MATH PHENOMENON A NEW DIMENSION

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PROBLEMS

ALGEBRA

A.001. If $a,b,c>0$ then:

$$\frac{(a^2+(a+b+c)^2)(b^2+(a+b+c)^2)(c^2+(a+b+c)^2)}{abc(a+b+c)^3}\geq\frac{1000}{27}$$

A.002. If $a,b,c>1$ then:

$$\log a\cdot\log b\cdot\log c\,(\log_a e+\log_b e+\log_c e)^2\geq 3\log(abc)$$

A.003. If $a,b,c>1$ then:

$$3\log a\cdot\log b\cdot\log c\,(\log_a e+\log_b e+\log_c e)\leq\log^2(abc)$$

A.004. Find $x,y,z\in[2,3]$ such that:

$$4xyz+25(x+y+z)=10(xy+yz+zx)+62$$

A.005. If $a,b,c>0; ab+bc+ca=2018$ then:

$$\frac{a}{a^2-bc+2019}+\frac{b}{b^2-ca+2019}+\frac{c}{c^2-ab+2019}>\frac{1}{a+b+c}$$

A.006. Solve for real numbers: $\begin{cases}4(xy+yz+zx)=3\\ 8xyz=1\end{cases}$

A.007. If $a,b,c>0; ab+bc+ca=3$ then:

$$\frac{(a^2+b^2)(ab+1)}{a+b}+\frac{(b^2+c^2)(bc+1)}{b+c}+\frac{(c^2+a^2)(ca+1)}{c+a}\geq 6$$

A.008. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(\prod_{cyc} \log(1 + 2\sin^2 x)\right)\left(\prod_{cyc} \log(1 + 2\cos^2 x)\right) \leq \log^6 2$$

A.009. If $a, b, c, d > 0; abcd = 1$ then:

$$a + b + c + d + \frac{1}{a + 8} + \frac{1}{b + 8} + \frac{1}{c + 8} + \frac{1}{d + 8} \geq \frac{40}{9}$$

A.010. If $a, b, c > 0; a^{a^2} b^{b^2} c^{c^2} = 1$ then:

$$(a^4 - 2a^2 + a) + (b^4 - 2b^2 + b) + (c^4 - 2c^2 + c) \geq 0$$

A.011. Solve for real numbers:

$$\frac{x^2 + x}{x^2 + x + 1} + \frac{y^2 + y}{y^2 + y + 1} + \frac{z^2 + z}{z^2 + z + 1} + 1 = 0$$

A.012. If $x, y, z \geq 2$ then:

$$\sum_{cyc} \frac{1}{x + 1} = 1 \Rightarrow \sum_{cyc} \frac{3x^2 + x + 4}{(x + 1)(x^4 + 2)} + 2 \leq 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

A.013. If $a, b, c > 0; abc = 4$ then:

$$\frac{2(a^5 + b^5) + c^5}{(a + b)^2} + \frac{2(b^5 + c^5) + a^5}{(b + c)^2} + \frac{2(c^5 + a^5) + b^5}{(c + a)^2} \geq 15$$

A.014. Solve for real numbers:

$$\begin{cases} x^4 + 2x^3 + 2y^4 = 0 \\ 3^x + 4^y = 5^z \end{cases}$$

A.015. Solve for real numbers:

$$\begin{cases} 0 \le x \le y \le z \le t \\ x + y + z + t = 4 \\ x \cdot \max(y) + y \cdot \max(z) + z \cdot \max(t) = \frac{22}{3} \\ x \cdot \max(z) + y \cdot \max(t) + z \cdot \max(x) = 7 \\ x \cdot \max(t) + y \cdot \max(x) + z \cdot \max(y) = \frac{19}{3} \end{cases}$$

A.016. If $a, b, c > 0; \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \frac{17}{6}$ then:

$$\sum_{cyc} \frac{abc + bc + 2a}{2bc + a + 1} > \frac{1}{3}$$

A.017. Solve for real numbers:

$$3^{\cos x + \cos y + \cos z} = 3^{\cos^2 x + \cos x} + 3^{\cos^2 y + \cos y} + 3^{\cos^2 z + \cos z}$$

A.018. If $x, y, z, t > 0$ then:

$$\frac{(xz - yt)^2 + (xz - yt)(xt + yz + yt) + (xt + yz + yt)^2}{xyzt} \ge 9$$

A.019. If $x, y, z > 0; xyz = 1$ then:

$$\left(x + y - \frac{1}{\sqrt{z}}\right)^2 + \left(y + z - \frac{1}{\sqrt{x}}\right)^2 + \left(z + x - \frac{1}{\sqrt{y}}\right)^2 \ge 3$$

A.020. If $a, b, c > 0; abc = 1$ then:

$$\left(\prod_{cyc} \frac{a + b}{\sqrt{a} + \sqrt{b}}\right)\left(\prod_{cyc} \frac{a + b}{\sqrt[4]{a} + \sqrt[4]{b}}\right)\left(\prod_{cyc} \frac{a + b}{\sqrt[8]{a} + \sqrt[8]{b}}\right) \ge 1$$

A.021. Solve for real numbers:

$$\frac{1}{1+|x|}+\frac{1}{1+|y|}+\frac{1}{1+|z|}+\frac{|x+y+z|}{1+|x+y+z|}=3$$

A.022. If $0 < a \le b \le c \le d$ **then:**

$$\frac{3\sqrt[3]{abc}-2\sqrt{ab}}{d} \le \frac{c}{4\sqrt[4]{abcd}-3\sqrt[3]{abc}}$$

A.023. If $f:(0,\infty)\to(0,\infty); \sqrt{f(a)f(b)}=f\left(\frac{a+b}{2}\right); (\forall)a,b>0$ **then:**

$$f(a)+f(b)+f(c)\ge 3f\left(\frac{a+b+c}{3}\right)$$

A.024. If $f:(0,\infty)\to(0,\infty); f(a)+f(b)=2f\left(\frac{a+b}{2}\right); (\forall)\ a,b>0$

then: $f(a)\cdot f(b)\cdot f(c)\le f^3\left(\frac{a+b+c}{3}\right); (\forall)a,b,c>0$

A.025. If $a,b,c>0; \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ **then:**

$$\sum_{cyc}\frac{a^3(b+1)+b^3(a+1)}{a+2ab+b}\ge 3abc$$

A.026. Solve for real numbers:

$$\sqrt[5]{x^2-5x+4}+\sqrt[5]{2+x-x^2}=\sqrt[5]{6-4x}$$

A.027. If $a,b,c>0; a+b+c=3$ **then:**

$$\sum_{cyc}\frac{a^3c(b+1)+b^3c(a+1)}{a^2b(b+1)+b^2a(a+1)}\ge 3$$

A.028. If $x, y, z > 0$ then:

$$\frac{x^4}{(2y+z)^2(2z+y)^2}+\frac{y^4}{(2z+x)^2(2x+z)^2}+\frac{z^4}{(2x+y)^2(2y+x)^2}\geq\frac{1}{27}$$

A.029. If $0 < a \leq b \leq c \leq d$ then:

$$\frac{\frac{9abc}{ab+bc+ca}-\frac{4ab}{a+b}}{d}\leq\frac{c}{\frac{16abcd}{bcd+cda+dab+abc}-\frac{9abc}{ab+bc+ca}}$$

A.030. If $x, y, z, t \geq 0$ then:

$$\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+\sqrt[3]{t}\leq\sqrt[3]{16(x+y+z+t)}$$

A.031. If $x, y, z \geq 1$ then:

$$\left(\sqrt[3]{\log x}+\sqrt[3]{\log y}\right)\left(\sqrt[3]{\log y}+\sqrt[3]{\log z}\right)\left(\sqrt[3]{\log z}+\sqrt[3]{\log x}\right)\leq 4\sqrt[3]{(\log xy)(\log yz)(\log zx)}$$

A.032. If $0 < a \leq b$ then: $\frac{b\left(a+\sqrt{ab}\right)^2}{\left(a+b+\sqrt{ab}\right)^3}\leq\frac{4}{27}$

A.033. If $A, B, C, D \in M_2(\mathbb{R})$; $AB=\begin{pmatrix}6&5\\8&7\end{pmatrix}$; $CD=\begin{pmatrix}7&8\\5&6\end{pmatrix}$ then:

$$\det(BA+2A^{-1}B^{-1})=\det(CD+2C^{-1}D^{-1})$$

A.034. Solve for real numbers:

$$\begin{cases} x+y+z=11 \\ \frac{yz+36x}{x(y-x)(z-x)}+\frac{zx+36y}{y(x-y)(z-y)}+\frac{xy+36z}{z(x-z)(y-z)}=1 \\ xyz=36 \end{cases}$$

A.035. If $X=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in M_2(\mathbb{C})$ **denote** $\overline{X}=\begin{pmatrix} \overline{p} & \overline{q} \\ \overline{r} & \overline{s} \end{pmatrix}\in M_2(\mathbb{C})$**. Prove that if:** $(\overline{X}_n)^T(\overline{X}_{n-1})^T\cdot\ldots\cdot(\overline{X}_2)^T\cdot(\overline{X}_1)^T\cdot X_1X_2\cdot\ldots\cdot X_n=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then: $a+b+c+d\geq 0; (\forall)X_1,X_2,\ldots,X_n\in M_2(\mathbb{C}); n\in\mathbb{N}; n\geq 2$

A.036. Solve for complex numbers:

$$3x^6-9x^5+18x^4-21x^3+15x^2-6x+1=0$$

A.037. Solve for real numbers:

$$\begin{cases} x,y,z,t>0 \\ [x]\cdot(x-[x])+y+t=x^2+2z; [*]-gif \\ \dfrac{6}{(y+z)^3+2}+\dfrac{6}{(z+t)^3+2}+\dfrac{6}{(t+y)^3+z}=\dfrac{1}{y}+\dfrac{1}{z}+\dfrac{1}{t} \end{cases}$$

A.038. If $a,b,c>0; ab+bc+ca=4abc(3-a-b-c)$ **then:**

$$\frac{a}{4a^2+2a+1}+\frac{b}{4b^2+2b+1}+\frac{c}{4c^2+2c+1}\leq\frac{1}{2}$$

When equality holds?

A.039. Solve for real numbers: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \sqrt{x} & \sqrt[3]{x} & \sqrt[4]{x} & 2 \\ x & \sqrt[3]{x^2} & \sqrt{x} & 4 \\ x^2 & x\sqrt[3]{x} & x & 16 \end{vmatrix}=0$

A.040. If $a,b,c>0; a+b+c+3=3\sqrt{3}$ **then:**

$$\frac{b(a^2+2)}{a+1}+\frac{c(b^2+2)}{b+1}+\frac{a(c^2+2)}{c+1}\geq 12\left(2-\sqrt{3}\right)$$

A.041. Solve for real numbers:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{x} & e^{2x} & e^{3x} & 2 \\ e^{3x} & e^{6x} & e^{9x} & 8 \\ e^{4x} & e^{8x} & e^{12x} & 16 \end{vmatrix} = 0$$

A.042. Solve for real numbers:

$$\begin{cases} xy + yz + zx = 26 \\ \dfrac{48 + yz(y+z)}{(x-y)(x-z)} + \dfrac{48 + zx(z+x)}{(y-x)(y-z)} + \dfrac{48 + xy(x+y)}{(z-x)(z-y)} = 9 \\ xyz = 24 \end{cases}$$

A.043. Solve for complex numbers:

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1 \\ \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{z} + \dfrac{y}{x} + \dfrac{z}{y} + 2 = 0 \end{cases}$$

A.044. Solve for real numbers:

$$32x^6 - 48x^4 + 36x^2 - 2 - \sqrt{3} = 0$$

A.045. If $a, b, c > 0$ then:

$$\frac{(3a + 2b + c + 6)(3b + 2c + a + 6)(3c + 2a + b + 6)}{(a+1)(b+1)(c+1)} \geq 216$$

A.046. If $a, b, c > 0$ then:

$$(a + 2c)\sqrt{a} + (b + 2a)\sqrt{b} + (c + 2b)\sqrt{c} \leq (a + b + c)\sqrt{3(a + b + c)}$$

A.047. If $a, b, c \geq 1$ then: $a^{\sqrt{a}} \cdot b^{\sqrt{b}} \cdot c^{\sqrt{c}} \geq a^{\sqrt[4]{bc}} \cdot b^{\sqrt[4]{ca}} \cdot c^{\sqrt[4]{ab}}$

A.048. If $x, y, z \geq 0$; $\{x\}^9 + \{y\}^9 + \{z\}^9 = \frac{1}{64}$ then:

$$x^7[x] \cdot \{x\} + y^7 \cdot [y] \cdot \{y\} + z^7 \cdot [z] \cdot \{z\} < 64([x]^9 + [y]^9 + [z]^9) + 1$$

$\{x\} = [x] - x$; $[*]$ **- great integer function**

A.049. Find $a, b, c, d \in \mathbb{R}$ such that:

$$\begin{cases} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^3 + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{cases}$$

A.050. Solve for real numbers:

$$\begin{cases} \sqrt{x^2 + y^2} + \sqrt{(x-4)^2 + y^2} + \sqrt{x^2 + (y-3)^2} + \sqrt{(x-4)^2 + (y-3)^2} = 10 \\ x + 2y = 5z \end{cases}$$

A.051. If $n \in \mathbb{N} - \{0\}$ then:

$$\tan^{-1} 1 + 3\tan^{-1} 3 \ldots + (4n-3)\tan^{-1}(4n-3) \geq (2n-1)^2 \tan^{-1}(2n-1)$$

A.052. If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{\left(\frac{a+b}{2}\right)^5 - \left(\sqrt{ab}\right)^5 + \left(\frac{2ab}{a+b}\right)^5}{\left(\frac{a+b}{2} - \sqrt{ab} + \frac{2ab}{a+b}\right)^5} \geq 3$$

A.053. If $x, y, z > 0$, $\frac{x}{(z+1)^2} + \frac{y}{(x+1)^2} + \frac{z}{(y+1)^2} = \frac{3}{4}$ then:

$$\frac{y}{x^2 - x + 1} + \frac{z}{y^2 - y + 1} + \frac{x}{z^2 - z + 1} \leq 3$$

A.054. If $x, y, z > 0$, different in pairs, then:

$$3+\Omega(x,y)+\Omega(y,z)+\Omega(z,x)>\log\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)$$

$$\Omega(x,y)=\sum_{k=1}^{\infty}\left(\frac{1}{2k}\left(\frac{x^2-2xy+y^2}{x^2+2xy+y^2}\right)^k\right)$$

A.055. If $a, b > 0$ then:

$$\frac{2(5a^2+5b^2+6ab)(5a^2+5b^2+8ab)}{(7a^2+7b^2+10ab)^2}\le\left(\frac{a^2+b^2}{2ab}\right)^2$$

A.056. Solve for $x, y, z, t > 0$:

$$\begin{cases} xt=4e \\ \dfrac{x^2+y^2}{x+y}+\dfrac{y^2+z^2}{y+z}+\dfrac{z^2+x^2}{z+x}=x+y+z \\ t^{\log y}=4 \end{cases}$$

A.057. If $a, b, x, y, z > 0$ then:

$$4ab\left(\frac{x}{a}+\frac{y}{\sqrt{ab}}+\frac{z}{b}\right)\le\frac{(x+y+z)^2(a+b)^2}{ax+\sqrt{ab}y+bz}$$

A.058. If $z \in \mathbb{C}-\{0\}$ then:

$$|z-1|^4+\left|z+\frac{1-i\sqrt{3}}{2}\right|^4+\left|z+\frac{1+i\sqrt{3}}{2}\right|^4\ge3(1+2|z|^2+|z|^4)^2$$

A.059. If $a, b, c, d > 0$, $[*]$ – great integer function, then:

$$\frac{2ab}{a+b}+\frac{c+d}{2}+\frac{a+b}{2ab}+\frac{2}{c+d}<\left[\frac{2ab}{a+b}\right]+\left[\frac{c+d}{2}\right]+\left[\frac{a+b}{2ab}\right]+\left[\frac{2}{c+d}\right]+3$$

A.060. If $a, b, c, d, e \geq 0$ then:

$$\left(4\sqrt[4]{abcd} - 3\sqrt[3]{abc}\right)\left(5\sqrt[5]{abcde} - 4\sqrt[4]{abcd}\right) \leq de$$

A.061. If $a, b, c > 0$ *then*:

$$\left(\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2}\right)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)^2 \geq \left(\frac{a}{b^2} + \frac{b}{c^2} + \frac{c}{a^2}\right)^3$$

A.062. If $a, b, c \geq 1, a, b, c \in \mathbb{N}$ then:

$$\sum_{cyc}\left((a+1)\left(\binom{2b}{b} + \binom{2c}{c}\right)\right) \geq 2\left(4^{\sqrt{ab}} + 4^{\sqrt{bc}} + 4^{\sqrt{ca}}\right)$$

A.063. Solve for real numbers:

$$\begin{cases} x^3 + \log_2 x + \log_4 y = 67 \\ x^{\sqrt{xy}} \cdot y^{\frac{1}{\sqrt{xy}}} + x^{\frac{1}{\sqrt{xy}}} \cdot y^{\sqrt{xy}} = \sqrt{x^{x+y}} \cdot y^{\frac{2}{x+y}} + \sqrt{y^{x+y}} \cdot x^{\frac{2}{x+y}} \end{cases}$$

A.064. Solve for real numbers: $\begin{cases} x, y, z > 0 \\ x + y + z + xyz \geq 4 \\ \sqrt{x} + \sqrt{y} + \sqrt{z} = x + y + z \end{cases}$

A.065. In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\frac{a|b-c|}{b+c} + \frac{b|c-a|}{c+a} - \frac{c|a-b|}{a+b}\right) \leq \frac{8abc|(a-b)(b-c)(c-a)|}{(a+b)(b+c)(c+a)}$$

A.066. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ \dfrac{3x+3y}{y+2z} + \dfrac{3y+3z}{x+2z} + \dfrac{3x+9z}{x+y+z} = 8 \\ x^x + y^y + z^z = 3 \end{cases}$$

A.067. If $a,b,c\geq 0$

$$27abc\leq\left(\sum_{cyc}\sqrt{ab}\right)\left(\sum_{cyc}\sqrt[3]{a^2b}\right)\left(\sum_{cyc}\sqrt[4]{a^3b}\right)\leq(a+b+c)^3$$

A.068. Solve for real numbers:

$$\begin{vmatrix}1 & 1 & 1 & 1\\ \log x & \log(ex) & \log(e^2x) & \log(e^3x)\\ \log^2 x & \log^2(ex) & \log^2(e^2x) & \log^2(e^3x)\\ \log^3 x & \log^3(ex) & \log^3(e^2x) & \log^3(e^3x)\end{vmatrix}=7+2^{x-10}+\log_{12}x$$

A.069. $If\ x,y\in\mathbb{R}, x^2+y^2-6x-8y+24\leq 0\ then$:

$$16\leq x^2+y^2\leq 36$$

A.070. $If\ x\epsilon\mathbb{R}\ then$:

$$log^2(1+sin^2x)+log^2(1+log^2x)+log((1+sin^2x)(1+cos^2x))^2<2$$

A.071. If $a,b,c\geq 1, a,b,c\in\mathbb{N}$ then:

$$\sum_{cyc}\left(\frac{1}{\binom{2a}{a}^2}\cdot\sum_{k=0}^{a}\binom{a}{k}^3\right)\geq\frac{9}{2^a+2^b+2^c}$$

A.072. $If\ a,b,c,d>0, a+b+c=3\ then$:

$$(a+b+c+d)^4+4\geq c(a+b)^2+256d$$

A.073. If $a,b>0$ then:

$$256\sqrt{\frac{a^2+b^2}{2}}\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)^3 \le 27\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)^4$$

A.074. If $a,b,c,d \ge 0$ then:

$$27(a+b+c+d)^4+(4-256d)(a+b+c)^3 \ge 27c(a+b)^2$$

A.075. Solve for $x,y,z>0$:

$$\begin{cases} x-y+z=\dfrac{1}{2} \\ 3\left(\dfrac{1}{1+2x+4xy}+\dfrac{1}{1+2y+4yz}+\dfrac{1}{1+2z+4zx}\right)=2(x+y+z) \\ 8xyz=1 \end{cases}$$

A.076. In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \le \frac{1}{(a+b)(b+c)(c+a)}$$

A.077. If $0<a\le b$ then:

$$\left(\frac{2a+b}{3}+\frac{3a+b}{4}+\frac{4a+b}{5}\right)\left(\frac{3}{2a+b}+\frac{4}{3a+b}+\frac{5}{4a+b}\right) \le \frac{2a}{b}+\frac{2b}{a}+5$$

A.078. Solve for $x,y,z>0$:

$$\begin{cases} 4(xy+yz+zx)=3 \\ 3\left(\dfrac{1}{1+2x+4xy}+\dfrac{1}{1+2y+4yz}+\dfrac{1}{1+2z+4zx}\right)=2(x+y+z) \\ 8xyz=1 \end{cases}$$

A.079. If $a,b,c>1$ then:

$$\sum_{cyc} a^{\frac{3}{a}}\left(b^{\frac{1}{b}}+c^{\frac{1}{c}}-a^{\frac{1}{a}}\right) \le a^{\frac{1}{a}}\cdot b^{\frac{1}{b}}\cdot c^{\frac{1}{c}}\left(a^{\frac{1}{a}}+b^{\frac{1}{b}}+c^{\frac{1}{c}}\right)$$

A.080. If $x, y, z > 0$ then:

$$\frac{y(8x+5)}{48x^3+1}+\frac{z(8y+5)}{48y^3+1}+\frac{x(8z+5)}{48z^3+1}\le\sqrt{(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)}$$

A.081.

$$A,B\in M_4(\mathbb{C}),B^3=I_4,A^3=AB^2+BA^2,$$

$$C=\begin{pmatrix}28 & 18 & 36 & 723\\ 120 & 121 & 45 & 891\\ 330 & 27 & 151 & 210\\ 450 & 150 & 180 & 181\end{pmatrix}$$

Prove that: $det\left((CA-CB)(A^2-B^2)\right)\neq 0$

A.082. If $a, b, c > 0$ then:

$$\frac{(a^2-ab+b^2)^6}{(a+b)^{12}}+\frac{(b^2-bc+c^2)^6}{(b+c)^{12}}+\frac{(c^2-ca+a^2)^6}{(c+a)^{12}}\ge\frac{3}{4096}$$

A.083. If $x, y, z \ge 0$ then:

$$\sum_{cyc}\frac{(x+1)(y+1)}{(x+2)(y+2)}=\frac{3}{4}\Rightarrow\sum_{cyc}\sqrt{(x+1)(y+1)}=3$$

A.084. If $0 < a \le b < \pi$ then:

$$log\left(\frac{sinb}{sina}\right)\ge\left(1+\sqrt{\frac{a}{b}}\right)log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

A.085. If $a, b, c > 0, abc = 1$ then:

$$\frac{(a+b)\sqrt{c}}{2}\left(1-\frac{(a+b)\sqrt{c}}{2}\right)\le\frac{2}{(a+b)\sqrt{c}}\left(\frac{2}{(a+b)\sqrt{c}}-1\right)$$

A.086. If $a, b > 0$ then:

$$\left(\frac{a+b}{2}+\sqrt{ab}+\frac{2ab}{a+b}\right)^4 \geq \frac{(a+b)^4}{16}+15a^2b^2+64\left(\frac{2ab}{a+b}\right)^4$$

A.087. Solve for real numbers:

$$\begin{cases} 6x+3y+2z=18 \\ 108(x+y+z)^{x+y+z}=xy^2z^3\cdot 6^{x+y+z} \end{cases}$$

A.088. If $a, b, c > 0$ then:

$$\frac{\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{1}{ab}\right)}{\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)} \geq \frac{\left(\sum_{cyc}\frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc}\frac{1}{\sqrt[3]{a^2}}\right)}{\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc}\frac{1}{a^2b^2}\right)}$$

A.089. If $A, B \in M_n(\mathbb{R}), n \in \mathbb{N}, n \geq 2, A+B=AB, det(AB) \neq 0$ then:

$$det\left((I_n-A^3-B^3+(AB)^3)(I_n-A^5-B^5+(AB)^5)(I_n-A^7-B^7+(AB)^7)\right) \geq 0$$

A.090. If $0 < x, y, z$ then: $x(y^2[x]+z^2\{x\}) \geq (y[x]+z\{x\})^2$,

$\{x\} = x-[x], [*]$ **−great integer function.**

A.091. If $a, b, c, d, e, f > 0, a+b+c=3, d+e+f=9$ then:

$$\frac{a^a\cdot b^b\cdot c^c\cdot d^d\cdot e^e\cdot f^f}{3(a+d)^{a+d}(b+e)^{b+e}(c+f)^{c+f}} \geq \left(\frac{3}{8}\right)^8$$

A.092. If $x, y, z > 0$; $\sqrt{x}+\sqrt{y}+\sqrt{z}=3$ then:

$$x+y+z \geq \sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}$$

A.093. If $a, b, c > 0; abc = 1$ then:

$$\sum_{cyc} \frac{c(a^2+b^2)+1}{a+b} \geq \frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

