}\right) cos\left(\frac{\pi x}{2}\right) = 2\pi^2$$

Therefore,

$$x^2 - \frac{4x}{\Omega(x)} + \frac{1}{\pi^4} = 0 \Leftrightarrow x^2 - \frac{4x}{2\pi^2} + \frac{1}{\pi^4} = 0 \Leftrightarrow \left(x - \frac{1}{\pi^2}\right)^2 = 0 \Rightarrow$$

$$x = \frac{1}{\pi^2}$$

AN.054. *Solution by Ravi Prakash-New Delhi-India*

$$x_0 = 1, x_1 = 0, x_n = (n-1)(x_{n-1} + x_{n-2}), n \geq 2, n \in \mathbb{N}; (1) \Rightarrow$$

$$x_n - nx_{n-1} = -[x_{n-1} - (n-)x_{n-2}]. \textit{ Put: } a_n = x_n - nx_{n-1}, \forall n \geq 1, a_1 = -1$$

Also, (1) gives $a_n = -a_{n-1}, \forall n \geq 2 \Rightarrow (a_n)_{n\geq 2}$ *−geometric progression with ratio* $q = -1$.

Thus, $a_n = (-1)^{n-1}a_1, \forall n \geq 1 \Rightarrow x_n - nx_{n-1} = (-1)^n, \forall n \geq 1$

$$\Rightarrow \frac{x_n}{n!} - \frac{x_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!} \Rightarrow \sum_{r=1}^{n}\left(\frac{x_r}{r!} - \frac{x_{r-1}}{(r-1)!}\right) = \sum_{r=1}^{n}\frac{(-1)^r}{r!}, \forall n \geq 1$$

$$\Rightarrow \frac{x_n}{n!} = \sum_{r=1}^{n}\frac{(-1)^r}{r!} + \frac{x_0}{0!}$$

$$\Omega = \lim_{n\to\infty}\frac{x_n}{n!} = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{(-1)^r}{r!} = \frac{1}{e}$$

AN.055. *Solution by Ravi Prakash-New Delhi-India*

$$xf(y) + yf(x) = 2f(xy)$$

Put $y = 1$ *so that* $xf(1) + f(x) = 2f(x)$

$$f(x) = xf(1) = 3x; \forall x \in \mathbb{R}$$

Similarly, $g(x) = 2x; \forall x \in \mathbb{R}$. *So,*

$$f\left(\frac{\sinh x}{3}\right) = \sinh x \textit{ and } g\left(\frac{\cosh x}{2}\right) = \cosh x$$

$$\Omega = \int_0^1 \frac{\frac{e^x - e^{-x}}{2}dx}{\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}} = \frac{1}{2}\int_0^1 \frac{e^x - e^{-x}}{e^x}dx$$

$$= \frac{1}{2}\int_0^1 (1 - e^{-2x})\,dx = \frac{1}{2}\left[x + \frac{1}{2}e^{-2x}\right]_0^1 = \frac{1}{4}\left(1 + \frac{1}{e^2}\right)$$

AN.056. *Solution by Yen Tung Chung-Taichung-Taiwan*

$$\Omega = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\pi^n}\cdot\left(\frac{\pi}{e}\right)^k = \sum_{n=0}^{\infty}\frac{1}{\pi^n}\left(\sum_{k=0}^{n}\left(\frac{\pi}{e}\right)^k\right) = \sum_{n=0}^{\infty}\frac{1}{\pi^n}\cdot\frac{1-\left(\frac{\pi}{e}\right)^{n+1}}{1-\frac{\pi}{e}} =$$

$$= \frac{e}{e-\pi}\sum_{n=0}^{\infty}\left(\frac{1}{\pi^n}-\frac{\pi}{e^{n+1}}\right) = \frac{e}{e-\pi}\left(\frac{1}{1-\frac{1}{\pi}}-\pi\cdot\frac{\frac{1}{e}}{1-\frac{1}{e}}\right)$$

$$= \frac{e}{e-\pi}\left(\frac{\pi}{\pi-1}-\frac{\pi}{e-1}\right) = \frac{\pi e}{(\pi-1)(e-1)}$$

AN.057. *Solution by Adrian Popa-Romania*

$$Let\ x \in (0,1)$$

$$1-x+x^2+\cdots+(-1)^n x^n = \frac{(-x)^{n+1}-1}{-x-1}, (n\to\infty) \Rightarrow$$

$$1-x+x^2+\cdots = \frac{1}{1+x}$$

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots = log(1+x)$$

$$1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots = \frac{log(1+x)}{x}$$

$$Let\ x = \frac{1}{\pi}$$

$$1-\frac{1}{\pi}\cdot\frac{1}{2}+\frac{1}{\pi^2}\cdot\frac{1}{3}-\frac{1}{\pi^3}\cdot\frac{1}{4}+\cdots = \frac{log\left(\frac{1}{\pi}+1\right)}{\frac{1}{\pi}}$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{\pi}\right)^n\cdot\frac{1}{n+1} = \pi\, log\left(\frac{1}{\pi}+1\right)$$

$$\Omega_1 = 1+\frac{\pi}{2}+1-\frac{1}{2\pi} = \pi\, log\left(\frac{1}{\pi}+1\right) = \pi\, log\left(\frac{1}{\pi}+1\right)+\frac{1}{2\pi}-\frac{\pi}{2}$$

$$Let\ x=\frac{1}{e}$$

$$1-\frac{1}{e}\cdot\frac{1}{2}+\frac{1}{e^2}\cdot\frac{1}{3}-\frac{1}{e^3}\cdot\frac{1}{4}+\cdots=\frac{log\left(1+\frac{1}{e}\right)}{\frac{1}{e}}$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{e}\right)^n\cdot\frac{1}{n+1}=e\ log\left(1+\frac{1}{e}\right)$$

$$\Omega_2=-1+\frac{\pi}{2}+1-\frac{1}{2e}=e\ log\left(1+\frac{1}{e}\right)=e\ log\left(1+\frac{1}{e}\right)+\frac{1}{2e}-\frac{\pi}{2}$$

$$Let\ f(x)=x\ log\left(1+\frac{1}{x}\right)+\frac{1}{2x},x>1$$

$$f'(x)=log\left(1+\frac{1}{x}\right)+\frac{x\cdot\left(-\frac{1}{x^2}\right)}{1+\frac{1}{x}}-\frac{1}{2x^2}=log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}-\frac{1}{2x^2}$$

$$f''(x)=\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}+\frac{1}{(x+1)^2}+\frac{1}{x^3}=-\frac{1}{x(x+1)}+\frac{1}{(x+1)^2}+\frac{1}{x^3}$$
$$=\frac{2x+1}{x^3(x+1)^2}>0$$

AN.058. *Solution by Asmat Qatea-Afghanistan*

$$\because\ tan^{-1}x-tan^{-1}y=tan^{-1}\left(\frac{x-y}{1+xy}\right),$$

$$\because\ n^3+6n^2+11n+6=(n+1)(n+2)(n+3)$$

$$\Omega=\sum_{n=0}^{\infty}tan^{-1}\left(\frac{(n^3+6n^2+11n+5)\cdot n!}{1+(n^3+6n^2+11n+6)\cdot(n!)^2}\right)=$$

$$=\sum_{n=0}^{\infty}tan^{-1}\left(\frac{(n^3+6n^2+11n+6)\cdot n!-n!}{1+(n^3+6n^2+11n+6)\cdot(n!)^2}\right)=$$

$$=\sum_{n=0}^{\infty} tan^{-1}\left((n^3+6n^2+11n+6)\cdot n!\right)-\sum_{n=0}^{\infty} tan^{-1}(n!)=$$

$$=\sum_{n=0}^{\infty} tan^{-1}\left((n+3)!\right)-\sum_{n=0}^{\infty} tan^{-1}(n!)=$$

$$=\sum_{n=0}^{\infty} tan^{-1}\left((n+3)!\right)-2\,tan^{-1}(1)-tan^{-1}(2)-\sum_{n=3}^{\infty} tan^{-1}(n!)$$

$$=-\left(\frac{\pi}{2}+tan^{-1}(2)\right)$$

AN.059. *Solution by Ruxandra Daniela Tonilă-Romania*

$$\int_0^1 f^2(x)dx=\int_0^1 f^2(x)dx\cdot\int_0^1 dx+\frac{1}{2}\int_0^1 f^2(x)dx\overset{CBS}{\geq}$$

$$\geq\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2+\frac{1}{2}\int_0^1 f^2(x)dx=\frac{81}{2}+\frac{1}{2}\int_0^1 f^2(x)dx\geq$$

$$\geq 1+\frac{8}{3}+\frac{1}{2}\int_0^1 f^2(x)dx=1+8\int_0^1 x^2dx+\frac{1}{2}\int_0^1 f(x)dx\geq$$

$$\geq 1+\frac{1}{2}\int_0^1\left(16x^2+f^2(x)\right)dx\overset{AGM}{\geq}1+\int_0^1 4xf(x)dx$$

Therefore,

$$\int_0^1 f^2(x)dx\geq 1+4\int_0^1 xf(x)\,dx$$

AN.060. *Solution by Ty Halpen-USA*

$$\Omega(n)=(1+2^2)(1+2^4)(1+2^8)\cdot\ldots\cdot\left(1+2^{2^{n-1}}\right)=$$

$$=\frac{1-2^2}{1-2^2}\cdot(1+2^2)(1+2^4)(1+2^8)\cdot\ldots\cdot\left(1+2^{2^{n-1}}\right)=$$

$$= -\frac{1}{3}(1-2^4)(1+2^4)(1+2^8)\cdot\ldots\cdot\left(1+2^{2^{n-1}}\right) = -\frac{1}{3}\left(1-2^{2^n}\right)$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{1}{3\Omega(n)}\right)^{2^{2^n}} = \lim_{n\to\infty}\left(1+\frac{1}{2^{2^n}-1}\right)^{\left(2^{2^n}-1\right)\cdot\frac{2^{2^n}}{2^{2^n}-1}}$$
$$= exp\left\{\lim_{n\to\infty}\frac{2^{2^n}}{2^{2^n}-1}\right\} = e$$