A.094. If $x, y, z > 0$ then:

$$\frac{(1+\sqrt{x})(1+\sqrt[3]{y})(1+\sqrt[6]{z})}{\sqrt{1+x}\cdot\sqrt[3]{1+y}\cdot\sqrt[6]{1+z}} \geq 4$$

A.095. If $a, b, c, d > 0, a+b+c+d = 12$ then:

$$\frac{ab}{12(a+b)} + \frac{abc}{8(ab+bc+ca)} + \frac{abcd}{6(abc+bcd+cda+dab)} < 1$$

A.096. Solve for real numbers:

$$\begin{cases} 0 \leq x, y, z \leq 2 \\ \dfrac{x}{y+z+1} + \dfrac{y}{z+x+1} + \dfrac{z}{x+y+1} + xye^z = \dfrac{6}{5} + 4e^2 \end{cases}$$

A.097. If $x, y \in \mathbb{R}, |x| < \frac{1}{2}, |y| < \frac{1}{2}$ then:

$$\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2} + \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2} + \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2} + \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2} > \frac{8}{\sqrt[4]{5}}$$

A.098. Solve for complex numbers:

$$3x^6 - 9x^5 + 18x^4 - 21x^3 + 15x^2 - 6x + 1 = 0$$

A.099. If $a > 2, 0 \leq x \leq y \leq z, x+y+z = 3$ then:

$$(x-1)logalog(a-1) + (y-1)log(a-1)log(a+1)$$

$$+(z-1)logalog(a+1) \geq 0$$

A.100. If $x, y \geq 0, 0 \leq z \leq 1$ then:

$$4\sqrt{xy} \leq 2(1-z)\left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy} + z\left(\sqrt{x}+\sqrt{y}\right)^2 \leq 2(x+y)$$

A.101. If $a, b > 0$ then:

$$\left(\frac{2\sqrt{ab}}{a+b}+\frac{a+b}{2\sqrt{ab}}\right)\left(\sqrt{\frac{a^2+b^2}{2ab}}+\sqrt{\frac{2ab}{a^2+b^2}}\right)\left(\frac{\sqrt{2(a^2+b^2)}}{a+b}+\frac{a+b}{\sqrt{2(a^2+b^2)}}\right) \leq \left(\frac{a}{b}+\frac{b}{a}\right)^3$$

A.102. ***If*** $a, b, c > 0$ ***then:***

$$\frac{(3a+2b+c+6)(3b+2c+a+6)(3c+2a+b+6)}{(a+1)(b+1)(c+1)} \geq 216$$

A.103. If $a, b, c > 0$ prove:

$$\frac{(a^2+a+1)^{\sqrt{3}}(b^2+b+1)^{\sqrt{3}}(a^2+a+1)^{\sqrt{3}}(c^2+c+1)^{\sqrt{3}}}{e^{2a}\cdot e^{2b}\cdot e^{2c}} \leq 1$$

A.104. Solve for real numbers:

$$\begin{cases} x, y \geq 0; [*] - great\ integer\ function \\ (x+2)(y+3) = 8 \\ \sqrt{[x]\cdot[y]} + \sqrt{(x-[x])(y-[y])} = \sqrt{xy} \end{cases}$$

A.105. If $a, b, c > 0, a^2b^2 + b^2c^2 + c^2a^2 = 12abc$ then:

$$\sqrt[3]{\frac{a}{4a+bc}} + \sqrt[3]{\frac{b}{4b+ca}} + \sqrt[3]{\frac{c}{4c+ab}} \geq \frac{3}{2}$$

A.106. Solve for real numbers:

$$5^{2x+1} + 20x^2 + 29x + 6 = 11\cdot 5^x + x\cdot 5^{x+2}$$

A.107. If $a, b, c > 0$ then:

$$\left(\sum_{cyc} \sqrt[4]{a^3}\right)^4 \left(\sum_{cyc} \sqrt[5]{a^4}\right)^5 \left(\sum_{cyc} \sqrt[6]{a^5}\right)^6 \leq 27\left(\sum_{cyc} a\right)^{12}$$

A.108. If $x, y, z, u, v, w > 0, uv + vw + wu = 3$ then:

$$\sum_{cyc} \frac{(x^2 + y^2 + z^2 + 2xy + 2zy)u^2}{xz} \geq 18 + u^2 + v^2 + w^2$$

A.109. If $x, y, z \geq 1$ then:

$$\left(\prod_{cyc}(x+1) + 8xyz\right)\prod_{cyc}(x+3) \geq 16\prod_{cyc}(3x+1)$$

A.110. If $a, b, c > 0, a + b + c = 3,$ then:

$$\left(\sum_{cyc} \sqrt{a+2b}\right)^2 \geq 9 + 6 \cdot \sqrt[3]{\prod_{cyc}(3 + a - b)}$$

A.111. If $0 < a \leq b, 0 \leq c \leq 1$ then:

$$\begin{cases} 2c(a+b)\sqrt{ab} + (1-c)(a+b)^2 \geq 4ab \\ 2c(a+b)\sqrt{ab} + (1-c)(a+b)^2 \leq (a+b)\sqrt{2(a^2+b^2)} \end{cases}$$

A.112. Solve for complex numbers:

$$\begin{cases} \dfrac{x^7}{y^{30}} + \dfrac{y^7}{z^{30}} + \dfrac{z^7}{x^{30}} = \dfrac{(x+y+z)^7}{(x^5+y^5+z^5)^6} \\ x^4 - 3y^3 - 2z^2 - 3y + 1 = 0 \end{cases}$$

A.113. If $\mathbf{0} \leq \mathbf{x}, \mathbf{y}, \mathbf{z} \leq \mathbf{1}, \boldsymbol{\alpha} > 0$ then:

$$2\sum_{cyc}\left(\sqrt{(1-x^2)(1-y^2)}-xy\right) < 3\alpha^2+6-4\alpha(x+y+z)$$

A.114. If $x, y, z, t > 0$ then:

$$\frac{(xz-yt)^2+(xz-yt)(xt+yz+yt)+(xt+yz+yt)^2}{xyzt} \geq 9$$

A.115. Solve for real numbers:

$$\begin{cases} x+y+z+u=4 \\ x^2(x^2-v^2)+y^2(y^2-v^2)+v^4=z^2(v^2-z^2)+u^2(v^2-u^2) \end{cases}$$

A.116. If $a, b, c > 0, abc = 1$ then:

$$\prod_{cyc}\frac{a+b}{\sqrt{a}+\sqrt{b}}\cdot\prod_{cyc}\frac{a+b}{\sqrt[4]{a}+\sqrt[4]{b}}\cdot\prod_{cyc}\frac{a+b}{\sqrt[8]{a}+\sqrt[8]{b}} \geq 1$$

A.117. Solve for real number:

$$32x^6-48x^4+36x^2-2-\sqrt{3}=0$$

A.118. If $a, b, c, d, e > 0, a+b+c+d+e = 41$ then:

$$\frac{a^2}{\sqrt[3]{4}}+\frac{b^2+c^2+d^2}{\sqrt[3]{2}}+e^2 \geq 41\left(8\sqrt[3]{4}-\sqrt[3]{2}-5\right)$$

When equality holds?

A.119. Solve for real numbers:

$$\begin{cases} x \geq 0, y, z, t > 0, [*] - great\ integer\ function \\ [x](x-[x])+y+t=x^2+2z \\ \dfrac{1}{y}+\dfrac{1}{z}+\dfrac{1}{t}=\dfrac{6}{(y+z)^3+2}+\dfrac{6}{(z+t)^3+2}+\dfrac{6}{(t+y)^3+2} \end{cases}$$

A.120. $\boldsymbol{If\ a,b,c>0, ab+bc+ca=4abc(3-a-b-c)\ then}$:

$$\frac{a}{4a^2+2a+1}+\frac{b}{4b^2+2b+1}+\frac{c}{4c^2+2c+1}\le\frac{1}{2}$$

When equality holds?

A.121. Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \dfrac{x}{x+1}+\dfrac{y}{(x+1)(y+1)}+\dfrac{z}{(x+1)(y+1)(z+1)}+\dfrac{1}{8\sqrt{xyz}}=1 \end{cases}$$

A.122. If $a,b\ge 0$ then:

$$\sqrt{ab}+\sqrt[7]{\left(\frac{2ab}{a+b}\right)^7-\left(\sqrt{ab}\right)^7+\left(\frac{a+b}{2}\right)^7}\ge\frac{2ab}{a+b}+\frac{a+b}{2}$$

A.123. If $x,y,z\ge 0$ then:

$$\left(\prod_{cyc}(x+1)+\prod_{cyc}(2x+1)\right)\prod_{cyc}(x+2)\ge 2\prod_{cyc}(3x+2)$$

A.124. If $x,y,z>0$ then:

$$\frac{1}{\sqrt{(x+y)(y+z)}}+\frac{1}{\sqrt{(y+z)(z+x)}}+\frac{1}{\sqrt{(z+x)(x+y)}}\le\frac{3}{2}\sqrt{\frac{3}{xy+yz+zx}}$$

A.125.

$$A=\left\{x\ \middle|\ x\in\mathbb{Z},\left[\frac{x^7-15x^5+49x^3-36x}{56}\right]=0,[*]-GIF\right\}$$

Find:

$$\Omega = \sum_{x \in A} x$$

A.126. If $0 < a \le b \le \frac{\sqrt{3}}{3}$ then:

$$a^2b^2(2-a-b)^2(2+a+b)^2 \le (1-a^2)(1-b^2)(a+b)^4$$

A.127. Solve for real numbers:

$$\begin{cases} 0 < x, y < \frac{\pi}{2}, x+y = \frac{5\pi}{6} \\ 4sin^2xsin^2y(sin^2x + sin^2y) + 4cos^2xcos^2y(cos^2x + cos^2y) = sin^2 2x + sin^2 2y \end{cases}$$

A.128. Find all roots:

$$64x^5(x-1) + 32x^2(x^2+x+1) - 64x + 19 = 0$$

A.129.

$$A = \left\{ x \mid x \in \mathbb{R}, \sqrt[7]{2+x} + \sqrt[7]{5-x} = \sqrt[7]{7} \right\}$$

$$B = \left\{ x \mid x \in \mathbb{R}, \sqrt[9]{3+x} + \sqrt[9]{6-x} = \sqrt[9]{9} \right\}$$

Find the sets Ω_1, Ω_2 such that:

$$A\Delta\Omega_1 = B \quad \Omega_2\Delta B = A, \quad \left(X\Delta Y = (X/Y) \cup (Y/X)\right)$$

A.130. If $x, y, z \in \mathbb{R}$ then:

$$\frac{(x^{12}+x^6+1)(y^{24}+y^{12}+1)(z^{36}+z^{18}+1)}{(x^8+1)(y^{16}+1)(z^{24}+1)} > x^2y^4z^6$$

A.131. Find $x, y, z > 0$ such that:

$$\frac{(1+x^2)(1+y^2)}{(1+x)(1+y)} + \frac{(1+y^2)(1+z^2)}{(1+y)(1+z)} + \frac{(1+z^2)(1+x^2)}{(1+z)(1+x)} + 24\sqrt{2} = 36$$

A.132. Solve for real numbers:

$$\begin{cases} x^4+2y^3-6z^2+1=0, x,y,z>0 \\ \frac{1}{42x+43(y+z)}+\frac{1}{42y+43(z+x)}+\frac{1}{42z+43(x+y)}=\frac{1}{128}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \end{cases}$$

A.133. $\begin{cases} f,g,h:\mathbb{R}\to\mathbb{R} \\ f(x)+g(x)+h(x)=3x+3, \forall x\in\mathbb{R} \\ f^2(x)+g^2(x)+h^2(x)=3x^2+6x+5, \forall x\in\mathbb{R} \\ f^3(x)+g^3(x)+h^3(x)=3x^3+9x^2+15x+9, \forall x\in\mathbb{R} \end{cases}$

Solve for real numbers: $f(x)\cdot g(x)\cdot h(x)=0$

A.134. If $a,b\geq 0$ **then:**

$$\frac{(a+b)^3}{8}+\frac{8a^3b^3}{(a+b)^3}\geq ab\sqrt{ab}+\left(\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}+\frac{2ab}{a+b}\right)^3$$

A.135. If $a,b,c>0, \sqrt{ab}+\sqrt{bc}+\sqrt{ca}=12$ **then:**

$$\frac{\left(a+b+\sqrt{ab}\right)^3}{(a+b)^2}+\frac{\left(b+c+\sqrt{bc}\right)^3}{(b+c)^2}+\frac{\left(c+a+\sqrt{ca}\right)^3}{(c+a)^2}\geq 81$$

A.136. If $a,b,c>0$ **then:**

$$\sum_{cyc}\frac{c+\sqrt{ab}}{\sqrt{ab}(a+b+2c)}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

A.137. If $m,n\in\mathbb{N}-\{0\}, F_n$ **−Fibonacci numbers,** L_n **−Lucas numbers then:**

$$\sqrt[5]{\frac{F_m^2F_n^3L_n^2L_m^3}{F_{m+n}^5}}+\sqrt[5]{\frac{F_m^3F_n^2L_n^3L_m^2}{F_{m+n}^5}}<2$$

A.138. If $a,b \geq 0, n \in \mathbb{N}, n \geq 2, A \in M_n(\mathbb{R}), A^2 = O_n$ then:

$$det\left(\sqrt{3}(a+b)A + (a^2+ab+b^2)I_n\right) \geq 0$$

A.139. If $m,n,p \in \mathbb{N}$ then:

$$3\sqrt{3}\left(\frac{m^3}{(m+3)!} + \frac{n^5}{(n+5)!} + \frac{p^7}{(p+7)!}\right) < \sqrt{(m!)^2 + (n!)^2 + (p!)^2}$$

A.140. In ΔABC the following relationship holds $(\forall z \in \mathbb{C})$:

$$|z - cosA - isinA| + |z - cosB - isinB| + |z - cosC - isinC| \geq 3(|z| - 1)$$

A.141. Solve for real numbers:

$$\begin{cases} 5\left(\sqrt[5]{x} + \sqrt[5]{y}\right) - 2\left(\sqrt{x} + \sqrt{y}\right) = 6 \\ xz(z^4 + 10z^2 + 5) = y(5z^4 + 10z^2 + 1) \end{cases}$$

A.142. If $a,b,c,d > 1, abcd = e^4$ then:

$$\frac{log\left(\frac{e^2}{a}\right) \cdot log\left(\frac{e^2}{b}\right) \cdot log\left(\frac{e^2}{c}\right) \cdot log\left(\frac{e^2}{d}\right)}{log(ab) \cdot log(bc) \cdot log(cd) \cdot log(da)} \leq \frac{1}{16}$$

A.143. Solve for real numbers:

$$\frac{3}{\sqrt[3]{1+x}} + \frac{x}{\sqrt[3]{1+x^3}} = 2\sqrt[3]{4}$$

A.144. Solve for real numbers:

$$\begin{cases} xy(4xy-1)^2 + 16xy = 16z^2 \\ yz(4yz-1)^2 + 16yz = 16x^2 \\ zx(4zx-1)^2 + 16zx = 16y^2 \end{cases}$$

A.145. If $x, y, z > 0, 3(xy + yz + zx) = 1$ then:

$$27\sum_{cyc} x^3y + 36\sum_{cyc} x^2y + 6\sum_{cyc} x \geq 11$$

A.146. If $a, b, c > 0, abc = 1$ then:

$$\sum_{cyc} \frac{(a^{10} + b^{10})(a^9 + b^9)}{(a^4 + b^4)(a^3 + b^3)} \geq 3$$

A.147. If $a, b, c, d \in \mathbb{R}$ then:

$$4(ad - bc)^6 + 4(ac + bd)^6 \geq (a^2 + b^2)^3(c^2 + d^2)^3$$

A.148. Solve for natural numbers:

$$\begin{cases} \left(\frac{x+y}{2}\right)^{\frac{1}{xy}} \cdot \left(\frac{z+x}{2}\right)^{\frac{1}{xy}} = x^{\frac{1}{x(x+y)}} \cdot y^{\frac{1}{y(y+z)}} \\ \left(\frac{y+z}{2}\right)^{\frac{1}{yz}} \cdot \left(\frac{x+y}{2}\right)^{\frac{1}{xy}} = y^{\frac{1}{y(y+z)}} \cdot z^{\frac{1}{z(z+x)}} \\ \left(\frac{z+x}{2}\right)^{\frac{1}{xy}} \cdot \left(\frac{y+z}{2}\right)^{\frac{1}{yz}} = z^{\frac{1}{x(z+x)}} \cdot x^{\frac{1}{x(x+y)}} \\ (x+1)(y+1)(z+1) = 336 \end{cases}$$

A.149. If $a, b, c, d \in \mathbb{R}$ then:

$$(ad - bc)^8(a^2 + b^2)(c^2 + d^2) + (ac + bd)^{10} \leq (a^2 + b^2)^5(c^2 + d^2)^5$$

A.150. Solve for real numbers:

$$\begin{cases} [x] \cdot \{x\} + 1 = y, \quad [*] - GIF \\ \sqrt{\frac{xyz}{x^2 - xy + y^2}} + \sqrt{\frac{xyz}{y^2 - yz + z^2}} + \sqrt{\frac{xyz}{z^2 - zx + x^2}} = \sqrt{x} + \sqrt{y} + \sqrt{z} \\ [y] \cdot \{y\} + 1 = z, \{*\} = * - [*] \end{cases}$$

A.151. Solve for natural numbers:

$$\frac{(x-2)!!\,(x-3)!!}{(x-4)!!\,(x-5)!!}+\frac{(x-3)!!\,(x-4)!!}{(x-5)!!\,(x-6)!!}+\frac{(x-4)!!\,(x-5)!!}{(x-6)!!\,(x-7)!!}=38$$

A.152. If $x,y,z\geq 0, x^3+y^3+z^3=24$ then:

$$\left(x^5y+y^5z+z^5x\right)(x^2y+y^2z+z^2x)\geq 576xyz$$

A.153. Solve for real numbers:

$$\begin{cases} 0\leq x,y,z\leq 1 \\ (x^2+1)(y^2+1)(z^2+1)=8+(x^2-1)(y^2-1)(z^2-1) \end{cases}$$

A.154. If $a,b>0$ then:

$$\sqrt[3]{\frac{a^3+b^3}{2}}\cdot\sqrt[4]{\frac{a^4+b^4}{2}}\cdot\sqrt[5]{\frac{a^5+b^5}{2}}\leq\frac{a^5+b^5}{a^2+b^2}$$

A.155. If $z\in\mathbb{C}, t\in[0,2\pi), |z|=1$ then:

$$\left|z^2+z-cost+isint\right|+\left|cost+isint-z^2+z\right|$$
$$+\left|cost+isint+z^2-z\right|\leq 6$$

A.156. If $m,n,p,q\in\mathbb{N}; m,n,p,q\geq 4$ then:

$$4^n(4^n+1)+4^m(4^m+1)+4^p(4^p+1)+4^q(4^q+1)$$
$$\geq 4mnpq(mnpq+1)$$

A.157. If $a,b,c>0, \frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}=\frac{3}{4}$ then:

$$16\sum_{cyc}\frac{\sqrt{ab}}{a+b}+\sum_{cyc}\frac{(a+b)^2}{ab}\geq 12+4\sum_{cyc}\frac{a+b}{\sqrt{ab}}$$

A.158. Solve for real numbers:

$$\begin{cases} x,y,z,t>0 \\ xyz+yzt+ztx+txy=1 \\ \dfrac{x^6}{yzt}+\dfrac{y^6}{ztx}+\dfrac{z^6}{txy}+\dfrac{t^6}{xyz}=1 \end{cases}$$

A.159. If $x,y,z\in\mathbb{R}, a,b,c>0$ and

$$\Omega_1=((x-a)^2+y^2+z^2)^2+(x^2+(y-b)^2+z^2)^2 \\ +(x^2+y^2+(z-c)^2)^2$$

$$\Omega_2=\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{2(a^2+b^2+c^2)}$$

Prove that: $\Omega_1\ge\Omega_2$. When equality holds?

A.160. Solve for real numbers:

$$\begin{cases} x,y,z,t>1 \\ \log_y x^{\frac{2}{x+y}}+\log_z y^{\frac{2}{y+z}}+\log_t z^{\frac{2}{z+t}}+\log_x t^{\frac{2}{t+x}}=\dfrac{16}{x+y+z+t} \\ (2x+3y+4z+100)^{10}=10^{12}x^2y^3z^4 \end{cases}$$

A.161. If $a,b,c,d>0, a+b+c+d=log_{10}^2 98$ then:

$$\left(\frac{1}{a}+\frac{1}{c}\right)log_{10}^4 9+\left(\frac{1}{b}+\frac{1}{d}\right)log_{10}^4 11>4$$

PROBLEMS

GEOMETRY

G.001. In $\Delta ABC, N$ −nine point center, the following relationship holds:

$$\left(\frac{a^2+R^2}{NB}\right)^2+\left(\frac{b^2+R^2}{NC}\right)^2+\left(\frac{c^2+R^2}{NA}\right)^2\geq 192r^2$$

G.002. Solve for real numbers:

$$|cosx|+|cosy|=\sqrt{(2+sinx+siny)(2-sinx-siny)}$$

G.003. If $\alpha,\beta,\gamma,\delta\in\left(0,\frac{\pi}{2}\right), 16sin\alpha\cdot sin\beta\cdot sin\gamma\cdot sin\delta=1$ then:

$$\frac{sin^2\alpha}{cos\alpha}+\frac{sin^2\beta}{cos\beta}+\frac{sin^2\gamma}{cos\gamma}+\frac{sin^2\delta}{cos\delta}\geq\frac{2\sqrt{3}}{3}$$

G.004. In ΔABC the following relationship holds:

$$648\sqrt{3}r^3\leq a(a-3b-3c)^2+b(3a-b-c)^2+c(3a-b-c)^2$$
$$\leq 81\sqrt{3}R^3$$

G.005. In any Δ ABC with $B=2A$ and $C=4A, \sum h_a^2>\frac{7\sqrt{21}R^2}{10}$

G.006. If in $\Delta ABC, m(\sphericalangle A)=90^\circ$ then:

$$\frac{bc}{a(b+c-a)}+\frac{2bc+(a+b+c)^2}{a(a+b+c)}<\frac{3\sqrt{3}+\sqrt{2}+2}{2}$$

G.007. If $0\leq x,y,z\leq\frac{\pi}{2}$ then:

$$cos^2x+cos^2y+cos^2z+(1+sin^2x)(1+sin^2y)(1+sin^2z)\leq 8$$

G.008. Solve for real numbers:

$$\begin{cases} sinx + siny = 1 \\ cosx + cosy = \sqrt{3} \\ \sqrt[4]{z + sin^{-6}x} + \sqrt[4]{z + sin^{-6}y} = 4\sqrt{2} \end{cases}$$

G.009. Solve for real numbers:

$$\begin{cases} tan^2x(1 - sin^8x) + cot^2x(1 - cos^8x) = \frac{15}{8} \\ x + y = \pi \\ tan^2y(1 - sin^{10}y) + cot^2y(1 - cos^{10}y) = \frac{31}{16} \end{cases}$$

G.010. In any acute-angled ΔABC:

$$\prod(\tan A + \cot A)(\cos A + \sec A) \geq \prod(\tan A + \cot B)(\cos A + \sec B)$$

G.011. Solve for $x, y, z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then:

$$\frac{cos(5x)}{cosx} + \frac{cos(5y)}{cosy} + \frac{cos(5z)}{cosz} = \frac{15}{4}$$

G.012. If $0 \leq x \leq \frac{\pi}{4}$ then:

$$tanh(sin2x) \leq tanh(sinxcosx) + \frac{sinxcosx}{cosh^2(sinxcosx)}$$

G.013. In any ΔABC: $\frac{h_a r_b^2}{m_a} + \frac{h_b r_c^2}{m_b} + \frac{h_c r_a^2}{m_c} \geq \frac{\sqrt{6}}{3} \cdot \frac{r(4R+r)^2}{R+(\sqrt{6}-2)r}$

G.014. Solve for real numbers:

$$sin^2x \cdot cos^2t + sin^2y \cdot cos^2x + sin^2z \cdot cos^2y + sin^2t \cdot cos^2z = 2$$

G.015. In ΔABC the following relationship holds:

$$\begin{vmatrix} 9R^2 & a^2 & b^2 & c^2 \\ a^2 & 9R^2 & c^2 & b^2 \\ b^2 & c^2 & 9R^2 & a^2 \\ c^2 & b^2 & a^2 & 9R^2 \end{vmatrix} > 0$$

G.016. In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{tanA}{\sqrt[3]{tanA + tanB + tanC}}\left(1 + \frac{tanB}{\sqrt[3]{tanA + tanB + tanC}}\right) \geq 6$$