AN.061. *Solution by Ruxandra Daniela Tonilă-Romania*

$$\Omega = \lim_{n\to\infty} n\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)^{\frac{1-n}{n^2}} \Leftrightarrow log\,\Omega$$
$$= \lim_{n\to\infty}\left(log\,n + \frac{1-n}{n^2}log\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)\right)$$

$$log\,\Omega = \lim_{n\to\infty}\left(log\,n - \frac{1}{n}log\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)\right) + \lim_{n\to\infty}\frac{log\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)}{n^2}$$
$$= \Omega_1 + \Omega_2$$

$$where\ \Omega_1 = \lim_{n\to\infty}\left(log\,n - \frac{1}{n}log\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)\right)\ and\ \Omega_2$$
$$= \lim_{n\to\infty}\frac{log\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)}{n^2}$$

$$\Omega_1 = \lim_{n\to\infty}\left(log\,n - \frac{1}{n}log\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\left(log\,n^n - log\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)\right) =$$

$$= \lim_{n\to\infty} log\left(\frac{n^n}{\prod_{k=2}^{n}\sqrt[k]{k!}}\right)^{\frac{1}{n}} = log\left(\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{\prod_{k=2}^{n}\sqrt[k]{k!}}}\right) \overset{C-D}{=}$$

$$= log\left(\lim_{n\to\infty}(\left(\frac{n+1}{n}\right)^n \cdot \frac{n+1}{\sqrt[n+!]{(n+1)!}}\right) = log\left(e \cdot \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right) =$$

$$= log\left(e \cdot \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}\right) \overset{C-D}{=} log\left(e \cdot \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right)$$

$$= log\left(e \cdot \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\right) = log\, e^2 = 2; (2)$$

$$\Omega_2 = \lim_{n\to\infty}\frac{log\left(\prod_{k=2}^n \sqrt[k]{k!}\right)}{n^2} = \lim_{n\to\infty}\frac{\sum_{k=2}^n log\left(\sqrt[k]{k!}\right)}{n^2} \overset{C-S}{=}$$

$$= \lim_{n\to\infty}\frac{log\left(\sqrt[n+1]{(n+1)!}\right)}{2n+1} = \lim_{n\to\infty}\frac{log(n+1)!}{(n+1)(2n+1)} =$$

$$= \lim_{n\to\infty}\frac{\sum_{k=1}^n log\, k}{2n^2+3n+1} \overset{C-S}{=} \lim_{n\to\infty}\frac{log(n+2)}{4n+5} \overset{L'H}{=} 0; (3)$$

From (1),(2) and (3), it follows that: $log\,\Omega = 2 \Rightarrow \Omega = e^2$.

AN.062. *Solution by Marian Ursărescu-Romania*

Put $9x - 4 = t \Rightarrow x = \frac{t+4}{9}, dx = \frac{1}{9}dt$

$$\Omega = \int_1^2 \frac{log(9x-4)}{3x^2+2}dx = \frac{1}{9}\int_5^{14}\frac{log\, t}{3\left(\frac{t+4}{9}\right)^2+2}dt$$

$$= \frac{1}{9}\int_5^{14}\frac{log\, t}{\frac{3(t^2+8t+16)}{81}+2}dt =$$

$$= 3\int_5^{14}\frac{log\, t}{t^2+8t+70}dt; (1)$$

$$Let: I = \int_5^{14}\frac{log\, t}{t^2+8t+70}dt \overset{t=\frac{70}{y}}{=} -70\int_{14}^{5}\frac{log\left(\frac{70}{y}\right)}{\frac{70^2}{y^2}+8\cdot\frac{70}{y}+70}dy =$$

$$= \int_5^4 \frac{log\,70 - log\,y}{y^2+8y+70}dy$$

$$= log\,70 \int_5^{14} \frac{1}{y^2+8y+70}dy - \int_5^{14} \frac{log\,y}{y^2+8y+70}dy$$
$$=$$

$$= \frac{1}{2} log\,70 \int_5^{14} \frac{1}{(y+y)^4 + \left(\sqrt{54}\right)^2}dy = \frac{1}{2\sqrt{54}} log\,70\, tan^{-1}\left(\frac{y+4}{\sqrt{54}}\right)\Bigg|_5^{14}$$
$$=$$

$$= \frac{1}{2\sqrt{54}} log\,70 \left(tan^{-1}\left(\frac{18}{\sqrt{54}}\right) - tan^{-1}\left(\frac{9}{\sqrt{54}}\right)\right) =$$

$$= \frac{1}{2\sqrt{54}} log\,70 \left(tan^{-1}\sqrt{6} - tan^{-1}\left(\frac{3}{\sqrt{6}}\right)\right); (2)$$

From (1),(2) it follows that:

$$\Omega = \int_1^2 \frac{log(9x-4)}{3x^2+2}dx = \frac{1}{2\sqrt{6}} log\,70 \left(tan^{-1}\sqrt{6} - tan^{-1}\left(\frac{3}{\sqrt{6}}\right)\right)$$

AN.063. *Solution by Cristi Iulian-Romania*

Using Bernoulli's inequality, we get:

$$x^y = \left(1+(x-1)\right)^y \geq 1 + y(x-1)$$

Integrating the previous relation w.r.t. x, respectively to y, and using the monotonicity property of integrals, we obtain:

$$2\int_a^b \int_a^b x^y\,dxdy \geq 2\int_a^b \int_a^b \left(1+y(x-1)\right)dxdy =$$

$$= 2\int_a^b (b-a)\left(1 + \frac{y(a+b)}{2} - y\right)dy$$

$$2\int_a^b \int_a^b x^y\,dxdy \geq 2(b-a)^2\left(1 + \frac{a+b-2}{4}(a+b)\right)$$

Lastly, we need to prove that:

$$2(b-a)^2\left(1+\frac{a+b-2}{4}(a+b)\right)\geq(2ab-a-b+2)(b-a)^2$$

That is equivalent to:

$$(b-a)^2\left(2+\frac{a+b-2}{2}(a+b)-2ab+a+b-2\right)\geq 0\Leftrightarrow$$

$$(a+b)^2\geq 4ab \text{ which is true } \forall a,b\geq 0.$$

AN.064. *Solution by Amrit Awasthi-India*

$$\text{We know: } K(n+1)=1^1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n \text{ and}$$

$$G(n+2)=1!\cdot 2!\cdot 3!\cdot\ldots\cdot n!$$

$$\Rightarrow G(n+2)=1^n\cdot 2^{n-1}\cdot 3^{n-2}\cdot\ldots\cdot(n-1)^2\cdot n^1$$

$$K(n+1)\cdot G(n+2)$$
$$=(1^1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n)\cdot(1^n\cdot 2^{n-1}\cdot 3^{n-2}\cdot\ldots\cdot(n-1)^2\cdot n^1)$$

$$=1^{n+1}\cdot 2^{n+1}\cdot\ldots\cdot(n-1)^{n+1}\cdot n^{n+1}=(n!)^{n+1}$$

Therefore,

$$\Omega=\sum_{n=2}^{\infty}\sqrt[n]{\frac{n!}{K(n+1)\cdot G(n+2)}}=\sum_{n=2}^{\infty}\sqrt[n]{\frac{n!}{(n!)^{n+1}}}=\sum_{n=2}^{\infty}\sqrt[n]{\frac{1}{(n!)^n}}=$$

$$=\sum_{n=2}^{\infty}\frac{1}{n!}=e-2$$

AN.065. *Solution by Ravi Prakash-New Delhi-India*

$$Let\ a_k=\frac{(2n+1)!}{(2n+1-k)!};0\leq k\leq n\ and$$

$$b_k=\frac{\left((2n+1)!\right)^2}{k!\,(2n+1-k)!}=a_k\cdot\frac{(2n+1)!}{k!\,(2n-k)!}=a_k\cdot\binom{2n+1}{k}\Rightarrow$$

$$\sum_{k=0}^{n} b_k = a_0 + a_1\binom{2n+1}{1} + a_2\binom{2n+1}{2} + \dots + a_n\binom{2n+1}{n} >$$

$$> (2n+1)(a_1 + a_2 + \dots + a_n) =$$

$$= 2n(a_0 + a_1 + \dots + a_n) - 2n + a_1 + \dots + a_n > 2n(a_0 + \dots + a_n)$$

$$\because a_1 - 2n = 2n + 1 - 2n = 1 > 0. \text{ Thus,}$$

$$0 < \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)^{-1} < \frac{1}{2n}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=0}^{n} \frac{(2n+1)!}{(2n+1-k)!}\right)\left(\sum_{k=0}^{n} \frac{\left((2n+1)!\right)^2}{k!\left((2n+1-k)!\right)^2}\right)^{-1} = 0$$

AN.066. *Solution by Ravi Prakash-New Delhi-India*

$$\sum_{k=1}^{n-1}\left(\binom{n}{k}\sum_{m=1}^{n}\frac{m^n}{m^k}\right) = \sum_{k=1}^{n-1}\left(\binom{n}{k}\sum_{m=1}^{n} m^{n-k}\right) = \sum_{m=1}^{n}\left(\sum_{k=1}^{n-1}\binom{n}{k} m^{n-k}\right) =$$

$$= \sum_{m=1}^{n}[(m+1)^n - m^n - 1) = (n+1)^n - 1 - n$$

Therefore,

$$\Omega = \lim_{n\to\infty}(1+n)^{-n}\sum_{k=1}^{n-1}\left(\binom{n}{k}\sum_{m=1}^{n}\frac{m^n}{m^k}\right) = \lim_{n\to\infty}\frac{(n+1)^n - n - 1}{(n+1)^n} = 1$$

AN.067. *Solution by Adrian Popa-Romania*

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1} \Rightarrow \sum_{k=0}^{n}\left(1 - \frac{2k}{n}\binom{n}{k}\right)^2 = \sum_{k=0}^{n}\left(1 - 2\binom{n-1}{k-1}\right)^2 =$$

$$=\sum_{k=0}^{n}\left(1-4\binom{n-1}{k-1}+4\binom{n-1}{k-1}^2\right)$$
$$=\sum_{k=0}^{n}1-4\sum_{k=0}^{n}\binom{n-1}{k-1}+4\sum_{k=0}^{n}\binom{n-1}{k-1}^2$$
$$=n-4\cdot 2^{n-1}+4\cdot\binom{2n-2}{n-1}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\binom{2n-2}{n-1}^{-1}\sum_{k=0}^{n}\left(1-\frac{2k}{n}\binom{n}{k}\right)^2$$
$$=\lim_{n\to\infty}\frac{n-4\cdot 2^{n-1}+4\cdot\binom{2n-2}{n-1}}{\binom{2n-2}{n-1}}=4$$

AN.068. *Solution by Adrian Popa-Romania*

$$\Omega(\alpha,\beta)=\int_{-1}^{1}\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx\overset{\frac{1-x}{1+x}=t}{=}$$

$$=\int_0^\infty\frac{2^{2\alpha+2\beta-2}t^{2\alpha-1}}{(1+t)^{2\alpha+2\beta-2}}\cdot\frac{(1+t)^{2\alpha+2\beta}}{2^{\alpha+\beta}(1+t^2)^{\alpha+\beta}}\cdot\left(\frac{2}{1+t^2}\right)dt\overset{u=t^2}{=}$$

$$=2^{\alpha+\beta-2}\int_0^\infty\frac{\left(\sqrt{u}\right)^{2\alpha-1}}{(1+u)^{\alpha+\beta}}\cdot\frac{du}{2\sqrt{u}}=2^{\alpha+\beta-2}\int_0^\infty\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du$$
$$=2^{\alpha+\beta-2}B(\alpha,\beta)$$

$$\Omega(\alpha,\beta)=2^{\alpha+\beta-2}B(\alpha,\beta)$$

$$\Omega(3,5)=2^6\cdot B(3,5)=2^6\cdot\frac{\Gamma(3)\Gamma(5)}{\Gamma(3+5)}=2^6\cdot\frac{2!\cdot 4!}{7!}$$

$$\Omega(4,5)=2^7\cdot B(4,5)=2^7\cdot\frac{\Gamma(4)\Gamma(5)}{\Gamma(4+5)}=2^7\cdot\frac{3!\cdot 4!}{8!}$$

$$\Omega(3,6)=2^7\cdot B(3,6)=2^7\cdot\frac{\Gamma(3)\Gamma(6)}{\Gamma(3+6)}=2^7\cdot\frac{2!\cdot 5!}{8!}$$

$$\sqrt{\Omega(4,5)\cdot\Omega(3,6)}=\frac{2^7}{8!}\sqrt{2!\cdot 3!\cdot 4!\cdot 5!}\overset{(?)}{\leq}\frac{2^6}{7!}\cdot 2!\cdot 4!\Leftrightarrow$$

$$\frac{1}{4}\sqrt{(2!)^2\cdot 3\cdot(4!)^2\cdot 5}\leq 2!\cdot 4!\Leftrightarrow\sqrt{15}\leq\sqrt{16}.$$

AN.069. *Solution by Adrian Popa-Romania*

$$\binom{2n}{n-k}=\frac{(2n)!}{(n-k)!\cdot(n+k)!}\Rightarrow\frac{1}{(n-k)!\cdot(n+k)!}=\frac{\binom{2n}{n-k}}{(2n)!}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}=\binom{2n}{n}+\binom{2n}{n-1}+\binom{2n}{n-2}+\cdots+\binom{2n}{0}$$

$$=\binom{2n}{n+1}+\binom{2n}{n+2}+\cdots+\binom{2n}{2n}=S$$

$$\because\binom{2n}{0}+\binom{2n}{1}+\binom{2n}{2}+\cdots+\binom{2n}{2n}=2^{2n}\Rightarrow 2S+\binom{2n}{n}=2^{2n}\Rightarrow$$

$$2S=4^n-\binom{2n}{n}$$

$$\Rightarrow S=\frac{4^n-\binom{2n}{n}}{2}\Rightarrow\sum_{k=0}^{n}\binom{2n}{n-k}=S+\binom{2n}{n}=\frac{\left(4^n+\binom{2n}{n}\right)}{2}$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{(2n)!\cdot\left(2\sum_{k=0}^{n}\frac{1}{(n-k)!\cdot(n+k)!}-\frac{4^n}{(2n)!}\right)}=$$

$$=\lim_{n\to\infty}\sqrt[n]{(2n)!\cdot\left(\frac{4^n+\binom{2n}{n}}{2(2n)!}\cdot 2-\frac{4^n}{(2n)!}\right)}=\lim_{n\to\infty}\sqrt[n]{\binom{2n}{n}}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n)!}{(n!)^2}}\overset{C-D}{=}$$

$$=\lim_{n\to\infty}\frac{(2n+2)!}{\left((n+1)!\right)^2}\cdot\frac{(n!)^2}{(2n)!}=\lim_{n\to\infty}\frac{(2n)!\,(2n+1)(2n+2)(n!)^2}{(n!)^2(n+1)^2(2n)!}=4$$

AN.070. *Solution by Ose Favour-Nigeria*

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{sin^{-1}\varepsilon}^{sin^{-1}(1-\varepsilon)} log\left((cos\,x)^{cot\,x}\cdot(sin\,x)^{\frac{cos\,x}{1+sin\,x}}\right)dx =$$

$$= \int_0^{\frac{\pi}{2}} cot\,x\,log(cos\,x)\,dx + \int_0^{\frac{\pi}{2}} \frac{cos\,x}{1+sin\,x} log(sin\,x)\,dx = A + B$$

$$A \overset{u=cos\,x}{=} \int_0^1 \frac{u}{1-u^2} log\,u\,du = \sum_{n=0}^{\infty}\int_0^1 u^{2n+1}\,log\,u\,du \overset{IBP}{=}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 u^{2n+1}\,du = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = -\frac{1}{4}\zeta(2) = -\frac{\pi^2}{24}$$

$$B \overset{u=sin\,x}{=} \int_0^1 \frac{log\,u}{1+u}du = Li_2(-1) = -\frac{\pi^2}{12} \Rightarrow \Omega = A + B = -\frac{\pi^2}{8}$$

AN.071. *Solution by Adrian Popa-Romania*

$$\Omega = \lim_{n\to\infty} n^7\cdot\Omega(n)\cdot sin\left(\frac{1}{n^4}\right)\cdot tan\left(\frac{1}{n^5}\right) =$$

$$= \lim_{n\to\infty} n^9\left(\frac{1}{n}\cdot\sum_{k=1}^{n} e^{4\left(2-\frac{k}{n}\right)}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n} e^{6\left(2-\frac{k}{n}\right)} - \left(\frac{1}{n}\sum_{k=1}^{n} e^{5\left(2-\frac{k}{n}\right)}\right)^2\right)\cdot sin\left(\frac{1}{n^4}\right)\cdot tan\left(\frac{1}{n^5}\right) =$$

$$= \lim_{n\to\infty}\frac{sin\left(\frac{1}{n^4}\right)}{\frac{1}{n^4}}\cdot\frac{tan\left(\frac{1}{n^5}\right)}{\frac{1}{n^5}}\left(\int_0^1 e^{4(2-x)}dx\cdot\int_0^1 e^{6(2-x)}dx - \left(\int_0^1 e^{5(2-x)}dx\right)^2\right) =$$

$$= \left(-e^8\frac{e^{-4x}}{4}\Big|_0^1\right)\cdot\left(-e^{12}\frac{e^{-6x}}{6}\Big|_0^1\right) - \left(-e^{10}\frac{e^{-5x}}{5}\Big|_0^1\right)^2 =$$

$$= e^{10}\left[\frac{(e^4-1)(e^6-1)}{24} - \frac{(e^5-1)^2}{25}\right]$$

AN.072. *Solution by Hikmat Mammadov-Azerbaijan*

$$\int_a^b\int_a^b\left(\frac{1}{1+x^2} - \frac{1}{1+xy}\right)dxdy = \int_a^b\int_a^b \frac{x(y-x)}{(1+x^2)(1+xy)}dxdy =$$

$$= \frac{1}{2}\left(\int_a^b\int_a^b \frac{x(y-x)}{1+x^2(1+xy)}dxdy + \int_a^b\int_a^b \frac{y(x-y)}{(1+y^2)(1+yx)}dydx\right) =$$

$$= \frac{1}{2}\int_a^b\int_a^b \frac{(y-x)[x(1+y^2)-y(1+x^2)]}{(1+x^2)(1+y^2)(1+xy)}dxdy =$$

$$= \frac{1}{2}\int_a^b\int_a^b \frac{(y-x)(x+xy^2-y-yx^2)}{(1+x^2)(1+y^2)(1+xy)}dxdy =$$

$$= \frac{1}{2}\int_a^b\int_a^b \frac{(y-x)(-1+xy)}{(1+x^2)(1+y^2)(1+xy)}dxdy \geq$$

$$\geq \frac{1}{2}\int_a^b \frac{(y-x^2)(-1+a^2)}{(1+x^2)(1+y^2)(1+xy)}dxdy \geq 0$$

Therefore,

$$\int_a^b\int_a^b \frac{dxdy}{1+xy} \leq \int_a^b\int_a^b \frac{1}{1+x^2}dxdy = y|_a^b tan^{-1} x|_a^b$$
$$= (b-a)(tan^{-1} b - tan^{-1} a)$$
$$= (b-a)\,tan^{-1}\left(\frac{b-a}{1+ab}\right)$$