G.017. If $\begin{cases} x^2 + xy + y^2 = 36, x, y, z > 0 \\ y^2 + yz + z^2 = 100 \\ z^2 + zx + x^2 = 64 \\ a + b + c = 8\sqrt[4]{3}, a, b, c > 0 \end{cases}$ **then:**

$$a^2x + b^2y + c^2z \geq 2xyz$$

G.018. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$, $sinxsinysinz = cosxcosycosz$ then:

$$8 + (1 + tan^3x)(1 + tan^3y)(1 + tan^3z) \geq \frac{2}{cos^2xcos^2ycos^2z}$$

G.019. Solve for real numbers:

$$4sin\frac{\pi}{26} + 4xsin\frac{3\pi}{26} + 4sin\frac{9\pi}{26} = x + \sqrt{13}$$

G.020. If $x, y, z \in [0, 1]$ then in ΔABC the following relationship holds:

$$(a^x + b^x + c^x)(a^y + b^y + c^y)(a^z + b^z + c^z) \leq \frac{(a + b + c)^3}{\sqrt[3]{(abc)^{3-x-y-z}}}$$

G.021. In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\frac{\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)\left(\sqrt{a'} + \sqrt{b'} + \sqrt{c'}\right)}{\sqrt[6]{aa'bb'cc'}} \leq \frac{2ss'}{\sqrt[3]{2RR'FF'}}$$

G.022. Solve for real numbers:

$$\sum_{0\le i<j\le 5}\frac{(-1)^{j-i}}{2}\binom{5}{i}\binom{5}{j}\cos(2j-2i)x+63=0$$

G.023. If in ΔABC, $\mu(\sphericalangle A)\ge 45^\circ$, $\mu(\sphericalangle B)\ge 45^\circ$, $\mu(\sphericalangle C)\ge 45^\circ$ then:

$$(\tan A+\tan B+\tan C)^2\left(\frac{1}{\cos^2 A}+\frac{1}{\cos^2 B}+\frac{1}{\cos^2 C}-3\right)\ge 243$$

G.024. Solve for real numbers:

$$\left|\left|\frac{\tan^2 x-1}{\tan^4 x}\right|+\frac{1}{\sin^2 x}-2\cot^3 x\right|+2(\cot x+\cot^3 x)=4$$

G.025. Solve for real numbers:

$$\begin{cases}\dfrac{\tan^2 x+\tan^2 y}{\cos^2 y}+\left(\dfrac{\tan y}{\cos z}\right)^2+\left(\dfrac{\tan z}{\cos x}\right)^2=29\\ \dfrac{\tan^2 y+\tan^2 z}{\cos^2 z}+\left(\dfrac{\tan z}{\cos x}\right)^2+\left(\dfrac{\tan x}{\cos y}\right)^2=19\\ \dfrac{\tan^2 z+\tan^2 x}{\cos^2 x}+\left(\dfrac{\tan x}{\cos y}\right)^2+\left(\dfrac{\tan y}{\cos z}\right)^2=23\end{cases}$$

G.026. Solve for real numbers:

$$\cos 2x+\frac{\sin^3 x-\cos^3 x}{\sin^3 x+\cos^3 x}=\tan\left(x-\frac{\pi}{4}\right)$$

G.027. If $a,b,c,d,e,f,g,h,i>0$,

$a^2+b^2+c^2=d^2+e^2+f^2=g^2+h^2+i^2=\sqrt[3]{2}$ then:

$$\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}\cdot\begin{vmatrix}a & d & g\\ b & e & h\\ c & f & i\end{vmatrix}\le 2$$

G.028. If $x,y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left(1+\frac{\sin^2 x}{\cos^2 y}\right)^{\cos^2 y}\cdot\left(1+\frac{\cos^2 x}{\sin^2 y}\right)^{\sin^2 y}\le 2$$

G.029. In ΔABC the following relationship holds:

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\leq\frac{\sqrt{3}}{72r^3}$$

G.030. In ΔABC the following relationship holds:

$$am_a+bm_b+cm_c\geq 6F+\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

G.031. If $0<x+y+z<\frac{\pi^2}{2}$ then:

$$\frac{x\cos\sqrt{z}+y\cos\sqrt{x}+z\cos\sqrt{y}}{\cos\sqrt{\frac{xy+yz+zx}{x+y+z}}}\geq x+y+z$$

G.032. In ΔABC the following relationship holds:

$$\frac{a^2}{8a^3+(a+b)b^2}+\frac{b^2}{8b^3+(b+c)c^2}+\frac{c^2}{8c^3+(c+a)a^2}$$
$$\leq\frac{5s^2+r^2+4Rr}{10s(s^2+r^2+2Rr)}$$

G.033. In ΔABC the following relationship holds:

$$72\sum_{cyc}\left(\frac{1}{b+c}+\frac{2}{c+a}\right)\left(\frac{1}{c+a}+\frac{2}{a+b}\right)\left(\frac{1}{a+b}+\frac{2}{b+c}\right)\leq\left(\frac{s}{rR}\right)^3$$

G.034. In ΔABC the following relationship hods:

$$81\sqrt{3}\prod_{cyc}(2b^2+2c^2+a^2)>4096s^3r_ar_br_c$$

G.035. In ΔABC the following relationship holds:

$$\left(\frac{4}{(b+c)^2}+\frac{9}{(c+a)^2}+\frac{1}{(a+b)^2}\right)\left(\frac{9}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{4}{(a+b)^2}\right)$$
$$> 49\sum_{cyc}\frac{1}{(a+b)^2(b+c)^2}$$

G.036. If $0 < x, y \le \frac{\pi}{4}$ then:

$$cos^2x\cdot cos^2y\cdot (tanx)^{tanx}\cdot (tany)^{tany}$$
$$\le (1-sinxcosx)(1-sinycosy)$$

G.037. . Solve for real numbers:

$$15cosx\cdot cosy\cdot cosz+4\sum_{cyc}cos5x\cdot cosy\cdot cosz=0, -\frac{\pi}{2}<x,y,z<\frac{\pi}{2}$$

PROBLEMS

ANALYSIS

AN.001. If $0 < a \leq b$ then:

$$\frac{1}{2}\int_a^b\int_a^b \frac{x+y}{\sqrt{xy}}dxdy + 2\int_a^b\int_a^b \frac{\sqrt{xy}}{x+y}dxdy \leq \log\left(\frac{b}{a}\right)^{b^2-a^2}$$

AN.002. $a, b, c > 0, abc = 1$

$$\Omega(a) = \int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x} + 6xe^{3x^2}log(1+e^x)\right)dx$$

Prove that: $\Omega(a) + \Omega(b) + \Omega(c) \geq 3e^{a^2+b^2+c^2}$

AN.003. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\left(\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}\right)\left(\sum_{k=1}^{n}\frac{k^3}{3k^2-3nk+n^2}\right)}$$

AN.004. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{n!}\cdot\int_{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}^{\frac{\pi^2}{6}} e^{x^2}dx\right)$$

AN.005. If $a, b, c > 0, a + b + c = 9$ then:

$$\int_0^3 e^{x^2}dx + \sum_{cyc}\frac{1}{a}\int_0^a e^{x^2}dx \geq 4\sum_{cyc}\frac{1}{9-a}\int_0^{\sqrt{bc}} e^{x^2}\,dx$$

AN.006. Find without any software:

$$\Omega = \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{x}{\sin 2x} dx$$

AN.007. Find:

$$\Omega = \lim_{n\to\infty} \left(n \cdot \int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} \frac{\sqrt[n]{e^x}}{x} dx \right)$$

AN.008. If $0 < a, b, c \leq 1$ then

$$\int_a^1 x^x dx + \int_b^1 x^x dx + \int_c^1 x^x dx \geq \log\big((2-a)(2-b)(2-c)\big)$$

AN.009. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{2 \sum_{0\leq i<j\leq n} \binom{n}{i}\binom{n}{j} + \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}}$$

AN.010. If $a \geq 1$ then:

$$4(\sqrt{a}-1)^2 + \left(\int_1^a \sqrt{1-\frac{1}{x}} dx\right)^2 \leq (a-1)^2$$

AN.011. If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \sqrt{\left(1+\frac{1}{x^4}\right)\left(1+\frac{1}{y^4}\right)} dxdy \geq \frac{2(b-a)^2}{ab}$$

AN.012. Find:

$$\Omega = \lim_{n\to\infty}\left(n^2\sum_{k=1}^{n}\frac{1}{k^4-k^2+1}\right)$$

AN.013. Find a closed form:

$$\Omega = \sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n}\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}^n, \Omega \in M_2(\mathbb{R})$$

AN.014. If $a,b,c>0, a+b+c=9, n\in\mathbb{N}^*, F_n$ −Fibonacci numbers then:

$$\frac{a^4}{sin^3(F_{2n+2})}+\frac{b^4}{sin^3(F_n^2)}+\frac{c^4}{cos^3(F_{n+2}^2)}>72$$

AN.015. Find a closed form:

$$\Omega = \prod_{n=1}^{\infty}\left(\frac{n^{\frac{1}{n+1}}}{2}\right)$$

AN.016. Find:

$$\Omega = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{e-1}{n}log\left(1+\frac{(e-1)k}{n}\right)\right)\right)$$

AN.017. Find a closed form:

$$\Omega = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+3)(2n+5)}$$

AN.018. If $a,b,c,d\ge e, e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, then:

$$5log(ae)\cdot log(be)\cdot log(ce)\cdot log(de)\ge log(abcde)^{16}$$

AN.019. If $0 < a, b, c \leq \frac{\pi}{2}$ then:

$$(1+cos^2 a)(1+cos^2 b)(1 + cos^2 c)(sina)^{2sin^2 a}(sinb)^{2sin^2 b}(sinc)^{2sin^2 c} \geq 1$$

AN.020. If $0 < a \leq b, n \in \mathbb{N} - \{0\}$ then:

$$(b-a)^{n-1}\int_a^b \left(\prod_{k=1}^n \mathrm{erf}(k)\right) dx \geq \prod_{k=1}^n \int_a^b \mathrm{erf}(kx)\, dx$$

AN.021. Find without softs:

$$\Omega = \int (10tan^3 x + 7tan^2 x + 12tanx + 9)e^x dx$$

AN.022. If $a, b, c > 1, \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 18$ then:

$$e^{9+\frac{\Gamma'(a)}{\Gamma(a)}+\frac{\Gamma'(b)}{\Gamma(b)}+\frac{\Gamma'(c)}{\Gamma(c)}} < abc$$

AN.023.

$$f \in C^1([a,b]), f(0) = 0, f\left(\frac{\pi}{2}\right) = 96, f'(x) = f'\left(\frac{\pi}{2} - x\right), \forall x \in [a,b]$$

Find:

$$\Omega = \int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2} - x\right) f(x) dx$$

AN.024. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{n \cdot \sqrt[n]{\prod_{k=1}^n \sin^2\left(\frac{k}{n}\right)}}{\sum_{1\leq i<j\leq n} \sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right)}\right)$$

AN.025.

$$If\ 0 < a \leq b < \frac{\pi}{10}\ then:$$

$$\tan\left(\frac{5}{2}(\sqrt{a}+\sqrt{b})\sqrt[4]{ab}\right)\cdot\tan\left(\frac{3}{2}(\sqrt{a}+\sqrt{b})^2\right)$$
$$\leq\tan\left(\frac{3}{2}(\sqrt{a}+\sqrt{b})\sqrt[4]{ab}\right)\cdot\tan\left(\frac{5}{2}(\sqrt{a}+\sqrt{b})^2\right)$$

AN.026. If $a,b\geq 0$ then:

$$\int_a^b\int_a^b|ay-ab+bx|dydx\leq a^2b^2$$

AN.027. Find a closed form:

$$\Omega=\prod_{n=1}^{\infty}\log\left(2+\frac{1}{n}\right)\log\left(2-\frac{1}{n+1}\right)$$

AN.028. If $f:\mathbb{R}\to(0,\infty)$, f –continuous, $a>0$, $f(x)=f(-x)$, $\forall x\in\mathbb{R}$ then:

$$\int_{\frac{1}{a}}^{a}\frac{x+\log x}{xf\left(x-\frac{1}{x}\right)}dx=\frac{1}{2}\int_{\frac{1+\sqrt{1+4a^2}}{2a}}^{\frac{a+\sqrt{1+4a^2}}{2}}\frac{dx}{f(x)}$$

AN.029. If $0<a,b\leq\frac{\pi}{2}$ then:

$$\frac{2}{\pi}\leq\frac{2\sin(\sqrt{ab})}{(a+b)\sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}\leq\frac{2\sin\left(\frac{a+b}{2}\right)}{a+b}$$

AN.030. If $1<a<b\leq e$ then:

$$125^a\cdot(4a+b)^{a+4b}\leq 125^b\cdot(a+4b)^{4a+b}$$

AN.031. If $2\leq a,b,c\leq 3$ then:

$$2\left(\sqrt{a^a}+\sqrt{b^b}+\sqrt{c^c}\right)+a+b+c\geq a^2+b^2+c^2+6$$

AN.032. If $0 < x, y < \frac{\pi}{2}$ then:

$$\left(\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^4\right)\left(\left(\frac{\sin y}{y}\right)^3 + \left(\frac{\tan y}{y}\right)^5\right) > 4$$

AN.033. Find:

$$\Omega = \lim_{n\to\infty} \exp\left(\frac{1}{n^3}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\sin k \cdot \sin j}{\sin k + \sin j} - 2 \cdot \sum_{1\le k<j\le n}\frac{\sin k \cdot \sin j}{\sin k + \sin j}\right)\right)$$

AN.034. If $f: [0,1] \to (0,\infty)$, f – continuous then:

$$2\left(\int_0^1 \sqrt{f(x)}\,dx\right)^2 \le \int_0^1 f(x)\,dx + \left(\int_0^1 \sqrt[3]{f(x)}\,dx\right)^3$$

AN.035. If $0 < a \le b < \frac{\pi}{4}$ then:

$$tan\left(2\sqrt{ab}\right) \cdot tan\left(\frac{a+b}{2}\right) \le tan\left(\sqrt{ab}\right) \cdot tan(a+b)$$

AN.036. Find:

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n}\left(1 + \frac{n\tan^{-1}\left(\frac{k}{n}\right)}{k^2+n^2}\right)$$

AN.037.

$$If\ x, y \in \mathbb{R}, x^2 + y^2 = 3\ then:$$

$$tan^{-1}\left(\frac{x+2y}{3}\right) + tan^{-1}\left(\frac{x+3y}{4}\right) + tan^{-1}\left(\frac{x+4y}{5}\right) < \pi$$

AN.038. Find $x, y, z \in \left(0, \frac{\pi}{2}\right]$ such that:

$$\frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin z}{z} = 3 + \frac{8-4\pi}{\pi^2}\left(x^2+y^2+z^2\right)$$

AN.039. Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k(k+1)e^{\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}}}{n(n+1)(n+2)}$$

AN.040. If $0 < a \le b$ then:

$$\left(1+\frac{2ab}{a+b}\right)^{\sqrt{ab}}\cdot\left(1+\sqrt{ab}\right)^{\frac{a+b}{2}} \le \left(1+\sqrt{ab}\right)^{\frac{2ab}{a+b}}\cdot\left(1+\frac{a+b}{2}\right)^{\sqrt{ab}}$$

AN.041. If $0 < a \le b$ then:

$$\int_a^b \frac{\tan^{-1}\left(e^{-x^2}\right)}{e^{x^2}}dx \ge \int_a^b \frac{1}{e^{x^2}}dx\cdot\tan^{-1}\left(\int_a^b\frac{1}{e^{x^2}}dx\right)$$

AN.042. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\frac{4}{3}(b^3-a^3)+\pi^3\int_a^b\frac{\sin x}{x}dx \ge 3\pi^2(b-a)$$

AN.043. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\int_e^n\frac{x^{n-1}e^x(x\log x+n\log x-n)}{\log^{n+1}x}dx}$$

AN.044. If $n\in\mathbb{N}, n\ge 1$ then:

$$\left(H_n^{(2)}+H_n^{(6)}\right)\left(H_n^{(4)}+H_n^{(8)}\right)\ge\left(H_n^{(3)}+H_n^{(7)}\right)^2$$

AN.045.

$$\Omega_1(t)=\int_0^1\left(\frac{x^{\sin^2 t}}{1+x^{\cos^2 t}}\right)dt, \Omega_2(t)=\int_0^1\left(\frac{x^{\cos^2 t}}{1+x^{\sin^2 t}}\right)dt$$

Prove that:

$$\Omega_1(t)+\Omega_2(t)\ge\frac{4-\sin 2t}{8+\sin 2t},t\in\left[0,\frac{\pi}{2}\right]$$

AN.046. If $f:\mathbb{R}\to\mathbb{R}, f$ – continuous, $f(x)+f(y)\ge 3f(x+y),\forall x,y\in\mathbb{R}$ then:

$$3\int_0^1\int_0^1\int_0^1 f(x+y+z)dx\,dy\,dz\le 5\int_0^1 f(x)\,dx$$

AN.047. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\log_n^n\left(\frac{(1+H_1)^2+(1+H_2)^n+\cdots+(1+H_n)^2}{n}\right)}{\log_n(1+H_1)\cdot\log_n(1+H_2)\cdot\ldots\cdot\log_n(1+H_n)}\right)$$

AN.048. If $0<a\le b$ then:

$$e^{\frac{(a^2+b^2)(b-a)}{ab(b+a)}}\ge\frac{b}{a}$$

AN.049. Find:

$$\Omega=\sum_{n=2}^{\infty}\left(\frac{H_n}{\left(\sum_{i=1}^{n-1}H_i\right)\left(\sum_{j=1}^{n}H_j\right)}\right)$$

AN.050. $f:\mathbb{R}\to\mathbb{R}, 2f^4(x)+2f^2(x)+2\le 3f^3(x)+3f(x),\forall x\in\mathbb{R}$

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{1\le i<j\le n}\left(f\left(e+\frac{\pi i}{n}\right)f\left(e+\frac{\pi j}{n}\right)\right)\right)$$

AN.051. Find:

$$\Omega=\sum_{n=2}^{\infty}\left(\frac{H_n}{\left(\sum_{i=1}^{n-1}H_i\right)\left(\sum_{j=1}^{n}H_j\right)}\right)$$

AN.052. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[4]{(2n+3)^3\cdot\sqrt[n+1]{(n+1)!}}-\sqrt[4]{(2n+1)^3\cdot\sqrt[n]{n!}}\right)$$

AN.053. Solve for $x > 0$:

$$e^2+\int_e^x\left(t^{\log t}(1+2\log t)\right)dt = x^4$$

AN.054. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n(n+1)}\sum_{k=1}^{n}\left(k\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right)\right)\right)$$

AN.055. If $f:\mathbb{R}\to(0,\infty)$, $a,b\in\mathbb{R}$, $a\le b$ then:

$$\int_a^b\int_a^b\left(\left(f(x)+f(y)\right)\tan^{-1}\left(\frac{f(x)+f(y)}{2}\right)\right)dx\,dy$$

$$\le 2(b-a)\int_a^b\left(f(x)\tan^{-1}\left(f(x)\right)\right)dx$$

AN.056. If $a,b\in\mathbb{R}$, $a\le b$, $f:\mathbb{R}\to\left(0,\frac{\pi}{2}\right)$, f − continuous then:

$$4\int_a^b\csc\left(2f(x)\right)dx+\int_a^b\cos\left(\frac{\pi}{4}-f(x)\right)dx\ge 5(b-a)$$

AN.057. Find:

$$\Omega = \lim_{n\to\infty}\left(n\left(\log 2-\sum_{i=1}^{n}\frac{(n+i)^4}{3+(n+i)^5+\cot^{-1}(n+i)}\right)\right)$$

AN.058. If $f:\mathbb{R}\to(0,\infty)$, f – continuous, $a,b\in\mathbb{R}, a\le b$ then:

$$\int_a^b\int_a^b\left(\log\left(\frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2}\right)\right)dx\,dy$$

$$\le(b-a)\int_a^b f^2(x)\,dx-\left(\int_a^b f(x)\,dx\right)^2$$

AN.059. If $0<a<b$ then:

$$a^2<\left(\frac{be^b-ae^a}{e^b-e^a}-1\right)\left(\frac{ae^b-be^a}{e^b-e^a}+1\right)<b^2$$

AN.060. If $0<a<b<c<d$ then:

$$\frac{1}{b-a}\int_a^b e^{x^2}\,dx+\frac{1}{c-b}\int_b^c e^{x^2}\,dx+\frac{1}{d-c}\int_c^d e^{x^2}\,dx$$

$$>3\sqrt[36]{e^{(a+2b+2c+d)^2}}$$

AN.061. Find:

$$\Omega=\int\left(\frac{1}{(2x+5)\sqrt{(x+2)(x+3)}\left(4\tan^{-1}\left(\sqrt{\frac{x+3}{x+2}}\right)-\pi\right)^2}\right)dx,x$$

$$>0$$

AN.062. If $f:(0,\infty)\to(0,\infty)$; $f$continuous; $\int_a^b f(x)\,dx=5(b-a)$; $0<a\le b$ then:

$$\int_a^b\left(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\right)dx\le 9(b-a)$$

AN.063. If $m, n, p, q, r, s \in \mathbb{N} \setminus \{0\}$ denote

$H_n^{(m)} = \frac{1}{1^m} + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$**. Prove that:**

$$\left(H_n^{(2p)} + H_n^{(2q)}\right)\left(H_n^{(2r)} + H_n^{(2s)}\right) \geq \left(H_n^{(p+r)} + H_n^{(q+s)}\right)^2$$

AN.064. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$(b-a)^2 + (b-a)\left(\int_a^b \frac{x\,dx}{\sin x}\right) \cdot e^{\frac{1}{b-a}\int_a^b \log\left(\frac{\sin x}{x}\right)dx}$$

$$\leq 2\left(\int_a^b \frac{\sin x\,dx}{x}\right)\left(\int_a^b \frac{x\,dx}{\sin x}\right)$$

AN.065. Find (ϕ – golden ratio):

$$\Omega = \lim_{n\to\infty}\left(1 + \frac{1}{n\phi} + \sqrt{1 + \frac{1}{n\phi^2}} + \sqrt[3]{1 + \frac{1}{n\phi^3}} + \cdots + \sqrt[3]{1 + \frac{1}{n\phi^n}} - n\right)$$

AN.066. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^2}\sqrt[n]{\sum_{k=0}^{n-1}(n^2 - nk)^2\binom{2n}{k}}\right)$$

AN.067. $f: [0,1] \to (0,\infty), f$ – continuous. Find: $\Omega = \min_{k\in\mathbb{R}} K$ such that:

$$\int_0^1\int_0^1\int_0^1 \left(\left(f(x)+f(y)\right)^{f(z)} + \left(f(y)+f(z)\right)^{f(x)}\left(f(z)+f(x)\right)^{f(y)}\right) dx\,dy\,dz$$

$$\leq K\int_0^1 f(x)\,dx$$

AN.068.

$$\Omega(a)=\lim_{b\to\infty}\left(\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)\cdot\ldots\cdot(n+a-1)}{(-b)^{n-1}}\right),a\in\mathbb{N}-\{0,1\}$$

Find:

$$\Omega=\sum_{a=2}^{\infty}\frac{1}{\Omega(a)}$$

AN.069. Find:

$$\Omega=\lim_{n\to\infty}\left(\int_0^1\left(e^{\frac{x^2}{n}}\right)dx\right)^n$$

AN.070. If $f:[a.b]\to(0,\infty)$, continuous, $a\le b$ then:

$$\int_a^b\int_a^b\left(\sqrt{\frac{f^2(x)+f^2(y)}{2}}+\sqrt{f(x)f(y)}\right)dx\,dy\le 2(b-a)\int_a^b f(x)\,dx$$

AN.071. If $0<a\le b$ then:

$$\int_a^b\int_a^b\tan^{-1}\left(\frac{ax+by}{a+b}\right)dx\,dy+(b-a)\int_a^b\log x\,dx\le$$

$$\le\int_a^b\int_a^b\log\left(\frac{ax+by}{a+b}\right)dx\,dy+(b-a)\int_a^b\tan^{-1}x\,dx$$

AN.072. If $p,q,r>0,n\in\mathbb{N},n\ge 2,0<a\le b<\pi$ then:

$$\int_a^b\int_a^b\int_a^b\sqrt[n]{\sin\left(\frac{px+qy+rz}{p+q+r}\right)}\,dx\,dy\,dz\ge(b-a)^2\int_a^b\sqrt[n]{\sin x}\,dx$$

AN.073. $x_n > 0, n \in \mathbb{N}, n \geq 1$

$$\lim_{n\to\infty}\left(\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)^{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right)=\lim_{n\to\infty}\left(\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^{n-1}\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)\right)=\omega$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\prod_{i=1}^{n}x_i\right)$$