AN.073. *Solution by Ravi Prakash-New Delhi-India*

$$For\ n \geq 1:\ \frac{n(n+1)^2(2n+1)}{n!} = \frac{(n+1)^2(2n+1)}{(n-1)!}$$

$$Write: (n+1)^2(2n+1)$$
$$= A(n-1)(n-2)(n-3) + B(n-1)(n-2)$$
$$+ C(n-1) + D$$

Equaling coefficient of n^3*, we get:* $A = 2$. *Put* $n = 1 \Rightarrow D = 12$; $n = 2 \Rightarrow C = 33$,

$n = 3 \Rightarrow B = 17$. *Thus, for* $n \geq 4$ *we have:*

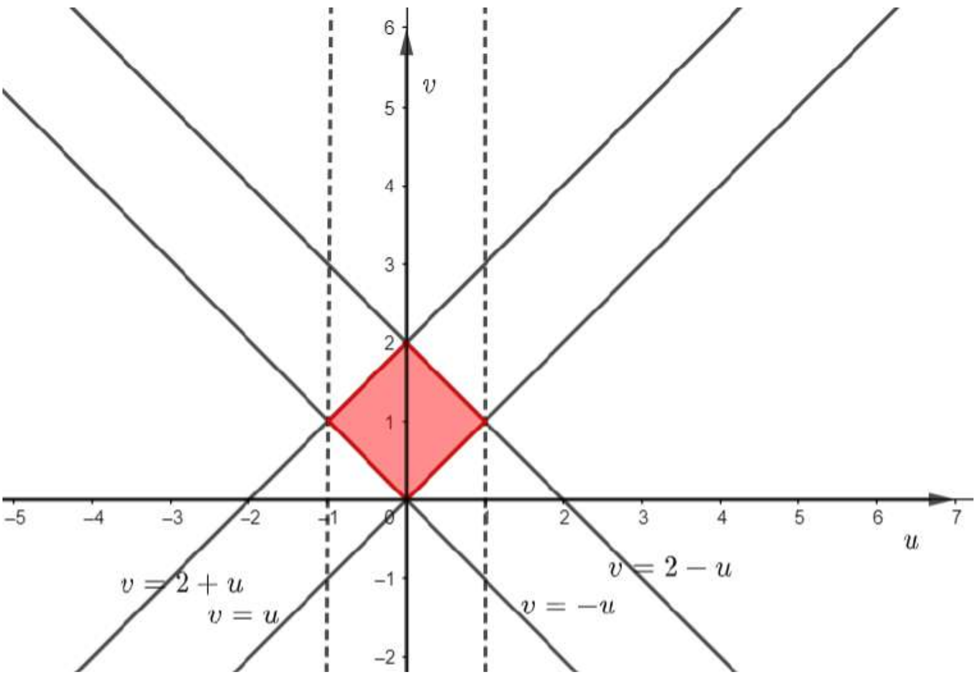
$$\frac{n(n+1)^2(2n+1)}{n!} = \frac{2}{(n-4)!} + \frac{17}{(n-3)!} + \frac{33}{(n-2)!} + \frac{12}{(n-1)!}$$

$$Now, \Omega = \sum_{n=1}^{\infty} \frac{n(n+1)^2(2n+1)}{4^n \cdot n!} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{(n+1)^2(2n+1)}{4^{n-1}(n-1)!} =$$

$$= \frac{1}{4}\cdot\frac{12}{0!} + \frac{1}{4^2}\cdot\frac{12}{1!} + \frac{1}{4^2}\cdot\frac{33}{0!} + \frac{1}{4^3}\cdot\frac{12}{2!} + \frac{1}{4^3}\cdot\frac{33}{1!} + \frac{1}{4^3}\cdot\frac{17}{0!} + \frac{1}{4^4}\cdot\frac{12}{3!} + \frac{1}{4^4}\cdot\frac{33}{2!} + \frac{1}{4^4}\cdot\frac{17}{1!} + \frac{1}{4^4}\cdot\frac{2}{0!} + \cdots$$

$$\Omega = \sum_{n=1}^{\infty} \frac{n(n+1)^2(2n+1)}{4^n \cdot n!} = \frac{12}{4}e^{\frac{1}{4}} + \frac{33}{16}e^{\frac{1}{4}} + \frac{17}{64}e^{\frac{1}{4}} + \frac{12}{256}e^{\frac{1}{4}} = \frac{43}{8}e^{\frac{1}{4}}$$

AN.074. *Solution by Said Cerbach-Algiers-Algerie*



Using Cauchy-Schwarz's inequality, we have:

$$\int_0^1 e^{x^2}\,dx\int_0^1 e^{-x^2}\,dx \geq 1\ then$$

$$I=\int_0^1 e^{x^2}\,dx\int_0^1 e^{-x^2}\,dx-1=\int_0^1\int_0^1\left(e^{x^2-y^2}-1\right)dxdy\,.\ We\ have:$$

$$\frac{(e-1)^2}{4e}=\frac{\cosh 1-1}{2}$$

We use a change of variables: $u=x-y$ *and* $v=x+y$ *then we have:*

$$I=\iint_D \frac{e^{uv}-1}{2}dudv\,,\qquad where\ D\ is\ red\ sector.$$

$$I=\int_0^1\int_{-u}^{2+u}\frac{e^{uv}-1}{2}dv\,du+\int_0^1\int_u^{2-u}\frac{e^{uv}-1}{2}dvdu=I_-+I_+, where$$

$$I_-=\int_0^1\int_{-u}^{2+u}\frac{e^{uv}-1}{2}dv\,du$$

$$I_+=\int_0^1\int_u^{2-u}\frac{e^{uv}-1}{2}dvdu$$

With change u *in* I_-*, we have:*

$$I_-=\int_0^1\int_u^{2-u}\frac{e^{-uv}-1}{2}dudv\,, then:$$

$$I=\int_0^1\int_u^{2-u}\frac{e^{uv}+e^{-uv}-2}{2}dvdu=\int_0^1\int_u^{2-u}\frac{\left(e^{\frac{uv}{2}}-e^{-\frac{uv}{2}}\right)^2}{2}dvdu=$$

$$=\int_0^1\int_u^{2-u}\sinh^2\left(\frac{uv}{2}\right)dvdu=\int_0^1\int_u^{2-u}\frac{\cosh v-1}{2}dv\,du\,;(0\leq v\leq 2)$$

We have:

$$I \leq \int_0^{\frac{1}{2}} \frac{\cosh x - 1}{2} dx \leq \int_0^1 \frac{\cosh 1 - 1}{2} dx = \frac{\cosh 1 - 1}{2}$$

AN.075. *Solution by Ravi Prakash-New Delhi-India*

$$f(1) = 11, g(0) = 2, h(0) = 3, t(0) = 4$$

$$f(x + y + z) = g(x) + h(y) + t(z), \forall x, y, z \in \mathbb{R}; (1)$$

$$f(x) \cdot g(x) = h(x) \cdot t(x); (2)$$

Put $y = z = 0$ *in (1),* $f(x) = g(x) + 3 + 4 \Rightarrow g(x) = f(x) - 7, \forall x \in \mathbb{R}$

Similarly, $h(x) = f(x) - 6, \forall x \in \mathbb{R}$ *and* $t(x) = f(x) - 5, \forall x \in \mathbb{R}$

Putting $z = 0$ *in (1), we get:*

$$f(x + y) = g(x) + h(y) + t(0) = f(x) - 7 + f(y) - 6 - 4 =$$

$$= f(x) + f(y) - 9, \forall x, y \in \mathbb{R}$$

$$f(x + y) - 9 = [f(x) - 9] + [f(y) - 9]; (3)$$

Let $\alpha(x) = f(x) - 9, \forall x \in \mathbb{R}$*, then (3) gives*

$\alpha(x + y) = \alpha(x) + \alpha)y), \forall x, y \in \mathbb{R}$*. Also,* α *is continuous on* $\mathbb{R}, \alpha(x) = x\alpha(1), \forall x \in \mathbb{R}$.

Now, $f(x)g(x) = h(x)t(x)$*, then*

$$f(x)[f(x) - 7] = [f(x) - 6][f(x) - 5]$$

$$\left(f(x)\right)^2 - 7f(x) = \left(f(x)\right)^2 - 11f(x) + 30$$

$$4f(x) = 30 \Rightarrow f(x) = 7.5 \qquad 2x + 9 = 7.7 \Rightarrow x = -0.75$$

AN.076. *Solution by Hikmat Mammadov-Azerbaijan*

$$\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right) = \frac{1}{x^{k+1}} \int_0^x t^k \cos\left(t + \frac{k\pi}{2}\right) dt \overset{t=ux}{\Longrightarrow}$$

$$\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right)=\frac{1}{x^{k+1}}\int_0^1 u^k x^k \cos\left(ux+\frac{k\pi}{2}\right)x\,du$$

$$=\frac{1}{x^{k+1}}\int_0^1 u^k x^{k+1}\cos\left(ux+\frac{k\pi}{2}\right)du$$

$$=\int_0^1 u^k\cdot\cos\left(ux+\frac{k\pi}{2}\right)du$$

$$\because -1\le \cos z\le 1,\forall z\in\mathbb{R}\ and\ \int_a^b f(x)dx\le\int_a^b g(x)dx\,,x\in[a,b]$$

$$-\int_0^1 u^k\,du\le\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\le\int_0^1 u^k\,du,0\le u\le 1$$

$$\left|\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\right|\le\int_0^1 u^k\,du$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right)\right|=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{1}{x^{k+1}}\int_0^x t^k\cos\left(t+\frac{k\pi}{2}\right)dt\right|$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\right|$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\right|\le\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1 u^k\,du$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k+1}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k+1}\overset{C-S}{=}\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}\frac{1}{k+1}-\sum_{k=1}^{n}\frac{1}{k+1}}{(n+1)-n}=\lim_{n\to\infty}\frac{1}{n+2}=0$$

$$0\le\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right)\right|\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k+1}$$

$$Therefore, \qquad \Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}\left|\left(\frac{\sin x}{x}\right)^{(k)}\right| = 0$$

AN.077. *Solution by Adrian Popa-Romania*

$$\left(\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}\right)^2 \overset{CBS}{\leq}$$

$$\leq\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)(H_1+H_2+\cdots+H_n) =$$

$$= H_n(H_1+H_2+\cdots+H_n)$$

$$\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n} \leq \sqrt{H_n(H_1+H_2+\cdots+H_n)}$$

$$0 \leq \frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}} \leq \frac{1}{n} \to 0$$

Therefore,

$$\Omega = \lim_{n\to\infty}\frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}}$$

AN.078. *Solution by Duc Nam-Vietnam*

By Euler's identity: $e^{ix} = \cos x + i\sin x$

$$e^{-ix} = \cos x - i\sin x$$

$$(*): e^{ix}+e^{2ix}+e^{3ix}+\frac{1}{e^{ix}}+\frac{1}{e^{2ix}}+\frac{1}{e^{3ix}} = -1 \Leftrightarrow$$

$$\cos x + i\sin x + \cos 2x + i\sin 2x + \cos 3x + i\sin 3x + \cos x - i\sin x + \cos 2x - i\sin 2x + \cos 3x - i\sin 3x = -1$$

$$cos\,x + cos\,2x + cos\,3x = -1$$

$$cos\,x + 2\,cos^2\,x - 1 + 4\,cos^3\,x - 3\,cos\,x + \frac{1}{2} = 0$$

$$4\,cos^3\,x + 2\,cos^2\,x - 2\,cos\,x - \frac{1}{2} = 0$$

$$8\,cos^3\,x + 4\,cos^2\,x - 4\,cos\,x - 1 = 0;\,(**)$$

Taking $\alpha = \frac{2\pi}{7}$ *and let* $x = cos\,\alpha$*, we have:*

$$cos\,\frac{4\pi}{7} = 2x^2 - 1, cos\,\frac{6\pi}{7} = 4x^3 - 3x, cos\,\frac{8\pi}{7} = 8x^4 - 8x^2 + 1$$

$$(x-1)(8x^3 + 4x^2 - 4x - 1) = 0$$

The polynomial $P = 8x^3 + 4x^2 - 4x - 1$ *has the same form as* $(**)$ *and have 3 roots*

$$cos\,\frac{2\pi}{7}, cos\,\frac{4\pi}{7}, cos\,\frac{6\pi}{7}$$

So,

$$cos\,x = cos\,\frac{2\pi}{7}, cos\,x = cos\,\frac{4\pi}{7}, cos\,x = cos\,\frac{6\pi}{7}$$

$$x \in \left\{\pm\frac{2\pi}{7} + 2k\pi, \pm\frac{4\pi}{7} + 2k\pi, \pm\frac{6\pi}{7} + 2k\pi \middle| k \in \mathbb{Z}\right\}$$

AN.079. *Solution by Ravi Prakash-New Delhi-India*

$$\Omega(n) = \int_0^n log\left(\sqrt{n+x} + \sqrt{n-x}\right) dx \overset{(x=n\,cos(2\theta))}{=}$$

$$= n\int_{\frac{\pi}{4}}^0 log\left(\sqrt{2n}(cos\,\theta + sin\,\theta)\right)(-2\,sin\,2\theta)d\theta =$$

$$= n\left[\frac{1}{2}log(2n) - I\right]$$

Where,

$$I=\int_{\frac{\pi}{4}}^{0} log(cos\,\theta+sin\,\theta)\frac{d}{d\theta}(cos\,2\theta)d\theta=$$

$$=log(cos\,\theta+sin\,\theta)\,cos(2\theta)\Big|_{\frac{\pi}{4}}^{0}+\int_0^{\frac{\pi}{4}}\frac{cos\,\theta-sin\,\theta}{cos\,\theta+sin\,\theta}(cos^2\theta-sin^2\theta)d\theta=$$

$$=\int_0^{\frac{\pi}{4}}(cos\,\theta-sin\,\theta)^2d\theta=\int_0^{\frac{\pi}{4}}(1-sin(2\theta))d\theta=$$

$$=\left(\theta+\frac{1}{2}cos(2\theta)\right)\Big|_0^{\frac{\pi}{4}}=\frac{\pi-2}{4}$$

Thus,

$$\Omega(n)=n\left[\frac{1}{2}log(2n)+\frac{\pi-2}{4}\right]=\frac{n}{4}[2\,log(2n)+\pi-2]$$

AN.080. *Solution by Tapas Das-India*

Let $f(y)=sin\,y-y$*, then* $f'(y)=cos\,y-1.$

Now $-1\le cos\,y\le 1, f'(y)<0;\forall y>0$*, thus* f *–is decreasing function.*

$$f(y)<f(0)\Rightarrow sin\,y-y><0\Rightarrow sin\,y<y$$

Now, $sin\,3x<3x$ *and* $sin\,5x<5x.$

$$\frac{sin^2x}{(1-cos\,x)^2}=\frac{1-cos^2x}{(1-cos\,x)^2}=\frac{1+cos\,x}{1-cos\,x}=cot^2\frac{x}{2}$$

Now, we know that

$$cot\,x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\cdots;(0<|x|<\pi)$$

$$cot\, x < \frac{1}{x}$$

$$cot\frac{x}{2} < \frac{2}{x} \Rightarrow cot^2\frac{x}{2} < \frac{4}{x^2}$$

$$\frac{sin\,3x \cdot sin\,5x \cdot sin^2 x}{(1 - cos\,x)^2} < 60$$

Therefore,

$$\int_a^b \frac{sin\,3x \cdot sin\,5x \cdot sin^2 x}{(1 - cos\,x)^2} dx < 60\int_a^b dx < 60(b - a)$$

AN.081. *Solution by Ravi Prakash-New Delhi-India*

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^n = \sum_{k=1}^{n}(-1)^n(-1)^k\binom{n}{n-k}k^n =$$

$$= (-1)^n\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^n; (1)$$

We have:

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^k = (1 - x)^n$$

Differentiating w.r.t. x, we get:

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k k x^{k-1} = (-1)(1 - x)^{n-1}(n) \Rightarrow$$

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k k x^k = (-1)(1 - x)^{n-1}(nx)$$

Differentiating w.r.t. x, we get:

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k k^2 x^{k-1}$$

$$= (-1)(1 - x)^{n-1}(n) + n(n - 1)(-1)^2(1 - x)^{n-2}(x)$$

Multiply again by x, we get:

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k k^2 x^k = (-1)(1-x)^{n-1}(nx) + n(n-1)(-1)^2(1-x)^{n-2}x^2$$

Differentiating again, we get:

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^3x^{k-1} = (-1)(1-x)^{*n-1} + n(n-1)(-1)^2(2x)(1-x)^{n-2} +$$

$$+n(n-1)(n-2)(-1)^3(1-x)^{n-3}x^2$$

Continuing this way, we get:

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^n x^{k-1} = (1-x)g(x) + n!\,(-1)^n x^n$$

Putting $x = 1$, we get

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^n = n!\,(-1)^n; (2)$$

From (1),(2) it follows

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}(n-k)^n - (-1)^n n!\,(-1)^n = n!$$

Now,

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^n} = \lim_{n\to\infty}\frac{1}{n}\sqrt[n]{n!} \overset{C-D}{=} \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

AN.082. *Solution by Tapas Das-India*

$$Let\ P = 5 + e^x\sin\left(x+\frac{\pi}{4}\right) + e^{-x}\cos\left(x+\frac{\pi}{4}\right)$$

$$dP = (e^x - e^{-x})\left[\sin\left(x+\frac{\pi}{4}\right) + \cos\left(x+\frac{\pi}{4}\right)\right] = dx$$

$$= (e^x - e^{-x})\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right]dx =$$

$$= (e^x - e^{-x})\sqrt{2}\cos x\,dx = 2\sinh x\sqrt{2}\cos x\,dx =$$

$$= 2\sqrt{2}\sinh x\cos x$$

$$\Omega = \int_0^{\frac{\pi}{4}}\frac{\cos x\cdot\sinh x}{5 + e^x\sin\left(x+\frac{\pi}{4}\right) + e^{-x}\cos\left(x+\frac{\pi}{4}\right)}dx =$$

$$= \frac{1}{2\sqrt{2}} \int_{5+\sqrt{2}}^{5+e^{\frac{\pi}{4}}} \frac{dP}{P} = \frac{1}{2\sqrt{2}} \log P \Big|_{5+\sqrt{2}}^{5+e^{\frac{\pi}{4}}} =$$

$$= \frac{1}{2\sqrt{2}} \log\left(\frac{5+e^{\frac{\pi}{4}}}{5+\sqrt{2}}\right)$$

AN.083. *Solution by Nikos Ntorvas-Greece*

Let be the function $f(t) = t \log t, t > 0, f$ *−is a strictly convex function on* $(0,\infty)$*, as a continuous function on* $(0,\infty)$ *where*

$$f''(t) = \frac{1}{t} > 0, \forall t > 0$$

For $t_1 + t_1 = 1$ *we have from Jensen's inequality that:*

$$f(t_1) + f(t_2) \geq 2f\left(\frac{t_1+t_2}{2}\right) \Leftrightarrow$$

$$t_1 \log t_1 + t_2 \log t_2 \geq 2f\left(\frac{1}{2}\right) \Leftrightarrow t_1 \log t_1 + t_2 \log t_2 \geq -\log 2 \Leftrightarrow$$

$$-t_1 \log t_1 - t_2 \log t_2 \leq \log 2 \Leftrightarrow t_1 \log\left(\frac{1}{t_1}\right) + t_2 \log\left(\frac{1}{t_2}\right) \leq \log 2 \,; (1)$$

For $t_1 = \frac{2a}{2a+3b}$; (2) *and* $t_2 = \frac{3b}{2a+3b}$; (3), $a, b > 0$ *we have:*

$$t_1 + t_2 = 1, 0 < t_1, t_2 < 1$$

Combining (1), (2) and (3) we have:

$$\frac{2a}{2a+3b} \log\left(1+\frac{3b}{2a}\right) + \frac{3b}{2a+3b} \log\left(1+\frac{2a}{3b}\right) \leq \log 2$$

Equality holds for $2a = 3b$.