AN.074. If $x \in \left(0, \frac{\pi}{2}\right)$ **then:**

$$\left(\left(\sqrt{\frac{a}{b}}\right)^{\frac{\sin x}{x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{\sin x}{x}}\right)\left(\left(\sqrt{\frac{a}{b}}\right)^{\frac{x}{\tan x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{x}{\tan x}}\right)$$

$$\leq\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^{2}$$

AN.075. If $0 \leq k \leq m; 0 \leq l \leq n$ **then:**

$$\left(\frac{4}{\pi}\right)^{k+l}\int_0^1(\tan^{-1}(1+x^2))^{m+n}\,dx \geq \int_0^1\frac{(\tan^{-1}(1+x^2))^{m+n}}{(\tan^{-1}(1+x^2))^{k+l}}dx$$

AN.076. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{(n+1)^n}+\frac{1}{2\cdot n^{n-1}}+\frac{1}{3^2\cdot(n-1)^{n-2}}+\cdots\right.$$
$$\left.+\frac{1}{(n-1)^{n-2}\cdot 3^2}+\frac{1}{n^{n-1}\cdot 2}+\frac{1}{(n+1)^n}\right)$$

AN.077. Find:

$$\lim_{n\to\infty}\left(\frac{1}{n}\left(\sqrt[n]{1+\frac{\sqrt[n]{e}}{e}}+\sqrt[n]{1-\frac{\sqrt[n]{e}}{e}}\right)\right)$$

AN.078. Solve for $x>1$:

$$\int_1^x\left(\frac{\log t-1}{t^2-\log^2 t}\right)dt=\frac{1}{2}\log\left(\frac{e-1}{e+1}\right)$$

AN.079. If $a,b\geq 1$; $f\in C^2([0,1])$, then f convexe:

$$2\int_0^1 f(x)\,dx\geq a^2\int_0^{\frac{1}{a}} f(x)\,dx+b^2\int_0^{\frac{1}{b}} f(x)\,dx$$

AN.080. Prove that:

$$\int_0^{\frac{1}{2}}\left(\log(1+x)\cdot\log\left(\frac{3}{2}+x\right)\right)dx\leq\frac{1}{2}\left(\int_0^1\log(1+x)\,dx\right)^2$$

AN.081. Prove that:

$$\int_0^1\left(\tan^{-1}x+\frac{x}{1+x^2}\right)^2dx+4\int_0^1\frac{1}{(1+x^2)^4}dx>\frac{(\pi+2)^2}{16}$$

AN.082. Solve for $x>0$:

$$e^2+\int_e^x\left(t^{\log t}(2\log t+1)\right)dt=x^4$$

AN.083. If $f:[0,1]\to(0,\infty)$; f continuous and $\int_0^1 f^2(x)\,dx=7$ then:

$$\int_0^1 f^5(x)\,dx>6+\int_0^1 f^3(x)\,dx$$

AN.084. If $a, b \in \mathbb{R}; a \leq b$ then:

$$\log 5 \cdot \int_a^b 5^{x^2}\,dx + \log 5 \cdot \int_a^b 5^{x^4}\,dx \geq 5^b - 5^a$$

AN.085. If $a, b \geq 1$ then:

$$a + b + 2\int_0^1 e^{x^2}\,dx \geq 2 + a^2\int_0^{\frac{1}{a}} e^{x^2}\,dx + b^2\int_0^{\frac{1}{b}} e^{x^2}\,dx$$

AN.086. If $x, y \in \mathbb{R}; x^2 + y^2 = 3$ then:

$$\tan^{-1}\left(\frac{x+2y}{3}\right) + \tan^{-1}\left(\frac{x+3y}{4}\right) + \tan^{-1}\left(\frac{x+4y}{5}\right) < \pi$$

AN.087. Solve for real numbers:

$$\sum_{i=1}^{p}\left(\sum_{j=1}^{q}\left(\log x - \frac{1}{i^5}\right)\left(\log x - \frac{1}{j^7}\right)\right) = 0; p, q \in \mathbb{N} \setminus \{0\}$$

AN.088. Let be $f: (1, \infty) \to (0, \infty)$; f continuous; $\int_e^{\pi} f(x)\,dx = e^2 - 1$

$$f(x) + f(y) = ef\left(\frac{x+y}{\pi}\right); (\forall) x, y \in \mathbb{R}$$

Find:

$$\Omega = \int_e^{\pi}\int_e^{\pi}\int_e^{\pi} f\left(\frac{x+y+z}{\pi^2 - 1}\right) dx\,dy\,dz$$

AN.089. Let be $f: [0, \infty) \to (0, \infty)$; $f(x) = \frac{4x+3}{x+2}$

Find:

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to 0}\left(\underbrace{f \circ f \circ \ldots \circ f}_{"n"\ times}\right)(x)\right)$$

AN.090. If $a \geq 0$ then:

$$2a^3\int_0^a e^{4x^2}\,dx+\left(\int_0^a e^{x^2}\,dx\right)^4 \geq 3a^2\left(\int_0^a e^{2x^2}\,dx\right)^2$$

AN.091. Prove that:

$$2\sqrt[4]{e}\int_0^{\frac{1}{2}}\left(e^{2x^2}+x\right)dx<\left(\int_0^1 e^{x^2}\,dx\right)^2$$

AN.092. If $a>1$ then:

$$\frac{4\log 2}{\pi}+\int_1^a \frac{2x\tan^{-1}x-\log(1+x^2)}{(1+x^2)(\tan^{-1}x)^2}-dx<\frac{a^2}{\tan^{-1}a}$$

AN.093. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{2}H_n+\log\left(\prod_k^n\frac{2k}{2k-1}\right)\right)$$

AN.094. If $0<a\leq b$ then:

$$\arctan\left(\frac{2ab}{a+b}\right)+\int_{\frac{2ab}{a+b}}^{\sqrt{ab}} e^{-t^2}\,dt\leq \arctan\left(\sqrt{ab}\right)$$

AN.095. If $0<a\leq b<e; 0<x\leq y\leq z; x+y+z=3$ then:

$$(x-1)\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}}+(y-1)\left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}+(z-1)\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}}\geq 0$$

AN.096. If $0<a\leq b<\frac{\pi}{2}$ then:

$$8\pi^3\int_a^b\int_a^b\left(\frac{sinxsinysin(x+y)}{xy(\pi-x-y)}\right)dxdy\leq 81\sqrt{3}(b-a)^2$$

AN.097. If $a \geq 1$ then:

$$\frac{8}{\pi-2}\int_1^a \frac{x-\tan^{-1}x}{(1+x^2)^2(\tan^{-1}x)^2}dx+\frac{16}{\pi^2}\geq\frac{1}{(\tan^{-1}a)^2}$$

AN.098. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{3^n}\sqrt[n]{\frac{1}{n!}\prod_{i=1}^{n}\left(\sum_{k=1}^{i}k\binom{2i}{2k}\right)}\right)$$

AN.099.

$$\Omega(p)=\lim_{n\to\infty}\left(\frac{1}{p}\sum_{k=1}^{n}\sqrt[n]{k}\right)^{np}, p\in\mathbb{N}-\{0\}$$

Find:

$$\Omega=\sum_{k=1}^{\infty}\frac{1}{\Omega(p)}$$

AN.100. Find:

$$\Omega=\lim_{n\to\infty}\left(n^6\sin\frac{1}{n^3}\tan\frac{1}{n^5}\sum_{1\leq k<l\leq n}\sin\left(\frac{k+l}{n}\right)\right)$$

AN.101. If $0<a\leq b<1$ then:

$$\sin\left(\frac{3a+b+2}{4}\right)\sin\left(\frac{a+3b+6}{4}\right)\leq\sin\left(\frac{a+3b+2}{4}\right)\sin\left(\frac{3a+b+6}{4}\right)$$

AN.102. If $x>1, p,q,r\in\mathbb{N}$ then:

$$\frac{(x+1)^{2(p+q+r)}(x^2-1)^3}{(x^{2p+2}-1)(x^{2q+2}-1)(x^{2r+2}-1)}\leq\frac{(2p)!\,(2q)!\,(2r)!}{p!\,q!\,r!}$$

AN.103. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{(n!)^{ne}}{e^{1+2^n+3^n+\cdots+n^n}}\cdot H_n^{-1}\right)$$

AN.104. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{|sinx \cdot sin(2) \cdot ... \cdot sin(2019x)|}{sin^{2019}x}dx \le 2019!\,(b-a)$$

AN.105. If $a, b \ge 0$ then:

$$\int_0^a \left(\int_0^t log^5(x^2+x+2)dx\right)dt + \int_0^b \left(\int_0^t log^5(x^2+x+2)dx\right)dt \ge ablog2$$

AN.106. Find without any software:

$$\Omega = \int \frac{3x^2+x}{1+6x(1+e^{3x})+2e^{3x}+e^{6x}+9x^2}dx$$

AN.107. If $0 < a \le b < 1, f: [0,1] \to (0,\infty), f$ −continuous then:

$$\int_a^b\int_a^b\int_a^b\int_a^b \big(f(x)+f(z)\big)\big(f(y)+f(t)\big)\,dxdydzdt$$

$$\ge 2(b-a)^3\left(2\int_a^b f(x)dx - 2b + 2a\right)$$

AN.108. Find without softs:

$$\Omega = \int_0^{\frac{\pi}{4}} sin\left(x-\frac{\pi}{4}\right)\sqrt{(1+sinx)(1+cosx)}dx$$

AN.109. $f: (0,\infty) \to \mathbb{R}, f$ −continuous, $f(x) - \log_3 x = 4 - f\left(5^{\log_3 x}\right), \forall x > 0$. Find:

$$\Omega = \int_2^3 (f(x)-2)\cdot \log_x 15\, dx$$

AN.110. If $a > 0$ then:

$$\int_{-a}^{a}\int_{-a}^{a}|(x+y)(1-xy)|\,dxdy \le \frac{2}{9}(3a+a^3)^2$$

AN.111. Find a closed form:

$$\Omega = \prod_{n=1}^{\infty}\left(1+\left(\frac{1}{\pi}\right)^{3^n}+\left(\frac{1}{\pi^2}\right)^{3^n}\right)$$

AN.112. Find:

$$\Omega = \lim_{n\to\infty}\left(n\left(\frac{log\left(1+\frac{\sqrt[n]{e}}{n}\right)^{n+1}}{log\left(1+\frac{\sqrt[n+1]{e}}{n+1}\right)^{n}}-1\right)\right)$$

AN.113. Find without softs:

$$\Omega = \lim_{(x,y)\to(0,0)}\left(\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y}\sqrt{tanx}\,dx\right)\left(\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y}\sqrt{cotx}\,dx\right)$$

AN.114. If $a,b,c > 0, a+b+c = 3$ then:

$$\int_0^{\frac{\pi}{2}} a^{sinx}dx + \int_0^{\frac{\pi}{2}} b^{sinx}dx + \int_0^{\frac{\pi}{2}} c^{sinx}dx \le \frac{3\pi}{2}$$

AN.115.

$$\Omega = \prod_{k=1}^{\infty}\left(\lim_{n\to\infty}\left(\frac{\sqrt[n]{k+1}+\sqrt[n]{k+3}}{\sqrt[n]{k+2}+\sqrt[n]{k+4}}\right)^n\right)^2$$

AN.116. Find:

$$\Omega=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{k\cdot(2n-2k-1)!!}{(k+1)!\cdot(2+2n-2k)!!}\right)$$

AN.117. If $a,b,c\in(0,1), a+b+c=1$ then:

$$\int_0^{\sqrt[4]{a}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx+\int_0^{\sqrt[4]{b}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx+\int_0^{\sqrt[4]{c}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx>1$$

AN.118. *If* $f:[a,b]\to\left(0,\frac{\pi}{2}\right), f$ −continuous, $a\le b$ then:

$$\int_a^b sinf(x)dx+\frac{1}{2}\int_a^b tanf(y)dy+\int_a^b cosf(t)dt+\frac{1}{2}\int_a^b cotf(z)dz\ge\left(\sqrt{2}+1\right)(b-a)$$

AN.119. Find:

$$\Omega=\lim_{n\to\infty}\left(tan\left(\gamma-H_n+\frac{\pi}{4}+logn\right)\right)^{\frac{1}{sin(\gamma-H_n+logn)}}$$

AN.120. If $2<a\le b$ then:

$$\frac{log(a+b)-log2}{log(a+b-2)-log2}\le\frac{log(ab)}{2log\left(\sqrt{ab}-1\right)}$$

AN.121.

$$\Omega(m)=\int_0^{\pi/2}\sqrt[m]{tanx}dx\,, m\in\mathbb{N}, m\ge 2$$

If $m, n, p \in \mathbb{N}, m, n, p \geq 2$ then:

$$\Omega(m) \cdot \Omega(n) \cdot \Omega(p) \geq \left(\frac{3\pi}{2\left(\cos\frac{\pi}{2m} + \cos\frac{\pi}{2n} + \cos\frac{\pi}{2p}\right)}\right)^3$$

AN.122. If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{(x+y)^4} \leq \frac{(b-a)^2(a^2+ab+b^2)}{48a^3b^3}$$

AN.123. *If* $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{(3x+2y)^2} + \int_a^b \int_a^b \frac{dxdy}{(2x+3y)^2} \geq \frac{8}{25}\left(\frac{b-a}{b+a}\right)^2$$

AN.124. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(\log\left(1+\frac{1}{n+1}\right)\right)^2}{\log\left(1+\frac{1}{n+2}\right)}\right)$$

AN.125. If $f: [a, b] \to \left(0, \frac{\pi}{2}\right), a \leq b, f$ −continuous, then:

$$\frac{2\sqrt{2}}{5}\int_a^b \sin f(x)dx + \frac{1}{10}\int_a^b \tan f(x)dx + \frac{2\sqrt{2}}{5}\int_a^b \cos f(x)dx$$
$$+\frac{1}{10}\int_a^b \cot f(x)dx \geq b - a$$

AN.126. If $x, y, z, a > 0$ then:

$$\int_0^a \int_0^a \int_0^a \frac{(x+y+z)^3}{(x+y)^2} dxdydz \geq \frac{27a^3}{8}$$

AN.127. If $a, b, c > 1, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ **then:**

$$\sum_{cyc}\frac{\Gamma'(a)}{\Gamma(a)} < log(abc)-\frac{1}{2} < \frac{1}{2}+\sum_{cyc}\frac{\Gamma'(a)}{\Gamma(a)}$$

AN.128.

$$x_0 = 7, y_0 = 5, 2x_n = x_{n-1}+y_{n-1}, y_n = \sqrt{x_{n-1}\cdot y_{n-1}}, n \in \mathbb{N}/\{0\}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{5x_n}{7y_n}\right)^{\frac{H_n}{n}}$$

AN.129.

$$\Omega_k(m) = 2\lim_{x\to 0}\left(\frac{1-(coskx)^{\frac{1}{k^{m+2}}}}{x^2}\right), k, m \in \mathbb{N}^*$$

Find a closed form for:

$$\Omega = \left(\sum_{k=1}^{\infty}\Omega_k(2)\right)\left(\sum_{k=1}^{\infty}\Omega_k(3)\right)$$

AN.130. If $x_i, y_i > 0, i = \overline{1,n}, n \in \mathbb{N}-\{0\}, 0 < a \le b$ **then:**

$$\underbrace{\int_a^b\int_a^b \dots \int_a^b \dots}_{for\ "2n"times}\prod_{i=1}^{n}\frac{dx_i dy_i}{x_i+y_i} \le \left(\frac{b-a}{2}log\frac{b}{a}\right)^n$$

AN.131. $f:(0,\infty)\to\mathbb{R}, f$ **−derivable,** $f(1)=2$,

$(1+2x^2log2)f(x)+xf'(x)=1, \forall x > 0$. **Find:**

$$\Omega = \lim_{n\to\infty}(nf(n))$$

AN.132.

$$\Omega(a)=\int_0^a \frac{\sinh t\cdot \cos t}{(\sin t+\cosh t)(\sinh t+\cos t)}dt,\ a>0$$

Find:

$$\Omega=\lim_{x\to 0}(2\Omega(x))^x$$

AN.133.

$$\Omega=\int_0^1 erf^3(x)dx+36\int_0^1 erf(x)dx-12\left(\int_0^1 erf(x)dx\right)^2$$

$$\boldsymbol{A}.\ \Omega<0 \qquad B.\ \Omega=0 \qquad C.\ \Omega>0$$

AN.134. If $0<a\le b<\frac{\pi}{2}$ then:

$$\log\left(\frac{\tan b}{\tan a}\right)+4\sqrt{2}(\cos a-\cos b)+4\sqrt{2}(\sin b-\sin a)\ge 10(b-a)$$

AN.135. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\binom{202}{0}}{\binom{n+2020}{n}}+\frac{\binom{202}{1}}{\binom{n+2020}{n+1}}+\cdots+\frac{\binom{202}{2020}}{\binom{n+2020}{n+2020}}\right)^n$$

AN.136. If $0\le x\le\frac{\pi}{2}, 0\le y\le 1$ then:

$$2x\le \pi xy\cos\frac{x}{2}+\pi(1-y)\sin x\le \pi x$$

AN.137. Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt{\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}}\sum_{k=1}^{n}\sqrt{\frac{k}{k+1}}\binom{n}{k}\right)$$

AN.138. If $0 \le x, y, z \le 1$ then:

$$\frac{9yz\sin^{-1}x}{1+\sin^{-1}x}+\frac{9zx\sin^{-1}y}{1+\sin^{-1}y}+\frac{9xy\sin^{-1}z}{1+\sin^{-1}z}\le (x+y+z)^3$$

AN.139. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{2n-1}}\sum_{k=1}^{n}\frac{1}{\sqrt{(2^k-1)k!}}\right)$$

AN.140. If $f: [0,1] \to \mathbb{R}$, f −continuous then:

$$\int_0^1\int_0^1\int_0^1 \sqrt[3]{(3f(x)f(y)f(z)-f^3(x)-f^3(y)-f^3(z))^2}\,dxdydz$$

$$\le 3\int_0^1 f^2(x)dx$$

AN.141. If $0 < a \le b$ then:

$$\int_a^b\int_a^b\int_a^b \left(\frac{(x+y+z)(xy+yz+zx)}{xyz}\right)dxdydz$$

$$\le \frac{(2a^2+5ab+2b^2)(b-a)^3}{ab}$$

AN.142. Find without softs:

$$\Omega = \int_{\frac{\pi^5}{1024}}^{\frac{\pi^5}{243}} \frac{\sin(\sqrt[5]{x})\cdot\sin(5\sqrt[5]{x})\cdot\sin(5\sqrt[5]{x})}{\sqrt[5]{x^4}}dx$$

AN.143.

$$\Omega(n,r) = \sum_{k=0}^{n}\frac{(-1)^k}{3r+3k-2}\binom{n}{k}, r\in\mathbb{N}, r-fixed$$

$$\textbf{Find}: \omega(r) = \lim_{n\to\infty} \sqrt[n]{\Omega(n,r)}$$

AN.144. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \frac{(\sin^3 x + \tan y)^2(\cos^3 x + \tan y)}{(\sin^2 x \cdot \cos x + \tan y)^3} dxdy \ge (b-a)^2$$

AN.145. Prove without any software:

$$\int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx < \left(\frac{1+e}{2\sqrt{e}}\right)^2$$

AN.146. $f: \mathbb{R} \to \mathbb{R}, f$ −continuous, $a, b \in \mathbb{R}, a \le b$. Prove that:

$$\int_a^b (f^8(x) + f^2(x))dx + b - a \ge \int_a^b \left(f^5(x) + f(x)\right)dx$$

AN.147. Find without any software:

$$\Omega = \int (4\cot^3 x - 5\cot^2 x + 7\cot x)e^x dx$$

AN.148.

$$\textit{If } a > 1 \textit{ then}:$$

$$\frac{4\log 2}{\pi} + \int_1^a \frac{2x\tan^{-1}x - \log(1+x^2)}{(1+x^2)(\tan^{-1}x)^2} dx < \frac{a^2}{\tan^{-1}a}$$

AN.149. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{2}H_n + \log\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right)\right)$$

AN.150. Find:

$$\Omega = \lim_{n\to\infty}\left(n\sum_{k=1}^{n}\frac{k(k+1)}{(k^2+n^2)(k^2+2k+1+n^2)}\right)$$

AN.151. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\sqrt{\frac{1}{n}\sum_{k=0}^{n}H_kH_{n+k}}\right)$$

AN.152. If $a,b,c>0$ then:

$$\int_a^{2a}\int_b^{3b}\int_c^{4c}\left(\sqrt[6]{\frac{x+1}{y+1}}+\sqrt[8]{\frac{y+1}{z+1}}+\sqrt[10]{\frac{z+1}{x+1}}\right)dxdydz \ge 15abc$$

AN.153. If $0<a\le b$ then:

$$(b-a)^2\int_a^b\frac{x^2dx}{1+x^2}+(b-a)\int_a^b\int_a^b\frac{y^2dxdy}{(1+x^2)(1+y^2)}$$

$$+\int_a^b\int_a^b\int_a^b\frac{z^2dxdydz}{(1+x^2)(1+y^2)(1+z^2)}+log^3\left(\sqrt{\frac{b}{a}}\right)$$

$$\ge (b-a)^3$$

AN.154. $f:\mathbb{R}\to\mathbb{R}, f$ **−continuous in** $x=0$,

$3f(3x)=3x+f(x),\forall x\in\mathbb{R}$. **Prove that** $\forall t\in\mathbb{R}, x,y\ge 0$:

$$\frac{3}{8}\cdot\left(\frac{8}{3}f(x)\right)^{sin^2t}\cdot\left(\frac{8}{3}f(y)\right)^{cos^2t}\le sin^2t\cdot f(x)+cos^2t\cdot f(y)$$

AN.155. Find:

$$\Omega(n)=\lim_{x\to 0}\left(\frac{5^x-1}{x^{n+1}}-\frac{log5}{x^n}-\frac{log^25}{2x^{n-2}}-\cdots-\frac{log^n5}{n!\cdot x}\right), n\in\mathbb{N}, n\ge 2$$

AN.156. In $\Delta ABC, M_n \in (BC), N_n \in (CA), P_n \in (AB), n \in \mathbb{N}, n \geq 2,$ $\frac{BM_n}{CM_n} = \frac{CN_n}{AN_n} = \frac{AP_n}{BP_n} = n$. Find:

$$\Omega = \frac{1}{a^2 + b^2 + c^2} \cdot \lim_{n\to\infty}(AM_n^2 + BN_n^2 + CP_n^2)$$

AN.157. Find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt{(1+n!)^{n!}}}{n \cdot (n!)!}$$

AN.158. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$2\int_a^b\int_a^b cos^2xcos^2y(1 + tanxtany)|tanx - tany|dxdy \leq (b-a)^2$$

AN.159. If $f, f': (0,\infty) \to (0,\infty), f$ –differentiable, $0 < a \leq b$ then:

$$\int_a^b\int_a^b \frac{(f(x)+f(y))f'(x)f'(y)}{\sqrt{1+f(x)f(y)}}dxdy \leq log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right)^{f^2(b)-f^2(a)}$$

AN.160. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{8^n}{n(2n+1)^2} \cdot \prod_{k=1}^{n-1} sin\left(\frac{k\pi}{n}\right) \cdot \prod_{k=1}^{n} sin^2\left(\frac{k\pi}{2n+1}\right) \cdot \prod_{k=n+1}^{2n} sin\left(\frac{k\pi}{2n+1}\right)\right)$$

AN.161. Find:

$$\Omega = \lim_{n\to\infty}\left((1-i)\prod_{k=1}^{n}\frac{k^2+k+1+i}{\sqrt{(k^2+1)(k^2+2k+2)}}\right)$$

AN.162. If $\frac{2}{3} < a \leq b$ then:

$$\int_a^b x \cdot sin\frac{\pi}{3x}dx \geq \sqrt{1+b^2} - \sqrt{1+a^2}$$

AN.163. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\left(\sum_{k=1}^{n}k^3\binom{n}{k}^2\right)\left(\sum_{k=1}^{n}k^2\binom{n}{k}^2\right)^{-1}}$$

AN.164. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\frac{3^{i-j}}{4^i}\right)$$

AN.165. If $0<a\le b\le\frac{\pi}{2}$ then:

$$\int_a^b\int_a^b\int_a^b(\tan x\tan y+1)(\tan y\tan z+1)(\tan z\tan x+1)dxdydz$$
$$\le(\tan b-\tan a)^3$$

AN.166. If $0<a\le b$ then prove:

$$\frac{b^b}{a^a}\ge\left(e\sqrt{ab}\right)^{b-a}$$

AN.167. If $0<a\le b, f:[a,b]\to(0,\infty), f$ –continuous, then:

$$(b-a)\left(\int_a^b f^3(x)dx\right)\left(\int_a^b\frac{dx}{f^2(x)}\right)\ge\left(\int_a^b\sqrt[3]{f^5(x)}dx\right)\left(\int_a^b\frac{dx}{\sqrt[3]{f(x)}}\right)^2$$

AN.168. If $0<a\le b$ then prove:

$$\left(erf(b)-erf(a)\right)^2\le\frac{16}{3}\left(erf\left(\frac{3a+b}{4}\right)-erf(a)\right)\left(erf\left(\frac{a+3b}{4}\right)-erf(a)\right)$$

AN.169. Find without any software:

$$\Omega=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sin^2(7x)+\cos^2(10x)}{\sin^2x}dx$$

AN.170. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2}\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right)^{n}$$

AN.171 Find:

$$\Omega = \lim_{n\to\infty}\left(n^{n-2}\cdot\left(\frac{2}{3}\right)^{2}\cdot\left(\frac{3}{5}\right)^{3}\cdot\ldots\cdot\left(\frac{n}{2n-1}\right)^{n}\right)$$

SOLUTIONS

ALGEBRA

A.001. *WLOG, we suppose* $a+b+c=1$

$$Inequality \Leftrightarrow \frac{(a^2+1)(b^2+1)(c^2+1)}{abc} \geq \frac{1000}{27}$$

$$\Leftrightarrow 27(a^2+1)(b^2+1)(c^2+1) \geq 1000abc$$

$$a^2+1 = a^2+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9} \geq 10\sqrt[10]{a^2\left(\frac{1}{9}\right)^9}$$

$$b^2+1 = b^2+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9} \geq 10\sqrt[10]{b^2\left(\frac{1}{9}\right)^9}$$

$$c^2+1 = c^2+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9} \geq 10\sqrt[10]{c^2\left(\frac{1}{9}\right)^9}$$

$$\Rightarrow 27(a^2+1)(b^2+1)(c^2+1) \geq 27\cdot 1000\sqrt[10]{(abc)^2\left(\frac{1}{9}\right)^{27}}$$

$$We\ must\ show\ that\ 27\cdot 1000\sqrt[10]{(abc)^2\left(\frac{1}{9}\right)^{27}} \geq 1000abc$$

$$\Leftrightarrow (27)^{10}\cdot\frac{1}{9^{27}} \geq (abc)^8 \Leftrightarrow abc \leq \frac{1}{27}$$

It is true because:

$$1 = a+b+c \geq 3\sqrt[3]{abc} \Leftrightarrow \frac{1}{3} \geq \sqrt[3]{abc} \Leftrightarrow \frac{1}{27} \geq abc$$

A.002. *If* $x_1, x_2, \dots, x_n > 0$ *denote:*

$$H_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}};\ G_n = \sqrt[n]{\prod_{i=1}^{n} x_i};\ A_n = \frac{\sum_{i=1}^{n} x_i}{n}$$

By Syerpinsky's inequality:

$$H_n^{n-1}A_n \le G^n \quad (1)$$

$$\text{For } n = 3 \text{ in (1): } H_3^2 A_3 \le G^3$$

$$\left(\frac{3}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}}\right)^2 \cdot \frac{x_1+x_2+x_3}{3} \le x_1x_2x_3 \quad (2)$$

Replacing in (2): $x_1 = \log a\,; x_2 = \log b\,; x_3 = \log c$

$$\frac{9}{(\log_e a + \log_e b + \log_e c)^2} \cdot \frac{\log(abc)}{3} \le \log a \cdot \log b \cdot \log c$$

$$\log a \cdot \log b \cdot \log c\,(\log_e a + \log_e b + \log_e c)^2 \ge 3\log(abc)$$

Equality holds for $a = b = c = e$.