AN.084. *Solution by Adrian Popa-Romania*

$$\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right) = \log\left(b+\sqrt{1+b^2}\right) - \log\left(a+\sqrt{1+a^2}\right) =$$

$$= \log\left(x+\sqrt{1+x^2}\right)\Big|_a^b = \int_a^b \frac{dx}{\sqrt{1+x^2}}$$

We must to prove:

$$\frac{\tan^{-1} x}{x} \geq \frac{1}{\sqrt{1+x^2}}, \forall x > 0$$

$$Let: f(x) = tan^{-1} x - \frac{x}{\sqrt{1+x^2}}, x > 0 \; then \; f'(x) = \frac{\sqrt{1+x^2}-1}{1+x^2} > 0, \forall x > 0$$

$$\Rightarrow f \nearrow \text{ and } f(0) = 0 \Rightarrow f(x) > 0, \forall x > 0 \Rightarrow tan^{-1} x \geq \frac{x}{\sqrt{1+x^2}}, \forall x > 0.$$

Therefore,

$$\int_a^b \left(\frac{1}{x} \cdot tan^{-1} x\right) dx \geq log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Equality holds for $a = b$.

AN.085. *Solution by Ravi Prakash-New Delhi-India*

For $x, y > 0$, let $x = r \sin\theta \,; y = r\cos\theta \,, r > 0, 0 \leq \theta \leq \frac{\pi}{2}$

$$sin^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = sin^{-1}(\sin\theta) + sin^{-1}(\cos\theta) = = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}$$

$$\Omega = \int_a^{a+1}\int_a^{a+1} sin^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) dxdy + \int_a^{a+1}\int_a^{a+1} sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) dxdy =$$

$$= \int_a^{a+1}\int_a^{a+1}\left[sin^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right] dxdy =$$

$$= \int_a^{a+1}\int_a^{a+1} \frac{\pi}{2} dxdy = \frac{\pi}{2}$$

AN.086. *Solution by Yen Tung Chung-Taichung-Taiwan*

$$\Omega = \int \frac{3xe^x + 2}{(2\log x + 3e^x - 1)(2\log x^x + 3xe^x + x)} dx =$$

$$= \int \frac{3xe^x + 2}{x(2\log x + 3e^x - 1)(2\log x + 3e^x + 1)} dx =$$

$$= \int \frac{3e^x + \frac{2}{x}}{(2\log x + 3e^x - 1)(2\log x + 3e^x + 1)} dx \overset{2\log x + 3e^x = y}{=}$$

$$\overset{2\log x + 3e^x = y}{=} \int \frac{1}{(y-1)(y+1)} dt = \frac{1}{2}\int \frac{dt}{y-1} - \frac{1}{2}\int \frac{dt}{y+1} =$$

$$=\frac{1}{2}log\left|\frac{y-1}{y+1}\right|+C=\frac{1}{2}log\left|\frac{2\,log\,x+3e^x-1}{2\,log\,x+3e^x+1}\right|+C$$

AN.087. *Solution by Mohammad Rostami-Afghanistan*

$$\Omega=\lim_{n\to\infty}\frac{1}{n!}\int_0^\infty\frac{x^n\,sin\left(x+\frac{\pi}{4}\right)}{e^x}dx$$

$$=\lim_{n\to\infty}\frac{1}{n!}\int_0^\infty\frac{x^n}{e^x}\cdot\frac{e^{ix+\frac{\pi}{4}i}-e^{-ix-\frac{\pi}{4}i}}{2i}dx=$$

$$=\lim_{n\to\infty}\frac{1}{n!}\left(\frac{e^{\frac{\pi}{4}i}}{2i}\int_0^\infty x^ne^{-(1-i)x}\,dx-\frac{e^{-\frac{pi}{4}i}}{2i}\int_0^\infty x^ne^{-(1+i)x}\,dx\right)\overset{(1\pm x)i=u}{=}$$

$$=\lim_{n\to\infty}\frac{1}{n!}\left(\frac{e^{\frac{\pi}{4}i}}{2i}\int_0^\infty\frac{u^{(n+1)-1}e^{-u}}{(1-i)^{n+1}}du-\frac{e^{-\frac{\pi}{4}i}}{2i}\int_0^\infty\frac{u^{(n+1)-1}e^{-u}}{(1+i)^{n+1}}du\right)=$$

$$=\lim_{n\to\infty}\frac{1}{\Gamma(n+1)}\cdot\Gamma(n+1)\left[\frac{e^{\frac{\pi}{4}i}}{2i(1-i)^{n+1}}-\frac{e^{-\frac{\pi}{4}i}}{2i(i+1)^{n+1}}\right]=$$

$$=\lim_{n\to\infty}\left[\frac{e^{\frac{\pi}{4}i}}{2i\left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^{n+1}}-\frac{e^{-\frac{\pi}{4}i}}{2i\left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{n+1}}\right]=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}\left(\frac{e^{\frac{\pi}{2}i+\frac{\pi}{4}ni}}{2i}-\frac{e^{-\frac{\pi}{2}i-\frac{\pi}{4}ni}}{2i}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}\left[\frac{\left(cos\frac{\pi}{2}+i\,sin\frac{\pi}{2}\right)e^{\left(\frac{\pi}{4}n\right)i}}{2i}-\frac{\left(cos\left(-\frac{\pi}{2}\right)+i\,sin\,\left(-\frac{\pi}{2}\right)\right)e^{\left(-\frac{\pi}{4}n\right)i}}{2i}\right]=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}\left(\frac{e^{\left(\frac{\pi}{4}n\right)i}+e^{\left(-\frac{\pi}{4}n\right)i}}{2}\right)=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}cos\left(\frac{\pi}{4}n\right)=0$$

AN.088. *Solution by Samar Das-India*

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx=\int\frac{\sin x+\sqrt{3}\cos x}{3\sin x-4\sin^3 x}dx=$$
$$=\int\frac{\sin x+\sqrt{3}\cos x}{\sin x\,(3-4\sin^2 x)}dx=\int\frac{\left(\sin x+\sqrt{3}\cos x\right)dx}{\sin x\,(3\sin^2 x+3\cos^2 x-4\sin^2 x)}$$
$$=\int\frac{\left(\sin x+\sqrt{3}\cos x\right)dx}{\sin x\cos^2 x\,(3-\tan^2 x)}=\int\frac{\left(1+\sqrt{3}\cos x\right)\sec^2 x\,dx}{3-\tan^2 x}\overset{y=\tan x}{=}$$
$$=\int\frac{\left(1+\frac{\sqrt{3}}{y}\right)dy}{3-y^2}=\int\frac{y+\sqrt{3}}{y(3-y^2)}dy=\frac{1}{\sqrt{3}}\int\left(\frac{1}{y}+\frac{1}{\sqrt{3}-y}\right)dy=$$
$$=\frac{1}{\sqrt{3}}\left(\log|y|-\log\left|\sqrt{3}-y\right|\right)+C=\frac{1}{\sqrt{3}}\log\left|\frac{y}{\sqrt{3}-y}\right|+C=$$
$$=\frac{1}{\sqrt{3}}\log\left|\frac{\sin x}{\sqrt{3}\cos x-\sin x}\right|+C$$

AN.089. *Solution by Adrian Popa-Romania*

$$\Omega=\lim_{n\to\infty}\frac{(2n-5)!}{(n+1)\cdot(2n)!\cdot B(6,2n-4)}=$$
$$=\lim_{n\to\infty}\frac{(2n-5)!}{(n+1)\cdot(2n)!\cdot\frac{\Gamma(6)\Gamma(2n-4)}{\Gamma(6n-4)}}=\lim_{n\to\infty}\frac{(2n-5)!\cdot\Gamma(2n+2)}{(n+1)(2n)!\,\Gamma(6)\Gamma(2n-4)}$$
$$=\lim_{n\to\infty}\frac{(2n-5)!\,(2n+1)!}{(n+1)(2n)!\cdot 5!\cdot(2n-5)!}=\lim_{n\to\infty}\frac{(2n)!\,(2n+1)}{(n+1)(2n)!\cdot 5!}=\frac{2}{5!}=\frac{1}{60}$$

AN.090. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$x,y,z\in[a,b]^3,0<a\le b\le 1\ and\ f(t)=t^2+1-t-t^t=$$
$$=t^2-t+1-t^t\ge 0\ in\ [0,1]\ and\ t^t\le 1$$
$$x^2+y^2+z^2+3=x^2+1+y^2+1+z^2+1$$
$$\ge x^x+x+y^y+y+z^z+z\Rightarrow$$
$$\frac{1}{x^2+y^2+z^2+3}\le\frac{1}{x^x+x+y^y+y+z^z+z}$$
$$\because (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\overset{CBS}{\ge}9\Rightarrow\frac{1}{a+b+c}\le\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c},\forall a,b,c>0$$

Hence, we have:

$$\frac{1}{x^2+y^2+z^2+3} \le \frac{1}{9(x^x+x)} + \frac{1}{9(y^y+y)} + \frac{1}{9(z^z+z)}$$

and then

$$\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{x^2+y^2+z^2+3}$$

$$\le \frac{1}{9}\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{x^x+x} + \frac{1}{9}\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{y^y+y} +$$

$$+\frac{1}{9}\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{z^z+z}$$

$$= \frac{(b-a)^2}{9}\int_a^b \frac{dx}{x^x+x} + \frac{(b-a)^2}{9}\int_a^b \frac{dy}{y^y+y}$$

$$+\frac{(b-a)^2}{9}\int_a^b \frac{dz}{z^z+z}$$

$$= 3\cdot\frac{(b-a)^2}{9}\int_a^b \frac{dx}{x^x+x} = \frac{(b-a)^2}{3}\int_a^b \frac{dx}{x^x+x}$$