A.003. *Denote* $x = \log a\,; y = \log b\,; z = \log c \Rightarrow x, y, z > 0$

Inequality can be written:

$$3xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \le (x+y+z)^2$$

$$3xyz \cdot \frac{xy+yz+zx}{xyz} \le (x+y+z)^2$$

$$3(xy+yz+zx) \le x^2+y^2+z^2+2(xy+yz+zx)$$

$$x^2+y^2+z^2-xy-yz-zx \ge 0$$

$$2x^2+2y^2+2z^2-2xy-2yz-2zx \ge 0$$

$$(x-y)^2+(y-z)^2+(z-x)^2 \ge 0$$

Equality holds for $x = y = z \Leftrightarrow a = b = c$.

A.004. $x \ge 2; y \ge 2; z \ge 2 \Rightarrow x-2 \ge 0; y-2 \ge 0; z-2 \ge 0$

$$(x-2)(y-2)(z-2) = (xy-2x-2y+4)(z-2) =$$

$$= xyz-2xy-2xz+4x-2yz+4y+4z-8 \ge 0 \quad (1)$$

$$x \ge 2; y \le 3; z \le 3 \Rightarrow x-2 \ge 0; y-3 \le 0; z-3 \le 0$$

$$(x-2)(y-3)(z-3) = (xy-3x-2y+6)(z-3) =$$

$$= xyz-3xy-3xz+9x-2yz+6y+6z-18 \ge 0 \quad (2)$$

$$x \le 3; y \ge 2; z \ge 3 \Rightarrow x-3 \le 0; y-2 \ge 0; z-3 \ge 0$$

$$(x-3)(y-2)(z-3) = (xy-2x-3y+6)(z-3) =$$

$$= xyz - 3xy - 2xz + 6x - 3yz + 9y + 6z - 18 \geq 0 \quad (3)$$

$$x \leq 3; y \leq 3; z \geq 2 \Rightarrow x - 3 \leq 0; y - 3 \leq 0; z - 2 \geq 0$$

$$(x-3)(y-3)(z-2) = (xy - 3x - 3y + 9)(z-2) =$$

$$= xyz - 2xy - 3xz + 6x - 3yz + 6y + 9z - 18 \geq 0 \quad (4)$$

By adding (1); (2); (3); (4):

$$4xyz + 25(x+y+z) - 10(xy+yz+zx) - 62 \geq 0$$

$$4xyz + 25(x+y+z) \geq 10(xy+yz+zx) + 62$$

Equality holds for $x = 2; y = 2; z = 2$.

For $x = 2$ *equality can be written:*

$$4 \cdot 2yz + 25(2+y+z) = 10(2y + yz + 2z) + 62$$

$$8yz - 50 + 25(y+z) = 20(y+z) + 10yz + 62$$

$$5(y+z) = 2yz + 12$$

$$y \geq 2; z \leq 3 \Rightarrow (y-2)(z-3) \leq 0$$

$$yz - 3y - 2z + 6 \leq 0 \Rightarrow yz + 6 \leq 3y + 2z \quad (5)$$

$$y \leq 3; z \geq 2 \Rightarrow (y-3)(z-2) \leq 0$$

$$yz - 2y - 3z + 6 \leq 0 \Rightarrow yz + 6 \leq 2y + 3z \quad (6)$$

By adding (5); (6): $5(y+z) \geq 2yz + 12$

Equality holds for $y = 2; z = 2$ *or* $y = z = 3$.

Solutions are: $(2,2,2); (2,3,3)$ *and permutations.*

A.005.

$$\sum_{cyc} \frac{a}{a^2 - bc + 2019} = \sum_{cyc} \frac{a}{a^2 + ab + ac - 2018 + 2019} =$$

$$= \sum_{cyc} \frac{a}{a(a+b+c)+1} = \sum_{cyc} \frac{a^2}{a^2(a+b+c)+a} \geq$$

$$\overset{Bergstrom}{\geq} \frac{(a+b+c)^2}{(a+b+c)(a^2+b^2+c^2)+(a+b+c)} =$$

$$= \frac{a+b+c}{a^2+b^2+c^2+1} > \frac{1}{a+b+c} \Leftrightarrow$$

$$\Leftrightarrow (a+b+c)^2 > a^2+b^2+c^2+1$$

$$a^2+b^2+c^2+2(ab+bc+ca) > a^2+b^2+c^2+1$$

$$2(ab+bc+ca) > 1, \quad 2\cdot 2019 > 1, \quad 4038 > 1$$

A.006. $xy+yz+zx \overset{AM-GM}{\geq} 3\sqrt[3]{x^2y^2z^2} = 3\sqrt[3]{\frac{1}{64}} = \frac{3}{4}$

$$\textit{But } xy+yz+zx = \frac{3}{4} \Rightarrow xy = yz = zx$$

$$8xyz = 1 \Rightarrow xy = \frac{1}{8z}; yz = \frac{1}{8x}; zx = \frac{1}{8y}$$

$$\frac{1}{8z} = \frac{1}{8x} = \frac{1}{8y} \Rightarrow x = y = z$$

$$xyz = \frac{1}{8} \Rightarrow x\cdot x\cdot x = \frac{1}{8} \Rightarrow x^3 = \frac{1}{8} \Rightarrow x = \frac{1}{2} \Rightarrow y = z = \frac{1}{2}$$

A.007. *First, we prove that for $a, b > 0$ holds:*

$$\frac{(a^2+b^2)(ab+1)}{a+b} \geq 2ab \quad (1)$$

$$(a^2+b^2)(ab+1) \geq 2ab(a+b)$$

$$a^3b + a^2 + ab^3 + b^2 - 2a^2b - 2ab^2 \geq 0$$

$$a^3b + 2a^2b^2 + ab^3 + a^2 + b^2 - 2a^2b - 2ab^2 + 2a^2b^2 \geq 0$$

$$ab(a^2+2ab+b^2) + a^2(1-2ab+b^2) + b^2(1-2a+a^2) \geq 0$$

$$ab(a+b)^2 + a^2(1-b)^2 + b^2(1-a)^2 \geq 0$$

Analogous: $\frac{(b^2+c^2)(bc+1)}{b+c} \geq 2bc$ *(2),* $\frac{(c^2+a^2)(ca+1)}{c+a} \geq 2ca$ *(3)*

By adding (1); (2); (3):

$$\frac{(a^2+b^2)(ab+1)}{a+b} + \frac{(b^2+c^2)(bc+1)}{b+c} + \frac{(c^2+a^2)(ca+1)}{c+a} \geq$$

$$\geq 2(ab+bc+ca) = 2\cdot 3 = 6.$$ *Equality holds for $a = b = c$.*

A.008. $\log(1+2\sin^2 x)\cdot\log(1+2\cos^2 x) \overset{AM-GM}{\leq}$

$$\leq \left(\frac{\log(1+2\sin^2 x) + \log(1+2\cos^2 x)}{2}\right)^2 =$$

$$=\frac{1}{4}(\log(1+2\cos^2 x+2\sin^2 x+4\sin^2 x\cos^2 x))^2=$$

$$=\frac{1}{4}(\log(3+4\sin^2 x\cos^2 x))^2\overset{AM-GM}{\leq}$$

$$\leq\frac{1}{4}\left(\log\left(3+4\cdot\left(\frac{\sin^2 x+\cos^2 x}{2}\right)^2\right)\right)^2=\frac{1}{4}\left(\log\left(3+4\cdot\frac{1}{4}\right)\right)^2=\frac{1}{4}\log^2 4$$

$$=\log^2 2$$

$$\left(\prod_{cyc}\log(1+2\sin^2 x)\right)\left(\prod_{cyc}\log(1+2\cos^2 x)\right)=$$

$$=\prod_{cyc}(\log(1+2\sin^2 x)\cdot\log(1+2\cos^2 x))\leq$$

$$\leq\prod_{cyc}\log^2 2=\log^2 2\cdot\log^2 2\cdot\log^2 2=\log^6 2$$

A.009. $a+8+\frac{81}{a+8}\overset{AM-GM}{\geq}2\sqrt{(a+8)\cdot\frac{81}{a+8}}=2\sqrt{81}=18$

$$a+8+\frac{81}{a+8}\geq 18\Rightarrow a+\frac{81}{a+8}\geq 10$$

$$\sum_{cyc}\left(a+\frac{81}{a+8}\right)\geq 10\cdot 4,\qquad \sum_{cyc}a+81\sum_{cyc}\frac{1}{a+8}\geq 40$$

$$81\sum_{cyc}a+81\sum_{cyc}\frac{1}{a+8}-80\sum_{cyc}a\geq 40$$

$$81\left(\sum_{cyc}\left(a+\frac{1}{a+8}\right)\right)\geq 40+80\sum_{cyc}a\overset{AM-GM}{\geq}$$

$$\geq 40+80\cdot 4\sqrt[4]{abcd}=40+320=360$$

$$81\sum_{cyc}\left(a+\frac{1}{a+8}\right)\geq 360,\qquad \sum_{cyc}\left(a+\frac{1}{a+8}\right)\geq\frac{360}{81}=\frac{40}{9}$$

A.010. *Let be* $f:(0,\infty)\to\mathbb{R}, f(x)=x^2+\frac{1}{x}-\ln x$

$$f'(x) = 2x - \frac{1}{x^2} - \frac{1}{x} = \frac{2x^3 - 1 - x}{x^2} =$$
$$= \frac{2x^3 - 2x + x - 1}{x^2} = \frac{2x(x-1)(x+1) + (x-1)}{x^2} =$$
$$= \frac{(x-1)(2x^2 + 2x + 1)}{x^2}$$
$$f'(x) = 0 \Rightarrow x = 1 \Rightarrow \max f(x) = f(1) = 2$$
$$\Rightarrow x^2 + \frac{1}{x} - \ln x \geq 2 \Rightarrow x^4 + x - x^2 \ln x \geq 2x^2$$

For $x = a; x = b$ respectively $x = c$: $a^4 + a - a^2 \ln a \geq 2a^2$ *(1)*

$b^4 + b - b^2 \ln b \geq 2b^2$ *(2),* $c^4 + c - c^2 \ln b \geq 2c^2$ *(3)*

By adding (1); (2); (3):

$$a^4 + b^4 + c^4 + a + b + c - \ln\left(a^{a^2} \cdot b^{b^2} \cdot c^{c^2}\right) \geq 2(a^2 + b^2 + c^2)$$
$$a^4 + b^4 + c^4 + a + b + c \geq 2a^2 + 2b^2 + 2c^2$$
$$(a^4 - 2a^2 + a) + (b^4 - 2b^2 + b) + (c^4 - 2c^2 + c) \geq 0$$

Equality holds for $a = b = c = 1$.

A.011. Solution (George Florin Şerban)

$$\sum_{cyc} \frac{x^2 + x}{x^2 + x + 1} + 1 = 0 \Rightarrow \sum_{cyc} \left(\frac{x^2 + x}{x^2 + x + 1} + \frac{1}{3}\right) = 0 \Rightarrow$$
$$\sum_{cyc} \frac{4x^2 + 4x + 1}{x^2 + x + 1} = 0 \Rightarrow \sum_{cyc} \frac{(2x+1)^2}{x^2 + x + 1} \geq 0, \forall x \in \mathbb{R},$$
$$(2x+1)^2 \geq 0 \text{ and} \Rightarrow x^2 + x + 1 > 0, \Delta = -3 < 0$$
$$\begin{cases} (2x+1)^2 = 0 \\ (2y+1)^2 = 0 \\ (2z+1)^2 = 0 \end{cases} \Rightarrow x = y = z = -\frac{1}{2}$$

A.012. For $x, y, z \geq 2$ and $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$. We may write Inequality :

$$\frac{3x^2 + x + 4}{(x+1)(x^4+2)} + \frac{3y^2 + y + 4}{(y+1)(y^4+2)} + \frac{3z^2 + z + 4}{(z+1)(x^4+2)}$$
$$+ 2\left(\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}\right) \leq 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right);$$

$$\leftrightarrow \sum\left(\frac{3x^2+x+4}{(x+1)(x^4+2)}+\frac{2}{x+1}\right)\le 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right);$$

$$\leftrightarrow \sum\left(\frac{2x^4+3x^2+x+8}{(x+1)(x^4+2)}\right)\le \sum\frac{2}{x};\quad (*)$$

Hence, we must show that:

$$\frac{2x^4+3x^2+x+8}{(x+1)(x^4+2)}\le\frac{2}{x};\ \ (\ \forall\, x\ge 2\ )$$

$$\leftrightarrow\ x(2x^4+3x^2+x+8)\le 2(x+1)(x^4+2)$$

$$\leftrightarrow 2x^4-3x^3-x^2-4x+4\ge 0 \leftrightarrow (x-2)(2x^3+x^2+x-2)\ge 0$$

Which is true because:

$$x\ge 2\ \rightarrow x-2\ge 0;\ 2x^3+x^2+x-2\ge 16+4+2-2=20>0$$

Similary:

$$\frac{2y^4+3y^2+y+8}{(y+1)(y^4+2)}\le\frac{2}{y};\ \ (\ \forall\, y\ge 2\ ),\frac{2z^4+3z^2+z+8}{(z+1)(z^4+2)}\le\frac{2}{z};\ \ (\ \forall\, z\ge 2\ )$$

A.013. $3a^5+b^5+c^5\overset{AM-GM}{\ge} 5\cdot\sqrt[5]{(a^5)^3b^5c^5}=5a^3bc\quad (1)$

$$a^5+3b^5+c^5\overset{AM-GM}{\ge} 5\cdot\sqrt[5]{a^5(b^5)^3c^5}=5ab^3c\quad (2)$$

$$2a^5+2b^5+c^5\overset{AM-GM}{\ge} 5\cdot\sqrt[5]{(a^5)^2\cdot(b^5)^2\cdot c^5}=5a^2b^2c\quad (3)$$

$$2a^5+2b^5+c^5\overset{AM-GM}{\ge} 5\cdot\sqrt[5]{(a^5)^2\cdot(b^5)\cdot c^5}=5a^2b^2c\quad (4)$$

By adding (1); (2); (3); (4):

$$8a^5+8b^5+4c^5\ge 5abc(a^2+2ab+b^2)$$

$$2a^5+2b^5+c^5\ge\frac{5}{4}abc(a+b)^2,\qquad \frac{2(a^5+b^5)+c^5}{(a+b)^2}\ge\frac{5}{4}abc$$

$$\sum_{cyc}\frac{2(a^5+b^5)+c^5}{(a+b)^2}\ge\frac{5}{4}\sum_{cyc}abc=\frac{5}{4}\cdot 3abc=\frac{15}{4}\cdot 4=15$$

Equality holds for $a=b=c=\sqrt[3]{4}$

A.014. $(x^2+xy-y^2)^2+(xy+y^2)^2=$

$$=x^4+x^2y^2+y^4+2x^3y-2x^2y^2-2xy^3+x^2y^2+y^4+2xy^3=$$

$$=x^4+2x^3+2y^4$$

$$x^4+2x^3+2y^4=0\Rightarrow(x^2+xy-y^2)+(xy+y^2)^2=0$$

$$\Rightarrow \begin{cases} x^2 + xy - y^2 = 0 \\ y(x+y) = 0 \end{cases} \Rightarrow x = y = 0$$

$$x + 7 = 0 \Rightarrow y = -x \Rightarrow x^2 - x^2 - y^2 = 0 \Rightarrow y = 0$$

$3^0 + 4^0 = 5^z \Rightarrow 5^z = 2 \Rightarrow z = \log_5 2$. *Solution:* $(0,0,\log_5 2)$

A.015.

$$\left.\begin{matrix} x \le y \\ x \le z \\ x \le t \\ x \le x \end{matrix}\right\} \Rightarrow 4x \le x + y + z + t = 4 \Rightarrow 4x \le 4 \Rightarrow x = 1 \Rightarrow \max(x) = 1$$

$$\left.\begin{matrix} y \le z \\ y \le t \\ y \le y \end{matrix}\right\} \Rightarrow 3y \le y + z + t \le x + y + z + t = 4$$

$$\Rightarrow 3y \le 4 \Rightarrow y \le \frac{4}{3} \Rightarrow \max(y) = \frac{4}{3}$$

$$\left.\begin{matrix} z \le t \\ z \le z \end{matrix}\right\} \Rightarrow 2z \le t + z \le x + y + z + t = 4$$

$$\Rightarrow 2z \le 4 \Rightarrow z \le 2 \Rightarrow \max(z) = 2$$

$$t \le x + y + z + t = 4 \Rightarrow t \le 4 \Rightarrow \max(t) = 4$$

$$\begin{cases} x \cdot \frac{4}{3} + y \cdot 2 + z \cdot 4 = \frac{22}{3} \\ x \cdot 2 + y \cdot 4 + z \cdot 1 = 7 \\ x \cdot 4 + y \cdot 1 + z \cdot \frac{4}{3} = \frac{19}{3} \end{cases}$$

With solution: $x = y = z = 1$.

A.016. *First, we prove that:* $\frac{abc+bc+2a}{2bc+a+1} \ge \frac{2a}{a+1}; a, b, c > 0$ *(1)*

$$(abc + bc + 2a)(a + 1) \ge 2a(2bc + a + 1)$$

$$a^2bc + abc + 2a^2 + abc + bc + 2a > 4abc + 2a^2 + 2a$$

$$a^2bc - 2abc + bc \ge 0$$

$$bc(a^2 - 2a + 1) \ge 0; bc(a-1)^2 \ge 0$$

Analogous: $\frac{abc+ca+2b}{2ca+b+1} \ge \frac{2b}{b+1}$ *(2)* $\frac{abc+ab+2c}{2ab+c+1} \ge \frac{2c}{c+1}$ *(3)*

By adding (1); (2); (3):

$$\sum_{cyc}\frac{abc+bc+2a}{2bc+a+1}>2\left(\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\right)=$$

$$=2\left(3-\frac{1}{a+1}-\frac{1}{b+1}-\frac{1}{c+1}\right)=2\left(3-\frac{17}{6}\right)=\frac{1}{3}$$

Inequality is strict because if

$$a=b=c=1;\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{3}{2}\neq\frac{17}{6}$$

A.017. $3^{\cos x+\cos y+\cos z}=3^{\cos^2 x+\cos x}+3^{\cos^2 y+\cos y}+3^{\cos^2 z+\cos z}\geq$

$$\overset{AM-GM}{\geq}3\sqrt[3]{3^{\cos^2 x+\cos^2 y+\cos^2 z+\cos x+\cos y+\cos z}}$$

$$\left(3^{\sum_{cyc}\cos x}\right)^3\geq 27\cdot 3^{\sum_{cyc}\cos^2 x+\sum_{cyc}\cos x}$$

$$3\sum_{cyc}\cos x\geq 3+\sum_{cyc}\cos^2 x+\sum_{cyc}\cos x,\qquad \sum_{cyc}\cos^2 x-2\sum_{cyc}\cos x+3\leq 0$$

$$(\cos x-1)^2+(\cos y-1)^2+(\cos z-1)^2\leq 0$$

$$\Rightarrow\begin{cases}\cos x=1\\ \cos y=1\\ \cos z=1\end{cases}\Rightarrow\begin{cases}x=\pm\frac{\pi}{2}+2m\pi;m\in\mathbb{Z}\\ y=\pm\frac{\pi}{2}+2n\pi;n\in\mathbb{Z}\\ z=\pm\frac{\pi}{2}+2p\pi;p\in\mathbb{Z}\end{cases}$$

A.018. $(xz-yt)^2+(xz-yt)(xt+yz+yt)+(xt+yz+yt)^2=$

$$=(x^2+xy+y^2)(z^2+zt+t^2)\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{x^3\cdot y^3}\cdot 3\sqrt[3]{z^3t^3}=9xyzt$$

$$\frac{(xz-yt)^2+(xz-yt)(xt+yz+yt)+(xt+yz+yt)^2}{xyzt}\geq 9$$

Equality holds for: $x=y=z=t$

A.019. *First we prove that:* $2\left(x+y-\sqrt{xy}\right)^2\geq x^2+y^2$ *(1)*

$$2\left(x^2+y^2+xy-2x\sqrt{xy}-2y\sqrt{xy}+2xy\right)\geq x^2+y^2$$

$$x^2+y^2+6xy-4x\sqrt{xy}-4y\sqrt{xy}\geq 0$$

$$x^2+2xy+y^2+4xy-4\sqrt{xy}(x+y)\geq 0$$

$$(x+y)^2-4\sqrt{xy}(x+y)+4xy\geq 0,\ \left(x+y-2\sqrt{xy}\right)^2\geq 0\quad (True)$$

By (1): $\left(x+y-\sqrt{xy}\right)^2\geq \frac{x^2+y^2}{2}$ *(2)*

$$\sum_{cyc}\left(x+y-\frac{1}{\sqrt{z}}\right)^2=\sum_{cyc}\left(x+y-\sqrt{xy}\right)^2\overset{(2)}{\geq}$$

$$\geq\sum_{cyc}\frac{x^2+y^2}{2}=x^2+y^2+z^2\overset{AM-GM}{\geq}3\sqrt[3]{(xyz)^2}=3\cdot\sqrt[3]{1^2}=3$$

Equality holds for $x=y=z=1$.