Therefore,

$$\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{x^2+y^2+z^2+3} \le \frac{(b-a)^2}{3}\int_a^b \frac{dx}{x+x^x}$$

AN.091. *Solution by Kamel Gandouli Rezgui-Tunisia*

$$e^x + e^y - e^e \overset{?}{\ge} e^{x+y-e}$$

Let $f(x) = e^x + e^y - e^e - e^{x+y-e}$ then $f'(x) = e^x - e^{x+y-e}$

$x + y - e \le x$ because $e \ge y \Rightarrow f'(x) \ge 0 \Rightarrow f \nearrow$.

$$\lim_{x\to 0_+} f(x) = e^y - e^e - e^{y-e}$$

Let $g(y) = e^y - e^e - y^{y-e}$ then $g'(y) = e^y - e^{y-e} \le 0 \Rightarrow g \searrow \Rightarrow 1 - \frac{1}{e^e} \le g(y) \le e^e$

$\Rightarrow g(y) \ge 0$ because $1 - \frac{1}{e^e} > 0 \Rightarrow f(x) \ge g(y) \ge 1 - \frac{1}{e^e} > 0$

$$\Rightarrow e^x + e^y - e^e \ge e^{x+y-e}$$

$$e^x + e^y + e^z + e^t = e^x + e^y - e^e + e^z + e^t + e^e$$

$$\ge e^{x+y-e} + 3\cdot\sqrt[3]{e^z e^t e^e} \ge$$

$$\ge e^{x+y-e} + 3\cdot\sqrt[3]{e^{z+t+e}}$$

AN.092. *Solution by Ravi Prakash-New Delhi-India*

$$For\ 1 \le k \le n, let\ a_k = \frac{(n-k+1)H_k}{k(n-k+1)^2+k} = \frac{(n-k+1)H_k}{((n-k+1)^2+1)k} =$$

$$= \frac{n-k+1}{(n-k+1)^2+1} \cdot \frac{H_k}{k} < \frac{1}{n-k+1} \cdot 1 = \frac{1}{n-k+1}$$

Hence,

$$0 < \frac{1}{n}\sum_{k=1}^{n} a_k < \frac{1}{n}\sum_{k=1}^{n} \frac{1}{n-k+1} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k}$$

$$As\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k} \overset{L.C-S}{=} 0$$

By the sandwich theorem, we get

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{(n-k+1)H_k}{k(n-k+1)^2+k} = 0$$

AN.093. *Solution by Ravi Prakash-New Delhi-India*

Put $n^2 - n + 1 = \cot\theta$, then

$$\cos^{-1}\left(\frac{n^2-n+1}{\sqrt{1+(n^2-n+1)^2}}\right) = \cos^{-1}(\cos\theta) = \theta =$$

$$= \cot^{-1}(n^2-n+1) = \tan^{-1}\left(\frac{n-(n-1)}{1+n(n-1)}\right) =$$

$$= \tan^{-1} n - \tan^{-1}(n-1)$$

$$\omega = \sum_{n=1}^{\infty} \cos^{-1}\left(\frac{n^2-n+1}{\sqrt{1+(n^2-n+1)^2}}\right) = \sum_{n=1}^{\infty} (\tan^{-1} n - \tan^{-1}(n-1))$$

$$= \frac{\pi}{2}$$

$$I_1 = \int_0^{4\omega} \frac{dx}{3+\sin x} = \int_0^{2\pi} \frac{dx}{3+\sin x}$$

$$= \int_0^{\pi} \frac{dx}{3+\sin x} + \int_{\pi}^{2\pi} \frac{dx}{3+\sin x} \overset{x=\pi+\theta}{=}$$

$$= \int_0^{\pi} \frac{dx}{3+\sin x} + \int_0^{\pi} \frac{d\theta}{3-\sin\theta} = \int_0^{\pi} \frac{6}{9-\sin^2 x} dx = 6\int_0^{\pi} \frac{dx}{8+\cos^2 x}$$

$$=$$

$$= 6\int_0^{\frac{\pi}{2}} \frac{dx}{8+\cos^2 x} + 6\int_{\frac{\pi}{2}}^{\pi} \frac{dx}{8+\cos^2 x} \overset{x=\frac{\pi}{2}+\theta}{=}$$

$$= 6\int_0^{\frac{\pi}{2}} \frac{dx}{8+\cos^2 x} + 6\int_0^{\frac{\pi}{2}} \frac{d\theta}{8+\cos^2 \theta} =$$

$$= 6\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{8\sec^2 x + 1} dx + 6\int_0^{\frac{\pi}{2}} \frac{\csc^2 \theta}{8\csc^2 \theta + 1} d\theta =$$

$$= 6\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{8\tan^2 x + 9} + 6\int_0^{\frac{\pi}{2}} \frac{\csc^2 x\, dx}{8\cot^2 x + 9} =$$

$$= \frac{6}{6\sqrt{2}} \tan^{-1}\left(\frac{2\sqrt{2}\tan x}{3}\right)\Bigg|_0^{\frac{\pi}{2}} - \frac{6}{6\sqrt{2}} \tan^{-1}\left(\frac{2\sqrt{2}\cot \theta}{3}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{\sqrt{2}}$$

$$I_2 = \int_0^{\frac{2\omega}{\pi}} \frac{x^3}{\sqrt{1+x+x^2}} dx = \int_0^1 \frac{x^3}{\sqrt{1+x+x^2}} dx =$$

$$= \int_0^1 \frac{x(x^2+x+1) - (x^2+x+1) + 1}{\sqrt{1+x+x^2}} dx =$$

$$= \int_0^1 (x-1)\sqrt{x^2+x+1}\, dx + \int_0^1 \frac{dx}{\sqrt{x^2+x+1}} =$$

$$= \frac{1}{2}\int_0^1 (2x+1-3)\sqrt{x^2+x+1}\, dx + \int_0^1 \frac{dx}{\sqrt{x^2+x+1}} =$$

$$= \frac{1}{2}\int_0^1 (2x+1)\sqrt{x^2+x+1}\, dx - \frac{3}{2}\int_0^1 \sqrt{x^2+x+1}\, dx + \int_0^1 \frac{dx}{\sqrt{x^2+x+1}} =$$

$$= \frac{1}{3}(x^2+x+1)^{\frac{3}{2}}\Big|_0^1 - \frac{3}{2}\int_0^1 \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$+ \int_0^1 \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} dx =$$

$$= \frac{1}{3}(3\sqrt{3}-1) - \frac{3}{2}\left[\frac{x+\frac{1}{2}}{2}\sqrt{x^2+x+1} + \frac{3}{4}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right]\Bigg|_0^1 +$$

$$+ \log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\Bigg|_0^1 =$$

$$= \frac{1}{24} + \frac{5}{8}\sqrt{3} + \frac{1}{8}\log\left(\frac{3}{2}\right) - \frac{1}{8}\log\left(\frac{3}{2} + \sqrt{3}\right)$$

Therefore,

$$\Omega = \frac{\pi}{\sqrt{2}}\left[\frac{1}{24} + \frac{5}{8}\sqrt{3} + \frac{1}{8}\log\left(\frac{3}{2}\right) - \frac{1}{8}\log\left(\frac{3}{2} + \sqrt{3}\right)\right]$$

AN.094. *Denote:* $s = f(x) + f(y) + f(z)$

$$\frac{f^2(x)}{f(y) + f(z)} + \frac{f^2(y)}{f(z) + f(x)} + \frac{f^2(z)}{f(x) + f(y)} = s$$

$$f(x) = s - f(y) - f(z)$$

$$f^2(x) = s^2 - 2s\big(f(y) + f(z)\big) + \big(f(y) + f(z)\big)^2$$

$$\frac{f^2(x)}{f(y) + f(z)} = \frac{s^2}{f(y) + f(z)} - 2s + f(y) + f(z); (1)$$

Analogous:

$$\frac{f^2(y)}{f(z) + f(x)} = \frac{s^2}{f(z) + f(x)} - 2s + f(z) + f(x); (2)$$

$$\frac{f^2(z)}{f(x) + f(y)} = \frac{s^2}{f(x) + f(y)} - 2s + f(x) + f(y); (3)$$

By adding (1),(2) and (3), we get:

$$\frac{f^2(x)}{f(y) + f(z)} + \frac{f^2(y)}{f(z) + f(x)} + \frac{f^2(z)}{f(x) + f(y)} =$$

$$= s^2\left(\frac{1}{f(x) + f(y)} + \frac{1}{f(y) + f(z)} + \frac{1}{f(z) + f(x)}\right) - 4s$$

$$s = s^2\left(\frac{1}{f(x) + f(y)} + \frac{1}{f(y) + f(z)} + \frac{1}{f(z) + f(x)}\right) - 4s$$

$$5s = s^2\left(\frac{1}{f(x) + f(y)} + \frac{1}{f(y) + f(z)} + \frac{1}{f(z) + f(x)}\right)$$

$$\frac{s}{f(x) + f(y)} + \frac{s}{f(y) + f(z)} + \frac{s}{f(z) + f(x)} = 5$$

$$\frac{f(x) + f(y) + f(z)}{f(x) + f(y)} + \frac{f(x) + f(y) + f(z)}{f(y) + f(z)} + \frac{f(x) + f(y) + f(z)}{f(z) + f(x)} = 5$$

$$\frac{f(x)}{f(y) + f(z)} + \frac{f(y)}{f(z) + f(x)} + \frac{f(z)}{f(x) + f(y)} = 2$$

Hence,

$$\int_a^b\int_a^b\int_a^b\left(\frac{f(x)}{f(y)+f(z)}+\frac{f(y)}{f(z)+f(x)}+\frac{f(z)}{f(x)+f(y)}\right)dxdydz$$
$$=2(b-a)^3$$
$$3\left(\int_a^b f(x)\,dx\right)\left(\int_a^b\int_a^b\frac{dydz}{f(y)+f(z)}\right)=2(b-a)^3$$