A.020. $\left(a-\sqrt{bc}\right)^2\geq 0,\ a^2+bc\geq 2a\sqrt{bc}$

$$a^2+ac+ab+bc\geq ac+ab+2a\sqrt{bc},(a+b)(a+c)\geq a\left(b+c+2\sqrt{bc}\right)$$

$$(a+b)(a+c)\geq a\left(\sqrt{a}+\sqrt{c}\right)^2\ \ (1).\ \ Analogous:$$

$$(b+c)(b+a)\geq b\left(\sqrt{c}+\sqrt{a}\right)^2\ \ (2),\ \ (c+a)(c+b)\geq c\left(\sqrt{a}+\sqrt{b}\right)^2\ \ (3)$$

By multiplying (1); (2); (3):

$$\left(\prod_{cyc}(a+b)\right)^2\geq abc\left(\prod_{cyc}\left(\sqrt{a}+\sqrt{b}\right)\right)^2=\left(\prod_{cyc}\left(\sqrt{a}+\sqrt{b}\right)\right)^2$$

$\prod_{cyc}(a+b)\geq\prod_{cyc}\left(\sqrt{a}+\sqrt{b}\right)$ *(4). Replacing* a,b,c *with* $\sqrt{a},\sqrt{b},\sqrt{c}$ *in (4):*

$$\prod_{cyc}\left(\sqrt{a}+\sqrt{b}\right)\geq\prod_{cyc}\left(\sqrt[4]{a}+\sqrt[4]{b}\right)\ \ (5)$$

Replacing a,b,c *with* $\sqrt{a},\sqrt{b},\sqrt{c}$ *in (5):*

$$\prod_{cyc}\left(\sqrt[4]{a}+\sqrt[4]{b}\right)\geq\prod_{cyc}\left(\sqrt[8]{a}+\sqrt[8]{b}\right)\ \ (6)$$

By (4); (5); (6):

$$\prod_{cyc}(a+b)\geq\prod_{cyc}\left(\sqrt{a}+\sqrt{b}\right)\geq\prod_{cyc}\left(\sqrt[4]{a}+\sqrt[4]{b}\right)\geq\prod_{cyc}\left(\sqrt[8]{a}+\sqrt[8]{b}\right)$$

$$\prod_{cyc}\left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)\cdot\prod_{cyc}\left(\frac{a+b}{\sqrt[4]{a}+\sqrt[4]{b}}\right)\cdot\prod_{cyc}\left(\frac{a+b}{\sqrt[8]{a}+\sqrt[8]{b}}\right)\geq 1$$

A.021.

$$\frac{|x+y|}{1+|x+y|}=\frac{1+|x+y|-1}{1+|x+y|}=1-\frac{1}{1+|x+y|}\leq 1-\frac{1}{1+|x|+|y|}\leq$$

$$\leq 1-\frac{1}{1+|x|+|y|+|x|\cdot|y|}=\frac{|x|+|y|+|x|\cdot|y|}{(1+|x|)(1+|y|)}\leq$$

$$\leq\frac{|x|+|y|+2|x|\cdot|y|}{(1+|x|)(1+|y|)}=\frac{|x|(1+|y|)+|y|(1+|x|)}{(1+|x|)(1+|y|)}=$$

$$=\frac{|x|}{1+|x|}+\frac{|y|}{1+|y|}=\frac{1+|x|-1}{1+|x|}+\frac{1+|y|-1}{1+|y|}=2-\frac{1}{1+|x|}-\frac{1}{1+|y|}$$

$$\frac{|x+y|}{1+|x+y|}\leq 2-\frac{1}{1+|x|}-\frac{1}{1+|y|}\quad (1)$$

$$\frac{|x+y+z|}{1+|x+y+z|}=\frac{|(x+y)+z|}{1+|(x+y)+z|}\overset{(1)}{\leq}$$

$$\leq 2-\frac{1}{1+|x+y|}-\frac{1}{1+|z|}=1+\frac{|x+y|}{1+|x+y|}-\frac{1}{1+|z|}\overset{(1)}{\leq}$$

$$\leq 1+2-\frac{1}{1+|x|}-\frac{1}{1+|y|}-\frac{1}{1+|z|}$$

$$\frac{|x+y+z|}{1+|x+y+z|}+\frac{1}{1+|x|}+\frac{1}{1+|y|}+\frac{1}{1+|z|}\leq 3$$

Equality holds for $x=y=z=0$.

A.022. $c\geq a; c\geq b\Rightarrow c^2\geq ab\Rightarrow a^2b^2c^2\geq a^3b^3\Rightarrow$

$$\Rightarrow\sqrt[6]{(abc)^2}\geq\sqrt[6]{(ab)^3}\Rightarrow\sqrt[3]{abc}\geq\sqrt{ab}\Rightarrow\frac{\sqrt[3]{abc}}{\sqrt{ab}}\geq 1\quad (1)$$

$$d\geq a; d\geq b, d\geq c\Rightarrow d^3\geq abc\Rightarrow a^3b^3c^3d^3\geq a^4b^4c^4$$

$$\Rightarrow\sqrt[12]{(abcd)^3}\geq\sqrt[12]{(abc)^4}\Rightarrow\sqrt[4]{abcd}\geq\sqrt[3]{abc}\Rightarrow\frac{\sqrt[4]{abcd}}{\sqrt[3]{abc}}\geq 1\quad (2)$$

$$c=\frac{abc}{ab}=\frac{\left(\sqrt[3]{abc}\right)^3}{\left(\sqrt{ab}\right)^2}=\sqrt{ab}\cdot\left(\frac{\sqrt[3]{abc}}{\sqrt{ab}}\right)^3\overset{Bernoulli;(1)}{\geq}$$

$$\geq\sqrt{ab}\left(1+3\left(\frac{\sqrt[3]{abc}}{\sqrt{ab}}-1\right)\right)=\sqrt{ab}+3\sqrt[3]{abc}-3\sqrt{ab}=$$

$$=3\sqrt[3]{abc}-2\sqrt{ab},\quad 3\sqrt[3]{abc}-2\sqrt{ab}\leq c\quad (3)$$

$$d=\frac{abcd}{abc}=\frac{\left(\sqrt[4]{abcd}\right)^4}{\left(\sqrt[3]{abc}\right)^3}=\sqrt[3]{abc}\cdot\left(\frac{\sqrt[4]{abcd}}{\sqrt[3]{abc}}\right)^4\overset{Bernoulli;(2)}{\geq}$$

$$\geq \sqrt[3]{abc}\left(1+4\left(\frac{\sqrt[4]{abcd}}{\sqrt[3]{abc}}-1\right)\right)=\sqrt[3]{abc}+4\sqrt[4]{abcd}-4\sqrt[3]{abc}=$$

$$=4\sqrt[4]{abcd}-3\sqrt[3]{abc}\quad,\quad 4\sqrt[4]{abcd}-3\sqrt[3]{abc}\leq d\quad (4)$$

By multiplying (3); (4): $\left(3\sqrt[3]{abc}-2\sqrt{ab}\right)\left(4\sqrt[4]{abcd}-3\sqrt[3]{abc}\right)\leq cd$

$$\frac{3\sqrt[3]{abc}-2\sqrt{ab}}{d}\leq\frac{c}{4\sqrt[4]{abcd}-3\sqrt[3]{abc}}$$

Equality holds for $a=b=c=d$.

A.023. $\sqrt{f(a)f(b)}=f\left(\frac{a+b}{2}\right)\Rightarrow f(a)\cdot f(b)=f^2\left(\frac{a+b}{2}\right)$

$$f(a)f(b)f(c)f(d)=f^2\left(\frac{a+b}{2}\right)\cdot f^2\left(\frac{c+d}{2}\right)=$$

$$=\left(f\left(\frac{a+b}{2}\right)\cdot f\left(\frac{c+d}{2}\right)\right)^2=\left(f\left(\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2}\right)\right)^4=\left(f\left(\frac{a+b+c+d}{4}\right)\right)^4$$

$$f(a)f(b)f(c)f(d)=f^4\left(\frac{a+b+c+d}{4}\right)$$

We take $d=\frac{a+b+c}{3}$

$$f(a)f(b)f(c)f\left(\frac{a+b+c}{3}\right)=f^4\left(\frac{a+b+c+\frac{a+b+c}{3}}{4}\right)$$

$$f(a)f(b)f(c)f\left(\frac{a+b+c}{3}\right)=f^4\left(\frac{3(a+b+c)+a+b+c}{3\cdot 4}\right)$$

$$f(a)f(b)f(c)f\left(\frac{a+b+c}{3}\right)=f^4\left(\frac{a+b+c}{3}\right)$$

$$f(a)f(b)f(c)=f^3\left(\frac{a+b+c}{3}\right)$$

$$f\left(\frac{a+b+c}{3}\right)=\sqrt[3]{f(a)f(b)f(c)}\overset{AM-GM}{\leq}\frac{f(a)+f(b)+f(c)}{3}$$

$$3f\left(\frac{a+b+c}{3}\right)\leq f(a)+f(b)+f(c)$$

Equality holds for $a=b=c$.

A.024. $f(a)+f(b)+f(c)+f(d)=2f\left(\frac{a+b}{2}\right)+2f\left(\frac{c+d}{2}\right)=$

$$=2\cdot 2\cdot f\left(\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2}\right)=4f\left(\frac{a+b+c+d}{4}\right)$$

$$Let\ be\ d=\frac{a+b+c}{3}$$

$$f(a)+f(b)+f(c)+f\left(\frac{a+b+c}{3}\right)=4f\left(\frac{a+b+c+\frac{a+b+c}{3}}{4}\right)=$$

$$=4f\left(\frac{3(a+b+c)+(a+b+c)}{4\cdot 3}\right)=4f\left(\frac{a+b+c}{3}\right)$$

$$f(a)+f(b)+f(c)=4f\left(\frac{a+b+c}{3}\right)-f\left(\frac{a+b+c}{3}\right)$$

$$f(a)+f(b)+f(c)=3f\left(\frac{a+b+c}{3}\right)$$

$$3f\left(\frac{a+b+c}{3}\right)=f(a)+f(b)+f(c)\overset{AM-GM}{\geq}3\sqrt[3]{f(a)\cdot f(b)\cdot f(c)}$$

$$3\sqrt[3]{f(a)\cdot f(b)\cdot f(c)}\leq 3f\left(\frac{a+b+c}{3}\right)$$

$$f(a)\cdot f(b)\cdot f(c)\leq f^3\left(\frac{a+b+c}{3}\right)$$

A.025. $(a-b)^2(ab+a+b)\geq 0,\ ab(a-b)^2+(a-b)^2(a+b)\geq 0$

$$ab(a-b)^2+(a^2-b^2)(a-b)\geq 0$$

$$ab(a^2-2ab+b^2)+a^3-a^2b-ab^2+b^3\geq 0$$

$$a^3b-2a^2b^2+ab^3+a^3-a^2b-ab^2+b^3\geq 0$$

$$a^3b+ab^3+a^3+b^3\geq a^2b+ab^2+a^2b^2+a^2b^2$$

$$a^3(b+1)+b^3(a+1)\geq a^2(b+b^2)+b^2(a+a^2)$$

$$\frac{a^3(b+1)+b^3(a+1)}{a^2b(b+1)+ab^2(a+1)}\geq 1.\ Multiplying\ with\ \frac{1}{c}:$$

$$\frac{a^3(b+1)+b^3(a+1)}{a^2bc(b+1)+ab^2c(a+1)}\geq\frac{1}{c}$$

$$\sum_{cyc}\frac{a^3(b+1)+b^3(a+1)}{abc(ab+a+ba+b)}\geq\sum_{cyc}\frac{1}{c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$$

$$\frac{1}{abc}\sum_{cyc}\frac{a^3(b+1)+b^3(a+1)}{a+2ab+b}\geq 3,\qquad \sum_{cyc}\frac{a^3(b+1)+b^3(a+1)}{a+2ab+b}\geq 3abc$$

Equality holds for $a=b=c=1.$

A.026. $a=\sqrt[5]{x^2-5x+4}, b=\sqrt[5]{2+x-x^2}, a^5+b^5=6-4x$

$$\sqrt[5]{x^2-5x+4}+\sqrt[5]{2+x-x^2}=\sqrt[5]{6-4x}$$

$$a+b=\sqrt[5]{a^5+b^5}\Rightarrow(a+b)^5=a^5+b^5\Longrightarrow(a+b)^5-a^5-b^5=0$$
$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5-a^5-b^5=0$$
$$5ab(a^3+2a^2b+2ab^2+b^3)=0$$
$$5ab(a^2(a+b)+ab(a+b)+b^2(a+b))=0$$
$$5ab(a+b)(a^2+ab+b^2)=0$$
$$a=0\Longrightarrow x^2-5x+4=0\Longrightarrow x_1=1, x_2=4$$
$$b=0\Rightarrow 2+x-x^2=0, x_3=-1, x_4=2$$
$$a+b=0\Rightarrow x^2-5x+4=x^2-x-2\Longrightarrow x_5=\frac{3}{2}$$
$$a^2+ab+b^2=\left(a+\frac{b}{2}\right)^3+\frac{3b^2}{4}\neq 0$$

A.027. $(a-b)^2(ab+a+b)\geq 0,\ ab(a-b)^2+(a-b)^2(a+b)\geq 0$

$$ab(a-b)^2+(a^2-b^2)(a-b)\geq 0$$

$$ab(a^2-2ab+b^2)+a^3-a^2b-b^2a+b^3\geq 0$$

$$a^3b+ab^3+a^3+b^3\geq 2a^2b^2+a^2b+b^2a$$

$$a^3(b+1)+b^3(a+1)\geq a^2(b+b^2)+b^2(a+a^2)$$

$\frac{a^3(b+1)+b^3(a+1)}{a^2b(1+b)+b^2a(1+a)}\geq 1.$ *Multiplying with c:*

$$\frac{a^3c(b+1)+b^3c(a+1)}{a^2b(b+1)+ab^2(a+1)}\geq c$$

$$\sum_{cyc}\frac{a^3c(b+1)+b^3c(a+1)}{a^2b(b+1)+ab^2(a+1)}\geq a+b+c=3$$

Equality holds for $a=b=c.$

A.028.

$$(2y+z)(2z+y)\overset{AM-GM}{\leq}\left(\frac{2y+z+2z+y}{2}\right)^2=\left(\frac{3(y+z)}{2}\right)^2=\frac{9}{4}(y+z)^2$$

$$\frac{1}{(2y+z)(2z+y)}\geq\frac{4}{9(y+z)^2}\Rightarrow\frac{1}{(2y+z)^2(2z+y)^2}\geq\frac{16}{81(y+z)^4}$$

$$\frac{x^4}{(2y+z)^2(2z+y^2)}\geq\frac{16x^4}{81(y+z)^4}$$

$$\sum_{cyc}\frac{x^4}{(2y+z)^2(2z+y)^2}\geq\frac{16}{81}\sum_{cyc}\left(\frac{x^2}{(y+z)^2}\right)^2\overset{CBS}{\geq}$$

$$\geq\frac{16}{81}\cdot\frac{1}{3}\left(\sum_{cyc}\left(\frac{x}{y+z}\right)^2\right)^2\overset{CBS}{\geq}\frac{16}{81\cdot 3}\cdot\left(\frac{1}{3}\left(\sum_{cyc}\frac{x}{y+z}\right)^2\right)^2\geq$$

$$\overset{NESSBIT}{\geq}\frac{16}{81\cdot 3}\cdot\frac{1}{9}\cdot\left(\frac{3}{2}\right)^4=\frac{16}{81\cdot 27}\cdot\frac{81}{16}=\frac{1}{27}$$

Equality holds for $x=y=z=1$.

A.029. *First, we prove that:* $\frac{9abc}{ab+bc+ca}-\frac{4ab}{a+b}\leq c \quad (1)$

$$\frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}-\frac{4}{\frac{1}{a}+\frac{1}{b}}\leq c.\text{ Denote }\frac{1}{a}+\frac{1}{b}=y;\frac{1}{c}=x$$

$$\frac{9}{y+x}-\frac{4}{y}\leq\frac{1}{x}\Leftrightarrow 9xy-4x(x+y)\leq y(x+y)$$

$$9xy-4x^2-4xy\leq xy+y^2\Leftrightarrow y^2+4x^2-4xy\geq 0\Leftrightarrow(y-2x)^2\geq 0$$

Second, we prove that:

$$\frac{16abcd}{bcd+cda+dab+abc}-\frac{9abc}{ab+bc+ca}\leq d \quad (2)$$

$$\frac{16}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}-\frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\leq d.\text{ Denote }\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=z;\frac{1}{d}=t$$

$$\frac{16}{z+t}-\frac{9}{z}\leq\frac{1}{t},\qquad 16zt-9t(z+t)\leq z(z+t)$$

$$16zt-9tz-9t^2\leq z^2+zt,\qquad z^2+9t^2-6tz\geq 0\Leftrightarrow(z-3t)^2\geq 0$$

By multiplying (1); (2):

$$\left(\frac{9abc}{ab+bc+ca}-\frac{4ab}{a+b}\right)\left(\frac{16abcd}{bcd+cda+dab+abc}-\frac{9abc}{ab+bc+ca}\right)\le cd$$

$$\frac{\frac{9abc}{ab+bc+ca}-\frac{4ab}{a+b}}{d}\le\frac{c}{\frac{16abcd}{bcd+cda+dab+abc}-\frac{9abc}{ab+bc+ca}}$$

Equality holds for $a=b=c=d$.

A.030. *Denote $x=a^3; y=b^3; ab\ge 0$. We prove that:*

$$\sqrt[3]{x}+\sqrt[3]{y}\le\sqrt[3]{4(x+y)}\quad (1)$$

$$\sqrt[3]{a^3}+\sqrt[3]{b^3}\le\sqrt[3]{4(a^3+b^3)}\le a+b\le\sqrt[3]{4(a^3+b^3)}\Leftrightarrow$$

$$\Leftrightarrow (a+b)^3\le 4(a^3+b^3)$$

$$4(a+b)(a^2-ab+b^2)-(a+b)^3\ge 0$$

$$(a+b)(4a^2-4ab+4b^2-a^2-b^2-ab)\ge 0$$

$$(a+b)(3a^2-6ab+3b^2)\ge 0,\qquad 3(a+b)(a-b)^2\ge 0$$

By (1): $\sqrt[3]{x}+\sqrt[3]{y}\le\sqrt[3]{4(x+y)},\ \sqrt[3]{z}+\sqrt[3]{t}\le\sqrt[3]{4(z+t)}$ *(2)*

By adding (1); (2):

$$\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+\sqrt[3]{t}\le\sqrt[3]{4}\left(\sqrt[3]{x+y}+\sqrt[3]{z+t}\right)\overset{(1)}{\le}$$

$$\le\sqrt[3]{4}\cdot\sqrt[3]{4(x+y+z+t)}=\sqrt[3]{16(x+y+z+t)}$$

Equality holds for $x=y=z=t$.

A.031. *First, we prove that if $a,b\ge 0$ then:*

$$\sqrt[3]{a}+\sqrt[3]{b}\le\sqrt[3]{4(a+b)}\quad (1)$$

Denote $u=\sqrt[3]{a}; v=\sqrt[3]{b}\Rightarrow u^3=a; v^3=b$

$$\sqrt[3]{u^3}+\sqrt[3]{v^3}\le\sqrt[3]{4(u^3+v^3)},\qquad u+v\le\sqrt[3]{4(u^3+v^3)}$$

$$(u+v)^3\le 4(u^3+v^3)$$

$$4(u+v)(u^2-uv+v^2)-(u+v)^3\ge 0$$

$$(u+v)(4u^2-4uv+4v^2-u^2-2uv-v^2)\ge 0$$

$$(u+v)(3u^2+3v^2-6uv)\ge 0,\qquad 3(u+v)(u^2+v^2-2uv)\ge 0$$

$$3(u+v)(u-v)^2\ge 0\ \ (true)$$

Replacing $a = \log x \geq 0; b = \log y \geq 0$ *in (1):*

$$\sqrt[3]{\log x} + \sqrt[3]{\log y} \leq \sqrt[3]{4(\log x + \log y)} = \sqrt[3]{4\log(xy)} \quad (2)$$

Analogous: $\sqrt[3]{\log y} + \sqrt[3]{\log z} \leq \sqrt[3]{4\log(yz)}$ *(3)*

$$\sqrt[3]{\log z} + \sqrt[3]{\log x} \leq \sqrt[3]{4\log(zx)} \quad (4)$$

By multiplying (2); (3); (4):

$$\left(\sqrt[3]{\log x} + \sqrt[3]{\log y}\right)\left(\sqrt[3]{\log y} + \sqrt[3]{\log z}\right)\left(\sqrt[3]{\log z} + \sqrt[3]{\log x}\right)$$

$$\leq 4\sqrt[3]{\log(xy) \cdot \log(yz) \cdot \log(zx)}$$

Equality holds for $x = y = z$.

A.032. *First, we prove that:* $\dfrac{\frac{a+b+\sqrt{ab}}{3}}{\frac{a+\sqrt{ab}}{2}} \geq 1$ *(1)*

$$2\left(a + b + \sqrt{ab}\right) \geq 3\left(a + \sqrt{ab}\right), \qquad 2a + 2b + 2\sqrt{ab} \geq 3a + 3\sqrt{ab}$$

$2b \geq a + \sqrt{ab}$. *But* $b \geq \sqrt{ab}$ *because* $b^2 \geq ab \Leftrightarrow b \geq a$

$$b \geq \sqrt{ab}; b \geq a \Rightarrow 2b \geq a + \sqrt{ab}$$

$$\frac{\left(\frac{a+b+\sqrt{ab}}{3}\right)^3}{\left(\frac{a+\sqrt{ab}}{2}\right)^2} = \frac{a+\sqrt{ab}}{2} \cdot \left(\frac{\frac{a+b+\sqrt{ab}}{3}}{\frac{a+\sqrt{ab}}{2}}\right)^3 \overset{Bernoulli;(1)}{\geq}$$

$$\geq \frac{a+\sqrt{ab}}{2}\left(1 + 3\left(\frac{\frac{a+b+\sqrt{ab}}{3}}{\frac{a+\sqrt{ab}}{2}} - 1\right)\right) =$$

$$= \frac{a+\sqrt{ab}}{2}\left(\frac{a+b+\sqrt{ab}}{\frac{a+\sqrt{ab}}{2}} - 2\right) = a + b + \sqrt{ab} - a - \sqrt{ab} = b$$

$$\frac{\left(\frac{a+b+\sqrt{ab}}{3}\right)^3}{\left(\frac{a+\sqrt{ab}}{2}\right)^2} \geq b \Rightarrow \frac{\left(a+b+\sqrt{ab}\right)^3}{27} \cdot \frac{4}{\left(a+\sqrt{ab}\right)^2} \geq b$$

$$\frac{\left(a+b+\sqrt{ab}\right)^3}{b\left(a+\sqrt{ab}\right)^2} \geq \frac{27}{4} \Rightarrow \frac{b\left(a+\sqrt{ab}\right)^2}{\left(a+b+\sqrt{ab}\right)^3} \leq \frac{4}{27}$$

A.033. $Tr(AB) = 13; \det(AB) = 42 - 40 = 2$

$$f_{AB}(X) = X^2 - \left(Tr(AB)\right)X + \det(AB) = X^2 - 13X + 2$$

$$f_{BA}(X) = f_{AB}(x) = X^2 - 13X + 2$$

$$f_{BA}(BA) = O_2 \Rightarrow (BA)^2 - 13(BA) + 2I_2 = O_2 \quad (1)$$

Multiplying (1) with $(BA)^{-1}$*:*

$$BA - 13I_2 + 2(BA)^{-1} = O_2 \Rightarrow BA = 2A^{-1}B^{-1} = 13I_2$$

$$\det(BA + 2A^{-1}B^{-1}) = \det(13I_2) = 13^2 = 169 \quad (2)$$

$$Tr(CD) = 13; \det(CD) = 2$$

$$f_{CD}(X) = X^2 - \left(Tr(CD)\right)X + \det(CD) = X^2 - 13X + 2$$

$$f_{DC}(X) = f_{CD}(X) = X^2 - 13X + 2$$

$$f_{DC}(DC) = O_2 \Rightarrow (DC)^2 - 13(DC) + 2I_2 = O_2 \quad (3)$$

Multiplying (3) with $(OC)^{-1}$*:*

$$DC - 13I_2 + 2(DC)^{-1} = O_2 \Rightarrow DC + 2C^{-1}D^{-1} = 13I_2$$

$$\det(DC + 2C^{-1}D^{-1}) = \det(13I_2) = 13^2 = 169 \quad (4)$$

By (3); (4) $\Rightarrow$

$$\det(BA + 2A^{-1}B^{-1}) = \det(CD + 2C^{-1}D^{-1})$$

A.034. Solution (Bedri Hajrizi)

Let $\sum x = 11 \quad (1); \ \prod x = 36 \quad (3)$

From second questions (+(1),(3)) we get: $-\sum \frac{y^2z^2+36^2}{36(x-y)(z-x)} = 1$

$$-36\prod(x-y) = y^3z^2 - y^2z^3 + 36^2y - 36^2z + z^3x^2 - z^2x^3 + 36^2z - 36^2x + x^3y^2 - x^2y^3 + 36^2x - 36^2y$$

$$-36\prod(x-y) = x^2z^2(z-x) + y^3(z+x)(z-x) - y^2(z^3-x^3)$$

$$-36\prod(x-y) = -(z-x)(y-z)(x-y)(xy+yz+zx)$$

$$-36\prod(x-y)=-\prod(x-y)\left(\sum xy\right)$$

If $x=y$ *or* $y=z$ *or* $z=x$ *no real solution!*

System become $\begin{cases} x+y+z=11 \\ xy+yz+zx=36 \\ xyz=36 \end{cases}$; x,y,z *solution of the equation:*

$$t^3-11t^2+36t-36=0$$

$$(t-2)(t-3)(t-6)=0$$

Solutions are $(2,3,6)$ *and permutations.*

A.035. *Lemma: If* $X=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in M_2(\mathbb{C})$ *and* $\bar{X}^T\cdot X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *then* $a+b+c+d\ge 0$; $X^T\cdot X=\begin{pmatrix} \bar{p} & \bar{q} \\ \bar{r} & \bar{s} \end{pmatrix}^T\cdot\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} \bar{p} & \bar{r} \\ \bar{q} & \bar{s} \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=$

$$=\begin{pmatrix} \bar{p}p+\bar{r}r & \bar{p}q+\bar{r}s \\ \bar{q}p+\bar{s}r & \bar{q}q+\bar{s}s \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$a+b+c+d=\bar{p}p+\bar{r}r+\bar{p}q+\bar{r}s+\bar{q}p+\bar{s}r+q\bar{q}+s\bar{s}=$$

$$=p(\bar{p}+\bar{q})+q(\bar{p}+\bar{q})+r(\bar{r}+\bar{s})+s(\bar{r}+\bar{s})=$$

$$=(\bar{p}+\bar{q})(p+q)+(\bar{r}+\bar{s})(r+s)=$$

$$=\overline{p+q}\cdot(p+q)+\overline{r+s}\cdot(r+s)=(p+q)^2+(r+s)^2\ge 0$$

Back to the problem: Denote $X=X_1X_2\cdot\ldots\cdot X_n\in M_2(\mathbb{C})$

By lemma if $(\bar{X})^T\cdot X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Rightarrow a+b+c+d\ge 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}=X^T\cdot X=(\overline{X_1X_2\cdot\ldots\cdot X_n})^T\cdot(X_1X_2\cdot\ldots\cdot X_n)=$$

$$=(\bar{X}_1\cdot\bar{X}_2\cdot\ldots\cdot\bar{X}_n)^T\cdot(X_1X_2\cdot\ldots\cdot X_n)=$$

$$=(\bar{X}_n)^T\cdot(\bar{X}_{n-1})^T\cdot\ldots\cdot(\bar{X}_2)^T\cdot(\bar{X}_1)^T\cdot X_1\cdot X_2\cdot\ldots\cdot X_n$$

$$\Rightarrow a+b+c+d\ge 0$$

A.036. *Let be* $x=\frac{1}{y}$. *The equation can be written:*

$$\frac{3}{y^6}-\frac{9}{y^5}+\frac{18}{y^4}-\frac{21}{y^3}+\frac{15}{y^2}-\frac{6}{y}+1=0$$

$$y^6-6y^5+15y^4-21y^3+18y^2-9y+3=0$$

$$(y^6-6y^5+15y^4-20y^3+15y^2-6y+1)-$$

$$-(y^3-3y^2+3y-1)+1=0,\qquad (y-1)^6-(y-1)^3+1=0$$

$$Denote\ (y-1)^3=z\Rightarrow z^2-z+1=0$$

$$z_1,z_2=\frac{1\pm i\sqrt{3}}{2}=\cos\frac{\pi}{3}\pm i\sin\frac{\pi}{3}$$

$$y-1\in\{z_0,z_1,z_2\};z_k=\frac{\frac{\pi}{3}+2k\pi}{3}+i\sin\frac{\frac{\pi}{3}+2k\pi}{3}$$

$$y-1\in\{z_0',z_1',z_2'\};z_k'=\cos\frac{\frac{5\pi}{3}+2k\pi}{3}+i\sin\frac{\frac{5\pi}{3}+2k\pi}{3}$$

$$k\in\{0,1,2\}$$

$$y\in\{z_0+1,z_1+1,z_2+1,z_0'+1,z_1+1,z_2'+1\}$$

$$x\in\left\{\frac{1}{z_0+1};\frac{1}{z_1+1};\frac{1}{z_2+1};\frac{1}{z_0'+1};\frac{1}{z_1'+1};\frac{1}{z_2'+1}\right\}$$

A.037. $y^2+2\overset{AM-GM}{\geq}3\sqrt[3]{y^3\cdot1\cdot1}=3y$

$$\tfrac{1}{y^3+2}\geq\tfrac{1}{3y}.\ Replace\ y\ with\ t+z$$

$$\frac{1}{(t+z)^3+2}\leq\frac{1}{3(t+z)}$$

$$6\sum_{cyc}\frac{1}{(t+z)^3+z}\leq6\cdot\frac{1}{3}\sum_{cyc}\frac{1}{t+z}\leq2\cdot\frac{1}{4}\sum_{cyc}\left(\frac{1}{t}+\frac{1}{z}\right)=\frac{1}{2}\cdot2\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)$$

$$Equality\ holds\ for\ y=z=t=\tfrac{1}{2}$$

$$[x]\cdot(x-[x])+2=x^2+2$$

$$Denote\ a=[x];b=\{x\}=x-[x],\ ab=(a+b)^2\Rightarrow a^2+ab+b^2=0$$

$$\Rightarrow\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}=0\Rightarrow b=a=0\Rightarrow x=0$$

$$Sol\begin{cases}x=0\\ y=z=t=\frac{1}{2}\end{cases}$$

A.038. $ab+bc+ca=12abc-4abc(a+b+c)$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=12-4(a+b+c), \qquad 4(a+b+c)+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=12$$

$$\sum_{cyc}\frac{a}{4a^2+2a+1}\le\frac{1}{2}\Leftrightarrow 12\sum_{cyc}\frac{a}{4a^2+2a+1}\le 6$$

$$12\sum_{cyc}\frac{a}{4a^2+2a+1}\le 12-6\Leftrightarrow$$

$$12\sum_{cyc}\frac{a}{4a^2+2a+1}\le 4(a+b+c)+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-6$$

$$\Leftrightarrow\sum_{cyc}\left(4a+\frac{1}{a}\right)\ge 6+\sum_{cyc}\frac{12a}{4a^2+2a+1}$$

$$\sum_{cyc}\left(\frac{4a^2+1}{a}\right)\ge\sum_{cyc}\left(\frac{12a}{4a^2+2a+1}+2\right)$$

We prove that: $\frac{4a^2+1}{a}\ge\frac{12a}{4a^2+2a+1}+2\Leftrightarrow\frac{4a^2+1}{a}+2\ge\frac{12a}{4a^2+2a+1}+4\Leftrightarrow$

$$\Leftrightarrow(4a^2+2a+1)^2\ge 12^2+4a(4a^2+2a+1)\Leftrightarrow(4a^2-1)^2\ge 0$$

Equality holds for $a=b=c=\frac{1}{2}$.

A.039.

$$\begin{vmatrix}1 & 1 & 1 & 1\\ \sqrt{x} & \sqrt[3]{x} & \sqrt[4]{x} & 2\\ (\sqrt{x})^2 & \left(\sqrt[3]{x^2}\right)^2 & (\sqrt[4]{x})^2 & 4\\ (\sqrt{x})^4 & (\sqrt[2]{x})^4 & (\sqrt[4]{x})^4 & 16\end{vmatrix}=0$$

$$\left(\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}+2\right)Vand\left(\sqrt{x},\sqrt[3]{x},\sqrt[4]{x},2\right)=0$$

$$\sqrt{x}=2\Rightarrow x=4, \qquad \sqrt[3]{x}=2\Rightarrow x=8, \qquad \sqrt[4]{x}=2\Rightarrow x=16$$

$\sqrt{x}=\sqrt[3]{x}\Rightarrow x^6=x^2\Rightarrow x\in\{0,1\}$. Observation: $x\ge 0$. $S=\{0,1,4,8,16\}$

A.040. $\left(a-(\sqrt{3}-1)\right)^2\ge 0\Rightarrow a^2-2a(\sqrt{3}-1)+(\sqrt{3}-1)^2\ge 0$

$$a^2+2a-2\sqrt{3}(a+1)+4\ge 0\Rightarrow\frac{a^2+2a+4}{a+1}\ge 2\sqrt{3}$$

$$\frac{a^2+2}{a+1}\ge 2\sqrt{3}-2\Rightarrow\frac{a^2+2}{a+1}\ge 2(\sqrt{3}-1), \qquad \frac{b(a^2+2)}{a+1}\ge 2(\sqrt{3}-1)b$$

$$\sum_{cyc}\frac{b(a^2+2)}{a+1}\geq 2(\sqrt{3}-1)\sum_{cyc}b=$$

$$=2(\sqrt{3}-1)(a+b+c)=2(\sqrt{3}-1)\cdot 3(\sqrt{3}-1)=$$

$$=6(3+1-2\sqrt{3})=12(2-\sqrt{3}).\ \textit{Equality holds for } a=b=c=\sqrt{3}-1$$

A.041.

$$\begin{vmatrix}1 & 1 & 1 & 1\\ e^x & e^{2x} & e^{3x} & 2\\ (e^x)^3 & (e^{2x})^3 & (e^{3x})^3 & 2^3\\ (e^x)^4 & (e^{2x})^4 & (e^{3x})^4 & 2^4\end{vmatrix}=0$$

$$\Rightarrow (e^x\cdot e^{2x}+e^x\cdot e^{3x}+e^x\cdot 2+e^{2x}\cdot e^{3x}+2e^{2x}+2e^{3x})\cdot$$

$$\cdot\, Vand\,(e^x,e^{2x},e^{2x},2)=0,\quad e^x=2\Rightarrow x=\log 2$$

$$e^{2x}=2\Rightarrow 2x=\log 2\Rightarrow x=\log\sqrt{2},e^{3x}=2\Rightarrow 3x=\log 2\Rightarrow x=\log\sqrt[3]{2}$$

$$e^x=e^{2x}\Rightarrow x=0$$

$$e^{3x}+e^{4x}+2e^x+e^{5x}+2e^{2x}+2e^{3x}\neq 0$$

$$S=\{0,\log 2,\log\sqrt{2},\log\sqrt[3]{2}\}$$

A.042. Solution (Tran Hong)

$$x\neq y;y\neq z;z\neq x,x,y,z\neq 0$$

$$xy+yz+zx=26;xyz=24$$

$$\frac{48+yz(y+z)}{(x-y)(x-z)}+\frac{48+zx(z+x)}{(y-x)(y-z)}+\frac{48+xy(x+y)}{(z-x)(z-y)}=9\Leftrightarrow$$

$$48\left(\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}\right)+$$

$$+\left(\frac{yz(y+z)}{(x-y)(x-z)}+\frac{zx(z+x)}{(y-x)(y-z)}+\frac{xy(x+y)}{(z-x)(z-y)}\right)=9\Leftrightarrow$$

$$48\left(\frac{-(y-z)-(z-x)-(x-y)}{(x-y)(y-z)(z-x)}\right)+$$

$$+\left(\frac{yz(y+z)}{(x-y)(x-z)}+\frac{zx(z+x)}{(y-x)(y-z)}+\frac{xy(x+y)}{(z-x)(z-y)}\right)=9$$

$$\left(\frac{-yz(y+z)(y-z)-zx(z+x)(z-x)-xy(x+y)(x-y)}{(x-y)(y-z)(z-x)}\right)=9$$

$$yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)=-9(x-y)(y-z)(z-x)$$

$$x^3y+y^3z+z^3x-(xy^3+yz^3+zx^3)$$
$$+9(x^2z+z^2y+y^2x-xz^2-zy^2-yx^2)=0$$

$$(x-y)(y-z)(z-x)(x+y+z-9)=0\overset{x\neq y;y\neq z;z\neq x}{\Longleftrightarrow}x+y+z-9=0$$

$$x+y+z=9$$

So, by Viete's Theorem: $X^3-9X^2+26X-24=0$

$(x;y;z)=(2;3;4)$ *and cyclic.*

A.043. Solution (Florentin Vişescu)

$$\begin{cases}\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1\\ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{x}{z}+\frac{y}{x}+\frac{z}{y}+2=0\end{cases}\Rightarrow\begin{cases}\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1\\ x\left(\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{z}+\frac{1}{x}\right)+z\left(\frac{1}{x}+\frac{1}{y}\right)+2=0\end{cases}\Rightarrow$$

$$\begin{cases}xy+yz+zx=xyz\\ x\left(1-\frac{1}{x}\right)+y\left(1-\frac{1}{y}\right)+z\left(1-\frac{1}{z}\right)+2=0\end{cases}$$

$$\Rightarrow\begin{cases}xy+yz+zx=xyz=c;c\in\mathbb{C}^*\\ x-1+y-1+z-1+2=0\end{cases}$$

$$\Rightarrow\begin{cases}xy+yz+zx=xyz=c;c\in\mathbb{C}^*\\ x+y+z=1\end{cases}$$

But: x,y,z —are solutions of the equation: $t^3-t^2+ct-c=0\Leftrightarrow$

$$(t-1)(t^2+c)=0\ \Leftrightarrow t_1=1;t_2=i\sqrt{|c|}=k\in\mathbb{C}^*;t_3=-i\sqrt{|c|}=-k$$

So, $(x,y,z)\in\{(1;k;-k)_{cyc}\ \mid\ k\in\mathbb{C}^*\}$

A.044. *The equation can be written:*

$18x^6-24x^4+18x^2-1=\frac{\sqrt{3}}{2}$. *Let be* $f(x)=18x^6-24x^4+18x^2-1$

$$f(\cos a)=18\cos^6 a-24\cos^4 a+18\cos^2 a-1=\frac{\sqrt{3}}{2},\qquad \cos 6a=\cos\frac{\pi}{6}$$

$$6a=\frac{\pi}{6}+2k\pi\Rightarrow a=\frac{\pi}{36}+\frac{k\pi}{3};k\in\overline{0,5}\ \Rightarrow x_k=\cos\left(\frac{\pi}{36}+\frac{k\pi}{3}\right);k\in\overline{0,5}$$

A.045. *Let be* $f:(0,\infty)\to\mathbb{R}; f(x)=\log(x+1)$

$$f'(x)=\frac{1}{x+1}; f''(x)=\frac{-1}{(x+1)^2}<0; f\ concave$$

$$f\left(\frac{1}{2}a+\frac{1}{3}b+\frac{1}{6}a\right)\ge\frac{1}{2}f(a)+\frac{1}{3}f(b)+\frac{1}{6}(c)$$

$$\log\left(\frac{3a+2b+c}{6}+1\right)\ge\frac{1}{2}\log(a+1)+\frac{1}{3}\log(b+1)+\frac{1}{6}\log(c+1)$$

$$\log\left(\frac{3a+2b+c+6}{6}\right)\ge\log\left((a+1)^{\frac{1}{2}}\cdot(b+1)^{\frac{1}{3}}\cdot(c+1)^{\frac{1}{6}}\right)$$

$$\frac{3a+2b+c+6}{6}\ge(a+1)^{\frac{1}{2}}\cdot(b+1)^{\frac{1}{3}}\cdot(c+1)^{\frac{1}{6}}$$

$$\prod_{cyc}\frac{3a+2b+c+6}{6}\ge\prod_{cyc}(a+1)^{\frac{1}{2}}(b+1)^{\frac{1}{3}}(c+1)^{\frac{1}{6}}$$

$$\frac{1}{216}\prod_{cyc}(3a+2b+c+6)\ge(a+1)(b+1)(c+1)$$

$$\frac{(3a+2b+c+6)(3b+2c+a+6)(3c+2a+b+6)}{(a+1)(b+1)(c+1)}\ge 216$$

A.046. $f:(0,\infty)\to\mathbb{R}; f(x)=\sqrt{x}; f'(x)=\frac{1}{2\sqrt{x}}$

$$f''(x)=\frac{-\frac{1}{2\sqrt{x}}}{4x}=-\frac{1}{8x\sqrt{x}}<0; f\ concave$$

By Jensen's inequality for $\lambda_1+\lambda_2+\lambda_3=1$*:*

$$f(\lambda_1x_1+\lambda_1x_2+\lambda_3x_3)\ge\lambda_1f(x_1)+\lambda_2f(x_2)+\lambda_3f(x_3)$$

$$For\ \lambda_1=\frac{a^2+2ac}{(a+b+c)^2};\ \lambda_2=\frac{b^2+2ba}{(a+b+c)^2};\ \lambda_3=\frac{c^2+2cb}{(a+b+c)^2}$$

$$\lambda_1+\lambda_2+\lambda_3=1; x_1=\frac{1}{a}; x_2=\frac{1}{b}; x_3=\frac{1}{c}$$

$$f\left(\sum_{cyc}\frac{a^2+2ac}{(a+b+c)^2}\cdot\frac{1}{a}\right)\ge\sum_{cyc}\frac{a^2+2ac}{(a+b+c)^2}\cdot\sqrt{\frac{1}{a}}$$

$$\sqrt{\sum_{cyc}\frac{a+2c}{(a+b+c)^2}}\ge\sum_{cyc}\frac{a(a+2c)}{(a+b+c)^2\cdot\sqrt{a}}$$

$$\sqrt{\frac{3}{(a+b+c)}} \geq \frac{1}{(a+b+c)^2}\sum_{cyc}\frac{(a+2c)\sqrt{a}}{1}$$

$$\sum_{cyc}(a+2c)\sqrt{a} \leq (a+b+c)^2\sqrt{\frac{3}{a+b+c}} = (a+b+c)\sqrt{3(a+b+c)}$$

A.047. *Let be* $x, y \geq 1$. *If* $x \geq y \Rightarrow \frac{x}{y} \geq 1; x-y \geq 0 \Rightarrow \left(\frac{x}{y}\right)^{x-y} \geq 1$

$$\text{If } x \leq y \Rightarrow \frac{y}{x} \geq 1; y-x \geq 0 \Rightarrow \left(\frac{y}{x}\right)^{y-x} \geq 1 \Rightarrow \left(\frac{x}{y}\right)^{x-y} \geq 1$$

$$\text{In both cases: } \left(\frac{x}{y}\right)^{x-y} \geq 1. \text{ Analogous: } \left(\frac{y}{z}\right)^{y-y} \geq 1; \left(\frac{z}{x}\right)^{z-x} \geq 1$$

$$\text{By multiplying: } \left(\frac{x}{y}\right)^{x-y} \cdot \left(\frac{y}{z}\right)^{y-z} \cdot \left(\frac{z}{x}\right)^{z-x} \geq 1$$

$$x^{x-y} \cdot y^{y-z} \cdot z^{z-x} \geq y^{x-y} \cdot z^{y-z} \cdot x^{z-x}$$

$$\frac{x^x}{x^y} \cdot \frac{y^y}{y^z} \cdot \frac{z^z}{z^x} \geq \frac{y^x}{y^y} \cdot \frac{z^y}{z^z} \cdot \frac{x^z}{x^x}, \qquad x^{2x} \cdot y^{2y} \cdot z^{2z} \geq x^{z+y} \cdot y^{x+z} \cdot z^{y+x}$$

$$\text{We take } x = \sqrt{a}; y = \sqrt{b}; z = \sqrt{c}$$

$$\left(\sqrt{a}\right)^{2\sqrt{a}} \cdot \left(\sqrt{b}\right)^{2\sqrt{b}} \cdot \left(\sqrt{c}\right)^{2\sqrt{c}} \geq \left(\sqrt{a}\right)^{\sqrt{b}+\sqrt{c}} \cdot \left(\sqrt{b}\right)^{\sqrt{c}+\sqrt{a}} \cdot \left(\sqrt{c}\right)^{\sqrt{a}+\sqrt{b}} \geq$$

$$\overset{AM-GM}{\geq} \left(\sqrt{a}\right)^{2\sqrt[4]{bc}} \cdot \left(\sqrt{b}\right)^{2\sqrt[4]{ca}} \cdot \left(\sqrt{c}\right)^{2\sqrt[4]{ab}}$$

$$a^{\sqrt{a}} \cdot b^{\sqrt{b}} \cdot c^{\sqrt{c}} \geq a^{\sqrt[4]{bc}} \cdot b^{\sqrt[4]{ca}} \cdot c^{\sqrt[4]{ab}}$$

$$\text{Equality holds for } a = b = c.$$

A.048. $\frac{[x]^9+\{x\}^9}{[x]\cdot\{x\}} = \frac{[x]^8}{\{x\}} + \frac{\{x\}^8}{[x]} \overset{BERGSTROM}{\geq} \frac{\left([x]^4+\{x\}^4\right)^2}{\{x\}+[x]} \geq$

$$\geq \frac{1}{x}\cdot\left(\frac{([x]^2+\{x\}^2)^2}{2}\right)^2 = \frac{1}{4x}([x]^2+\{x\}^2)^4 \geq$$

$$\geq \frac{1}{4x}\left(\left(\frac{[x]+\{x\}}{2}\right)^2\right)^4 = \frac{x^8}{16\cdot 4x} = \frac{x^7}{64}$$

$$64([x]^9+\{x\}^9) \geq x^7\cdot[x]\cdot\{x\}; x \geq 0,\ x^7\cdot[x]\cdot\{x\} \leq 64([x]^9+\{x\}^9\} \quad (1)$$

Analogous:

$$y^7\cdot[y]\cdot\{y\}\le 64([y]^9+\{y\}^9\}\quad (2),\ z^7\cdot[z]\cdot\{z\}\le 64([z]^9+\{z\}^9\}\quad (3)$$

By adding (1); (2); (3):

$$x^7\cdot[x]\cdot\{x\}+y^7\cdot[y]\cdot\{y\}+z^7\cdot[z]\cdot\{z\}\le$$
$$\le 64([x]^9+[y]^9+[z]^9\}+64(\{x\}^9+\{y\}^9+\{z\}^9)=$$
$$=64([x]^9+[y]^9+[z]^9\}+1$$

Inequality is strict because (1); (2); (3) are equalities only for $x=y=z=0$ *and in our case* $\{x\}^9+\{y\}^9+\{z^9\}=\frac{1}{66}\neq 0$

A.049. *Let be* $K=\left\{\begin{pmatrix} a & b\\ -b & a\end{pmatrix}\middle|\, a,b\in\mathbb{R}\right\}, f:(K,+,\cdot)\to(\mathbb{C},+,\cdot); f\left(\begin{pmatrix} a & b\\ -b & a\end{pmatrix}\right)=a+ib$*, isomorphism*

$$f\left(\begin{pmatrix} a & b\\ -b & a\end{pmatrix}+\begin{pmatrix} c & d\\ -d & c\end{pmatrix}\right)=f\left(\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right)=1$$

$$f\left(\begin{pmatrix} a & b\\ -b & a\end{pmatrix}^3+\begin{pmatrix} c & d\\ -d & c\end{pmatrix}^3\right)=f\left(\begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix}\right)=2$$

$$f\left(\begin{pmatrix} a & b\\ -b & a\end{pmatrix}\right)=a+ib=z_1; f\left(\begin{pmatrix} c & d\\ -d & c\end{pmatrix}\right)=c+id=z_2$$

$$\begin{cases} z_1+z_2=1\\ z_1^3+z_2^3=2\end{cases}; (z_1+z_2)(z_1^2-z_1z_2+z_2^2)=2$$

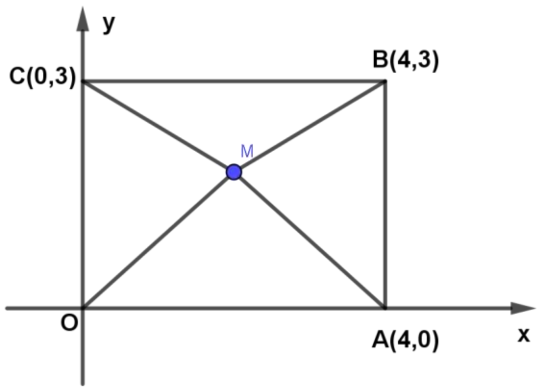
$$S=z_1+z_2; P=z_1z_2; z_1^2-z_1z_2+z_2^2=2$$

$$S^2-3P=2; 1-3P=2\Rightarrow P=-\frac{1}{3}$$

$$u^2-Su+P=0\Rightarrow u^2-u-\frac{1}{3}=0, z_1=\frac{1}{2}\left(1+\sqrt{\frac{7}{3}}\right); z_2=\frac{1}{2}\left(1-\sqrt{\frac{7}{3}}\right)$$

$$\begin{pmatrix} a & b\\ -b & a\end{pmatrix}=\begin{pmatrix} \frac{1}{2}\left(1+\sqrt{\frac{7}{3}}\right) & 0\\ 0 & \frac{1}{2}\left(1+\sqrt{\frac{7}{3}}\right)\end{pmatrix},$$

$$\begin{pmatrix} c & d\\ -d & c\end{pmatrix}=\begin{pmatrix} \frac{1}{2}\left(1-\sqrt{\frac{7}{3}}\right) & 0\\ 0 & \frac{1}{2}\left(1-\sqrt{\frac{7}{3}}\right)\end{pmatrix}$$



A.050.