Therefore,

$$\Omega(a,b)=\left(\int_a^b f(x)\,dx\right)\left(\int_a^b\int_a^b\frac{dydz}{f(y)+f(z)}\right)=\frac{2}{3}(b-a)^3$$

AN.095. *Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco*

$$The\ desired\ inequality\ is\ equivalent\ to:$$
$$\frac{x}{\sqrt{x+y}\left(4-\sqrt{y}\right)}+\frac{y}{\sqrt{x+y}\left(4-\sqrt{x}\right)}\le\frac{x+y}{\sqrt{x+y}\left(4-\sqrt{\frac{2xy}{x+y}}\right)}$$
$$Or\ \ \frac{x}{x+y}.\frac{1}{4-\sqrt{y}}+\frac{y}{x+y}.\frac{1}{4-\sqrt{x}}\le\frac{1}{4-\sqrt{\frac{2xy}{x+y}}}$$
$$Or\ \ \frac{x}{x+y}.f(y)+\frac{y}{x+y}.f(x)\le f\left(\frac{2xy}{x+y}\right)\ where\ f(x)=\frac{1}{4-\sqrt{x}},\ \ x\in(0,1).$$
$$We\ have:\ \ f'(x)=\frac{1}{2\sqrt{x}\left(4-\sqrt{x}\right)^2}\ \ and\ \ f''(x)=-\frac{4-3\sqrt{x}}{4\sqrt{x^3}\left(4-\sqrt{x}\right)^3}\le 0,\ \ \forall x\in(0,1).$$
$$So\ f\ is\ concave\ on\ (0,1)\ and\ by\ Jensen's\ inequality\ we\ get:$$
$$\frac{x}{x+y}.f(y)+\frac{y}{x+y}.f(x)\le f\left(\frac{x}{x+y}.y+\frac{y}{x+y}.x\right)=f\left(\frac{2xy}{x+y}\right)$$
$$So\ the\ proof\ is\ completed.\ Equality\ holds\ iff\ x=y.$$

AN.096. *Solution by Tapas Das-India*

Let $f(x)=e^{-x^2(1+x^2)}-1$, *then*

$f'(x)=-2x^3e^{-x^2}\le 0\Rightarrow f$ *−decreasing, then* $f(x)<f(0)\Rightarrow$

$$e^{-x^2}(1+x^2)-1\le 0\Rightarrow e^{-x^2}(1+x^2)\le 1\Rightarrow e^{-x^2}\le\frac{1}{x^2+1}$$

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx \leq \int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{1}{1+x^2}dx = tan^{-1} x|_{\sqrt{ab}}^{\frac{a+b}{2}} = tan^{-1}\left(\frac{a+b}{2}\right) - tan^{-1}(\sqrt{ab})$$

AN.097. *Solution by Ravi Prakash-New Delhi-India*

Let $b > a$:

$$\int_0^1 \frac{dx}{xa+(1-x)b} = \int_0^1 \frac{dx}{x(a-b)+b} = \frac{1}{a-b} log[(x(a-b)+b]\Big|_0^1 =$$

$$= \frac{1}{a-b}(log\, b - log\, a)$$

$$\int_0^\infty \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\int_0^\infty \left(\frac{1}{x+b} - \frac{1}{x+a}\right)dx = \frac{1}{a-b} log\left(\frac{x+b}{x+a}\right)\Big|_0^\infty =$$

$$= \frac{1}{a-b}\left(log\, 1 - log\left(\frac{b}{a}\right)\right) = \frac{log\, a - log\, b}{a-b}$$

Thus, we must to show that:

$$\frac{4}{a+b} \leq 2\cdot\frac{log\, a - log\, b}{a-b} \leq \frac{2}{\sqrt{ab}} \Leftrightarrow \frac{2}{a+b} \leq \frac{log\, a - log\, b}{a-b} \leq \frac{1}{\sqrt{ab}}; (1)$$

For $t \geq 0$, we have:

$$t^2 + 2t\sqrt{ab} + ab \leq t^2 + (a+b)t + ab$$

$$\leq t^2 + (a+b)t + ab + \left(\frac{a-b}{2}\right)^2 =$$

$$= t^2 + (a+b)t + \left(\frac{a+b}{2}\right)^2 \leq \left(t + \frac{a+b}{2}\right)^2$$

$$\frac{1}{\left(t+\frac{a+b}{2}\right)^2} \leq \frac{1}{(t+a)(t+b)} \leq \frac{1}{\left(t+\sqrt{ab}\right)^2}$$

$$\int_0^\infty \frac{dt}{\left(t+\frac{a+b}{2}\right)^2} \leq \int_0^\infty \frac{dt}{(t+a)(t+b)} \leq \int_0^\infty \frac{dt}{\left(t+\sqrt{ab}\right)^2}$$

$$-\frac{1}{t+\frac{a+b}{2}}\Bigg|_0^\infty \leq \frac{log\, a - log\, b}{a-b} \leq -\frac{1}{t+\sqrt{ab}}\Bigg|_0^\infty$$

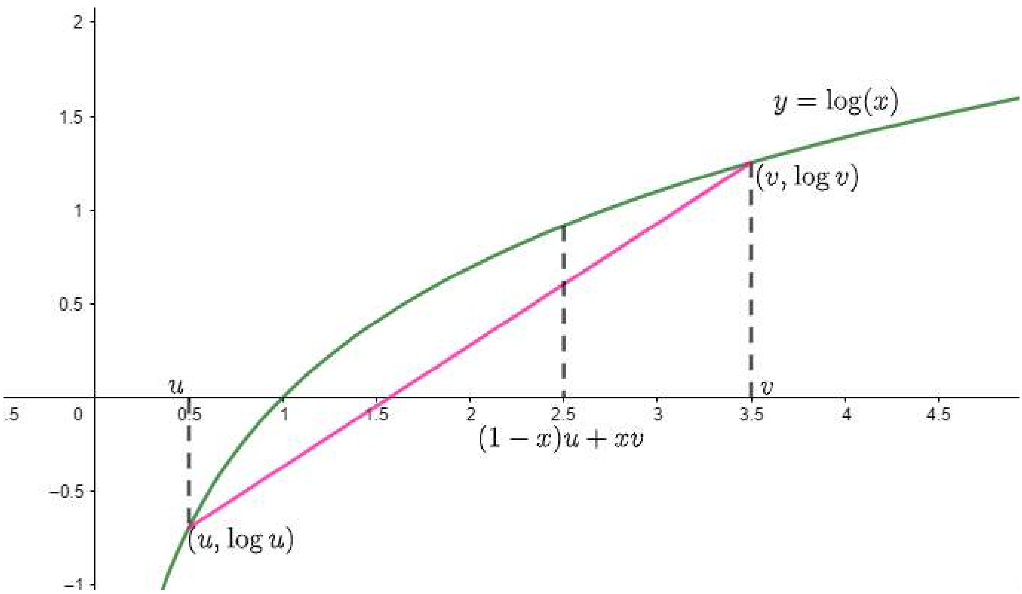
$$\frac{2}{a+b} \leq \frac{log\, a - log\, b}{a-b} \leq \frac{1}{\sqrt{ab}}$$

For $a = b$, *all the three expressions are equal to* $\frac{2}{a}$.

AN.098. *Solution by Ruxandra Daniela Tonilă-Romania*

$$\Omega = \int \cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)dx =$$
$$= \int \frac{e^{3x}+e^{-3x}}{2}\cdot\frac{e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}}{2}\cdot\frac{e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}}{2}dx =$$
$$= \frac{1}{8}\int (e^{3x}+e^{-3x})\left(e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}\right)\left(e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}\right)dx =$$
$$= \frac{1}{8}\int \left(e^{\frac{9x}{2}}+e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}+e^{-\frac{9x}{2}}\right)\left(e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}\right)dx =$$
$$= \frac{1}{8}\int (e^{11x}+e^{-2x}+e^{8x}+e^{-5x}+e^{5x}+e^{-8x}+e^{2x}+e^{-11x})\,dx =$$
$$= \frac{1}{8}\left(\frac{e^{11x}-e^{-11x}}{2}+\frac{e^{2x}-e^{-2x}}{2}+\frac{e^{5x}-e^{-5x}}{5}+\frac{e^{8x}-e^{-8x}}{8}\right)+C =$$
$$= \frac{\sinh(11x)}{44}+\frac{\sinh(2x)}{8}+\frac{\sinh(5x)}{20}+\frac{\sinh(8x)}{32}+C$$

AN.099. *Solution by Ravi Prakash-New Delhi-India*



Let $0 \le x \le 1$. *From the figure, it is clear that for* $u, v > 0, 0 \le x \le 1$:

$$(1-x)\ log\, u + x\ log\, v \le log[(1-x)u + xv]$$
$$u^{1-x}v^{x} \le (1-x)u + xv;\ \ (1)$$
Now, put in (1) $u = a + c, v = b + d$ *to obtain:*
$$(a+c)^{1-x}(b+d)^{x} \le (1-x)(a+c) + x(b+d);\ \ (2)$$
Now, put in (1), $u = a - c, v = b - d$ *to obtain:*
$$(a-c)^{1-x}(b-d)^{x} \le (1-x)(a-c) + x(b-d); (3)$$
Multiply (2) and (3) to obtain:
$$(a^2-c^2)^{1-x}(b^2-d^2)^{x} \le [(1-x)(a+c) + x(b+d)][(1-x)(a-c) + x(b-d)] =$$
$$= [(1-x)a + xb + (1-x)c + xd][(1-x)a + xb - (1-x)c - xd]$$
$$= [(1-x)a + xb]^2 - [(1-x)c + xd]^2$$
Therefore,
$$\left((1-x)a + xc\right)^2 - \left((1-x)b + xd\right)^2 \ge (a^2-b^2)^{1-x}(c^2-d^2)^{x}$$

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