$$M(x,y); MO=\sqrt{x^2+y^2}; MA=\sqrt{(x-4)^2+y^2}$$

$$MB=\sqrt{(x-4)^2+(y-3)^2}; MC=\sqrt{x^2+(y-3)^2}$$

$$MO+MB\geq OB=5,\qquad MC+MA\geq AC=5$$

$$MO+MB+MC+MA\geq 10$$

$$\text{Equality holds for } \{M\}=AC\cap OB,\ AC=OB=5; M\left(2;\frac{3}{2}\right)$$

$$x=2; y=\frac{3}{2},\qquad 2+2\cdot\frac{3}{2}=5z\Rightarrow z=1$$

A.051. Solution (Jalil Hajimir)

$$F(x)=\tan^{-1}x \text{ is a convex and increasing function:}$$

$$1\tan^{-1}1+3\tan^{-1}3+\cdots+(4n-3)$$

$$=F(1)+3F(3)+\cdots(4n-3)F(4n-3)\geq$$

$$\overset{Chebyshev}{\geq}\frac{1+3+\cdots++(4n-3)}{2n-1}[F(1)+F(3)+\cdots+F(4n-3)]\overset{Jensen}{\geq}$$

$$\geq(2n-1)^2F\left(\frac{1+3+\ldots++(4n-3)}{2n-1}\right)=(2n-1)^2\tan^{-1}(2n-1)$$

A.052. Solution (George Florin Serban)

We use the following formula:

$$(x+y+z)^5-x^5-y^5-z^5=5(x+y)(y+z)(x+z)\cdot$$

$$\cdot(x^2+y^2+z^2+xy+yz+xz).\ \text{Denote } y\to -y$$

$$(x-y+z)^5-x^5+y^5-z^5$$
$$=5(x-y)(z-y)(z+x)\cdot(x^2+y^2+z^2-xy-yz+xz)$$
$$Denote\ x=M_a=\frac{a+b}{2}, y=\sqrt{ab}=M_g, z=M_h=\frac{2ab}{a+b}$$
$$x\geq y\left(M_a\geq M_g\right)\Rightarrow x-y\geq 0, z\leq y\left(M_h\leq M_g\right)\Rightarrow z-y\leq 0, z+x>0\Rightarrow$$
$$\Rightarrow 5(x-y)(z-y)(z+x)\leq 0$$
$$We\ prove\ that\ E=x^2+y^2+z^2-xy-yz+xz>0$$
$$2\cdot E=2x^2+2y^2+2z^2-2xy-2yz+2xz(-x+y-z)^2+x^2+y^2+z^2>0$$
$$\Rightarrow(x-y+z)^5-x^5+y^5-z^5$$
$$=5(x-y)(z-y)(z+x)(x^2+y^2+z^2-xy-yz+xz)\leq 0$$
$$\Rightarrow(x-y+z)^5-x^5+y^5-z^5\leq 0\Rightarrow(x-y+z)^5\leq x^5-y^5+z^5$$
$$\Rightarrow\frac{x^5-y^5+z^5}{(x-y+z)^5}\geq 1\Rightarrow\sum_{cyc}\frac{x^5-y^5+z^5}{(x-y+z)^5}\geq\sum_{cyc}1=3$$

A.053. Solution (Tran Hong)

$$If\ a>0\ then:\ 4(a^2-a+1)\geq(a+1)^2$$
$$\Leftrightarrow 4a^2-4a+4\geq a^2+2a+1\Leftrightarrow 3a^2-6a+3\geq 0$$
$$\Leftrightarrow 3(a-1)^2\geq 0\ \ (true).\ Equality\Leftrightarrow a=1$$
$$\frac{y}{x^2-x+1}+\frac{z}{y^2-y+1}+\frac{x}{z^2-z+1}\leq\frac{4y}{(x+1)^2}+\frac{4z}{(y+1)^2}+\frac{4x}{(z+1)^2}$$
$$=4\left(\frac{y}{(x+1)^2}+\frac{z}{(y+1)^2}+\frac{x}{(z+1)^2}\right)=4\cdot\frac{3}{4}=3.\ Equality\Leftrightarrow x=y=1.$$

A.054. Solution (Remus Florin Stanca)

$$\Omega(x,y)=\sum_{k=1}^{\infty}\left(\frac{1}{2k}\left(\frac{x-y}{x+y}\right)^{2k}\right),\ let\ \frac{x-y}{x+y}=\alpha,\ we\ also\ know\ that\ x,y,z>0\Rightarrow$$
$$\Rightarrow -y<y\Rightarrow x-y<x+y\Rightarrow\frac{x-y}{x+y}<1\ because\ x+y>0,\ so\ \alpha<1\ \ (1)$$
$$\Omega(x,y)=\sum_{k=1}^{\infty}\left(\frac{\alpha^{2k}}{2k}\right)=\sum_{k=1}^{\infty}\left(\int\left(\frac{\alpha^{2k}}{2k}\right)'d\alpha\right)=$$

$$= \sum_{k=1}^{\infty}\left(\int \alpha^{2k-1}\,d\alpha\right) = \int\left(\sum_{k=1}^{\infty}\alpha^{2k-1}\right)d\alpha = \int \lim_{n\to\infty}\left(\alpha\cdot\frac{(\alpha^2)^n-1}{\alpha^2-1}\right)d\alpha$$

$$= -\int \frac{\alpha}{\alpha^2-1}\,d\alpha =$$

$$= -\frac{1}{2}\int\frac{2\alpha}{\alpha^2-1}\,d\alpha = -\frac{1}{2}\ln(|\alpha^2-1|) \overset{\alpha<1}{=} -\frac{1}{2}\ln(1-\alpha^2)$$

$$= -\frac{1}{2}\ln\left(1-\frac{x^2-2xy+y^2}{x^2+2xy+y^2}\right) =$$

$$= -\frac{1}{2}\ln\left(\frac{4xy}{(x+y)^2}\right) = -\ln\left(\frac{2\sqrt{xy}}{x+y}\right) \textit{ because } x,y>0$$

$$\Rightarrow \Omega(x,y) = -\ln\left(\frac{2\sqrt{xy}}{x+y}\right) \Rightarrow 3+\Omega(x,y)+\Omega(y,z)+\Omega(z,x) =$$

$$= 3-\sum_{cyc}\ln\left(\frac{2\sqrt{xy}}{x+y}\right) = 3+\ln\left(\frac{\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)}{8}\right) =$$

$$= 3-\ln(8)+\ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right) \ (2)$$

$$\textit{we know that } 2<e \Rightarrow 8<e^3 \Rightarrow \ln(8)<3 \Rightarrow 3-\ln(8)>0 \Rightarrow$$

$$\Rightarrow 3+\ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)-\ln(8) > \ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right) \overset{(2)}{\Rightarrow}$$

$$\Rightarrow 3+\Omega(x,y)+\Omega(y,z)+\Omega(z,x) > \ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)$$

A.055. Solution (Soumava Chakraborty)

$$QM \textit{ of } a,b = \sqrt{\frac{a^2+b^2}{2}} = Q \textit{ (say) and } GM \textit{ of } a,b = \sqrt{ab} = G \textit{ (say)}$$

$$\therefore a^2+b^2 = 2Q^2 \textit{ and } ab = G^2 \rightarrow \textit{using this, proposed inequality} \Leftrightarrow$$

$$\frac{2(10Q^2+6G^2)(10Q^2+8G^2)}{(14Q^2+10G^2)^2} \le \left(\frac{2Q^2}{2G^2}\right)^2$$

$$\Leftrightarrow Q^4(14Q^2+10G^2)^2-2G^4(10Q^2+6G^2)(10Q^2+8G^2)\geq 0$$

$$\Leftrightarrow 49t^8+70t^6-25t^4-70t^2-24\geq 0\ \left(t=\frac{Q}{G}\right)$$

$$\Leftrightarrow (t-1)(49t^7+49t^6+119t^5+119t^4+94t^3+94t^2+24t+24)\geq$$

$$0\to \textit{true},\because\ t=\frac{Q}{G}\geq 1\Rightarrow \textit{proposed inequality is true (Proved)}$$

A.056. Solution (Tran Hong)

$$\textit{For all } a,b>0 \textit{ we have: } \frac{a^2+b^2}{a+b}\geq\frac{a+b}{2}\quad (*)$$

$$\Leftrightarrow 2(a^2+b^2)\geq(a+b)^2\Leftrightarrow a^2-2ab+b^2\geq 0$$

$$\Leftrightarrow (a-b)^2\geq 0\quad \textit{(true)}$$

$$\textit{Equality}\Leftrightarrow a=b$$

$$\textit{So, }\frac{x^2+y^2}{x+y}+\frac{y^2+z^2}{y+z}+\frac{z^2+x^2}{z+x}\geq\frac{x+y}{2}+\frac{y+z}{2}+\frac{z+x}{2}=x+y+z$$

$$\text{“="}\Leftrightarrow x=y=z$$

$$t^{\log y}=4\Rightarrow \log\left(t^{\log y}\right)=\log(4)$$

$$\Rightarrow(\log y)(\log t)=\log(4)\Rightarrow(\log x)(\log t)=\log(4)$$

$$\overset{xt=4e}{\Rightarrow}\ t=\frac{4e}{x}\Rightarrow(\log x)\left(\log\frac{4e}{x}\right)=\log(4)$$

$$\Leftrightarrow(\log x)(\log(4e)-\log x)=\log(4)$$

$$\Leftrightarrow(\log x)(1+\log(4)-\log x)=\log(4)$$

$$\overset{a=\log x}{\Leftrightarrow}\ \alpha(1+\log(4)-\alpha)-\log(4)=0$$

$$\Leftrightarrow -\alpha^2+\alpha+(\alpha-1)\log(4)=0\Leftrightarrow -\alpha(\alpha-1)+(\alpha-1)\log(4)=0$$

$$\Leftrightarrow(\alpha-1)(\log(4)-\alpha)=0\Leftrightarrow \alpha=1 \textit{ or } \alpha=\log(4)$$

$$(*)\ \alpha=1\Rightarrow x=y=z=e\Rightarrow t=e$$

$$(*)\ \alpha=\log(4)\Rightarrow x=y=z=4\Rightarrow t=e$$

A.057. Solution (Serban George Florin)

$$4ab\left(\frac{x}{a}+\frac{y}{\sqrt{ab}}+\frac{z}{b}\right)=4\left(bx+\sqrt{ab}y+az\right)\leq\frac{(x+y+z)^2(a+b)^2}{ax+\sqrt{ab}y+bz}\Leftrightarrow$$

$$\Leftrightarrow \left(bx+\sqrt{ab}y+az\right)\left(ax+\sqrt{ab}y+bz\right) \le \frac{(x+y+z)^2(a+b)^2}{4}$$

$$\left(bx+\sqrt{ab}y+az\right)\left(ax+\sqrt{ab}y+bz\right) \overset{GM-AM}{\le} \frac{\left((a+b)x+2\sqrt{ab}y+(a+b)z\right)^2}{4} \le$$

$$\le \frac{(x+y+z)^2(a+b)^2}{4} \Rightarrow$$

$$(a+b)(x+z)+2\sqrt{ab}y \le (x+y+z)(a+b) = (a+b)(x+z)+y(a+b)$$

$$\Rightarrow 2\sqrt{ab}y \le y(a+b) \Leftrightarrow y\left(a-2\sqrt{ab}+b\right) \ge 0 \Leftrightarrow y\left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0 \ true$$

A.058. Solution (Ravi Prakash)

$$Let\ a = |z-1|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{z}) + 1$$

$$b = \left|z+\frac{1+\sqrt{3}i}{2}\right|^2 = |z-w^2|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{z}\omega^2) + 1$$

$$c = \left|z+\frac{1-\sqrt{3}i}{2}\right|^2 = |z-\omega|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{z}\omega) + 1$$

$$a+b+c = 3|z|^2 - 2\,\mathrm{Re}\left(\bar{z}(1+\omega+\omega^2)\right) + 3 = 3|z|^2+3 = 3(|z|^2+1)$$

$$Now, \frac{a^2+b^2+c^2}{3} \ge \left(\frac{a+b+c}{3}\right)^2 = (|z|^2+1)^2 \vDash |z-1|^4+|z-\omega|^4+|z-\omega^2|^4$$

$$\ge 3(|z|^2+1)^2 = 3(|z|^4+2|z|^2+1)$$

A.059. Solution (Petre Daniel Alexandru)

$$a,b,c,d > 0$$

$$1)\ \frac{2ab}{a+b}+\frac{c+d}{2}+\frac{a+b}{2ab}+\frac{2}{c+d} < \left[\frac{2ab}{a+b}\right]+\left[\frac{c+d}{2}\right]+\left[\frac{a+b}{2ab}\right]+\left[\frac{2}{c+d}\right]+3$$

$$x = [x]+\{x\}$$

$$(1) \Leftrightarrow \left\{\frac{2ab}{a+b}\right\}+\left\{\frac{a+b}{2ab}\right\}+\left\{\frac{c+d}{2}\right\}+\left\{\frac{2}{c+d}\right\} < 3$$

$$\frac{2ab}{a+b} = m, \frac{c+d}{2} = n, m,n > 0 \Rightarrow \{m\}+\left\{\frac{1}{m}\right\}+\{n\}+\left\{\frac{1}{n}\right\} < 3$$

$$or\ \{m\}+\left\{\frac{1}{m}\right\} < \frac{3}{2}$$

$$If\ 0 < m < 1 \Rightarrow \left.\begin{matrix}\{m\} = m\\ \left\{\frac{1}{m}\right\} = 0\end{matrix}\right\} \to \{m\}+\left\{\frac{1}{m}\right\} < 1$$

$$If\ m \ge 1; m = [m]+\{m\} = k+t, k \in \mathbb{N}\ and\ t \in (0,1)$$

$$\left\{\frac{1}{m}\right\}=\frac{1}{m}-\left[\frac{1}{m}\right]=\frac{1}{m}=\frac{1}{k+t}$$

$$\{m\}+\left\{\frac{1}{m}\right\}=t+\frac{1}{k+t}<t+\frac{1}{1+t}\overset{?}{<}\frac{3}{2}$$

$$f(t)=t+\frac{1}{1+t}\uparrow t\in(0,1)\Rightarrow f(t)<f(1)=1+\frac{1}{1+1}=\frac{3}{2}$$

A.060. Solution (Adrian Popa)

$$3\sqrt[3]{abc}+d=\sqrt[3]{abc}+\sqrt[3]{abc}+\sqrt[3]{abc}+d\overset{MA>MG}{\geq}$$

$$\geq 4\sqrt[4]{\left(\sqrt[3]{abc}\right)^3\cdot d}=4\sqrt[4]{abcd}\Rightarrow 4\sqrt[4]{abcd}-3\sqrt[3]{abc}<d\quad (1)$$

$$4\sqrt[4]{abcd}+e\overset{MA>MG}{>}5\sqrt[5]{\left(\sqrt[4]{abcd}\right)^4\cdot e}=\sqrt[5]{abcde}\Rightarrow$$

$$\Rightarrow 5\sqrt[5]{abcde}-4\sqrt[4]{abcd}<e\quad (2)$$

$$(1)\cdot(2)\Rightarrow\left(4\sqrt[4]{abcd}-3\sqrt[3]{abc}\right)\left(5\sqrt[5]{abcde}-4\sqrt[4]{abcd}\right)<de$$

A.061. Solution (Adrian Popa)

$$\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2=$$

$$=\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}\right)\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}\right)\overset{J.Holder}{\geq}$$

$$\geq\left(\frac{\sqrt[3]{a^3\cdot 1\cdot 1}}{\sqrt[3]{b^2\cdot b^2\cdot b^2}}+\frac{\sqrt[3]{b^3\cdot 1\cdot 1}}{\sqrt[3]{c^2\cdot c^2\cdot c^2}}+\frac{\sqrt[3]{c^3\cdot 1\cdot 1}}{\sqrt[3]{a^2\cdot a^2\cdot a^2}}\right)^3=\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)^3$$

A.062. Solution (Soumava Chakraborty)

$$\textit{We shall first demonstrate, that, } \forall n\geq 1, \binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2\overset{(1)}{=}\binom{2n}{n}$$

$$\textit{We have } (1+x)^n\overset{(i)}{=}\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+x\binom{n}{n}x^n$$

$$\textit{and} \therefore \left(1+\tfrac{1}{x}\right)^n\overset{(ii)}{=}\binom{n}{0}+\binom{n}{1}\tfrac{1}{x}+\binom{n}{2}\tfrac{1}{x^2}+\cdots+\binom{n}{n}\tfrac{1}{x^n}$$

$$\textit{Now, } \binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2$$

$$=\textit{coefficient of the term independent of } x \textit{ in}$$

$$\left[\binom{n}{0}+\binom{n}{1}x+\cdots+\binom{n}{n}x^n\right]\left[\binom{n}{0}+\binom{n}{1}\frac{1}{x}+\cdots+\binom{n}{n}\frac{1}{x^n}\right]$$

$$\overset{by\ (i).(ii)}{=}\ \textit{coefficient of term independent of } x \textit{ in}$$

$$(1+x)^n\left(1+\frac{1}{x}\right)^n=\frac{(1+x)^{2n}}{x^n}, = \textit{coefficient of } x^n \textit{ in } (1+x)^{2n}=\binom{2n}{n}$$

$\Rightarrow$ *(1) is true.* $\because f(t)=t^2$ *is convex,* $\therefore$

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2\ge(n+1)\left(\frac{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}}{n+1}\right)^2$$

$$=\frac{(2^n)^2}{n+1}\ [\textit{putting } x=1 \textit{ in (i)}, \binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n]$$

$$\overset{by\ (1)}{\Rightarrow}\binom{2n}{n}\overset{(2)}{\ge}\frac{2^{2n}}{n+1}.\ \textit{Using (2), LHS of (a)}$$

$$\ge(a+1)\left(\frac{2^{2b}}{b+1}+\frac{2^{2c}}{c+1}\right)+(b+1)\left(\frac{2^{2c}}{c+1}+\frac{2^{2a}}{a+1}\right)$$

$$+(c+1)\left(\frac{2^{2a}}{a+1}+\frac{2^{2b}}{b+1}\right)$$

$$=\left[\left(\frac{a+1}{b+1}\right)2^{2b}+\left(\frac{b+1}{a+1}\right)2^{2a}\right]+\left[\left(\frac{a+1}{c+1}\right)2^{2c}+\left(\frac{c+1}{a+1}\right)2^{2a}\right]+$$

$$+\left[\left(\frac{b+1}{c+1}\right)2^{2c}+\left(\frac{c+1}{b+1}\right)2^{2b}\right]$$

$$\underset{(3)}{\overset{A-G}{\ge}}2\sqrt{2^{2b}\cdot2^{2a}}+2\sqrt{2^{2c}\cdot2^{2a}}+2\sqrt{2^{2c}\cdot2^{2b}}=2\left(2^{a+b}+2^{c+a}+2^{b+c}\right)$$

$$\textit{Now, } (\ln 2)\left(b+c-2\sqrt{bc}\right)\ge0\Rightarrow(b+c)(\ln 2)\ge\left(2\sqrt{bc}\right)(\ln 2)$$

$$\Rightarrow\ln\left(2^{b+c}\right)\ge\ln\left(2^{2\sqrt{bc}}\right)\Rightarrow2^{b+c}\ge2^{\sqrt{bc}}\ \textit{and analogs} \therefore \textit{(3)} \Rightarrow \textit{LHS of (a)}$$

$$\ge2\left(2^{2\sqrt{ab}}+2^{2\sqrt{bc}}+2^{2\sqrt{ca}}\right)=2\left(4^{\sqrt{ab}}+4^{\sqrt{bc}}+4^{\sqrt{ca}}\right)\ \textit{(Proved)}$$

A.063. Solution (Khaled Abd Imouti)

$$x^{\sqrt{xy}}\cdot y^{\frac{1}{\sqrt{xy}}}+x^{\frac{1}{\sqrt{xy}}}\cdot y^{\sqrt{xy}}=x^{\frac{x+y}{2}}\cdot y^{\frac{2}{x+y}}+y^{\frac{x+y}{2}}\cdot x^{\frac{2}{x+y}}$$

$$x^G\cdot y^{\frac{1}{G}}+x^{\frac{1}{G}}\cdot y^G=x^M\cdot y^{\frac{1}{M}}+y^M\cdot x^{\frac{1}{M}}, M=\frac{x+y}{2}, G=\sqrt{xy}, M\ge G$$

$$e^{G\ln(x)+\frac{1}{G}\ln(y)}+e^{\frac{1}{G}\ln(x)+G\ln(y)}=e^{M\ln(x)+\frac{1}{M}\ln(y)}+e^{M\ln(y)+\frac{1}{M}\ln(x)} \quad (*)$$

(*) satisfying when $G=M$

When $G=M=4$ *and* $x=4, y=4$ *then:*

$$(u)^3+\frac{\ln(4)}{\ln(2)}+\frac{\ln(4)}{\ln(4)}=64+2+1=67, \qquad S=\{(x,y)=(4,4)\}$$

From (*):

$$e^{G\ln(x)+\frac{1}{G}\ln(y)}-e^{M\ln(x)+\frac{1}{M}\ln(y)}=e^{M\ln(x)+\frac{1}{M}\ln(y)}-e^{G\ln(y)+\frac{1}{G}\ln(x)}$$

Suppose: $f(t)=e^{t\ln(x)+\frac{1}{t}\ln(y)}, t>0$

$$f'(t)=\left(\ln(x)-\frac{1}{t^2}\ln(y)\right)\cdot e^{t\ln(x)+\frac{1}{t}\ln(y)}$$

$$\left.\begin{array}{c}\exists c_1\in(M,G) \text{ such that:}\\ f(G)-f(M)=\left(\ln(x)-\frac{1}{C_1^2}\ln(y)\right)(G-M)\\ \text{in a similarly way:}\\ \exists c_2\in\,]G,M[\text{ such that:}\\ f(M)-f(G)=\left(\ln(x)-\frac{1}{c_2^2}\ln(y)\right)(M-G)\end{array}\right\}\Rightarrow$$

$$\left(\ln(x)-\frac{1}{c_1^2}\ln(y)\right)(G-M)+\left(\ln(x)-\frac{1}{c_2^2}\ln(y)\right)(M-G)=0$$

$$\left(\ln(x)-\frac{1}{c_1^2}\ln(y)\right)(G-M)-\left(\ln(x)-\frac{1}{c_2^2}\ln(y)\right)(G-M)=0$$

$$\left[\ln(x)-\frac{1}{c_1^2}\ln(y)-\ln(x)+\frac{1}{c_2^2}\ln(y)\right](G-M)=0$$

$$\underbrace{\left(\frac{1}{c_2^2}-\frac{1}{c_1^2}\right)\cdot\ln(y)}_{\neq 0}(G-M)=0. \text{ So: } G-M=0\Rightarrow G=M.$$

A.064. Solution (Tran Hong) *With* $x,y,z>0$ *we have:*

$$x+y+z=1\cdot\sqrt{x}+1\cdot\sqrt{y}+1\cdot\sqrt{z}\overset{B.C.S}{\leq}\sqrt{3}\cdot\sqrt{x+y+z}$$

$$\Rightarrow\sqrt{x+y+z}\leq\sqrt{3}\Rightarrow x+y+z\leq 3 \quad (*)$$

$$xyz\leq\frac{(x+y+z)^3}{27}\Rightarrow 4\leq xyz+x+y+z\leq\frac{(x+y+z)^3}{27}+(x+y+z)$$

$$\Leftrightarrow (x+y+z)^3+27(x+y+z)-108\geq 0$$

$$\overset{t=x+y+z>0}{\Leftrightarrow} t^3+27t-108\geq 0 \Leftrightarrow (t-3)(t^2+3t+36)\geq 0$$

$$\Leftrightarrow t\geq 3 \Leftrightarrow x+y+z\geq 3 \quad (**)$$

$$\overset{(*),(**)}{\Rightarrow} x+y+z=3 \Leftrightarrow x=y=z=1 \Rightarrow (x;y;z)=(1;1;1) \text{ (Answer)}$$

A.065. Solution (Soumava Chakraborty)

Let $\frac{a|b-c|}{b+c}=x, \frac{b|c-a|}{c+a}=y, \frac{c|a-b|}{a+b}=z$. *Then, the proposed inequality*

transforms into: $\prod(x+y-z)\overset{(1)}{\leq} 8xyz$

Now, $xyz-\prod(x+y-z)=\sum x^3+3xyz-\sum x^2y-\sum xy^2 \overset{Schur}{\geq} 0$

(∵ x,y,z are non-negative) ∴ $\prod(x+y-z)\leq xyz\leq 8xyz$

(∵ $xyz\geq 0$ *as* $x,y,z\geq 0$) ⇒ *(1)* ⇒ *proposed inequality is true (Proved)*

A.066. Solution (George Florin Şerban)

$$8=\frac{3x+3y}{y+2z}+\frac{3y+3z}{x+2z}+\frac{3x+9z}{x+y+z}=$$

$$=\frac{3x}{y+2z}+\frac{3y}{x+2z}+\frac{3z}{x+y+z}+\frac{3y}{y+2z}+\frac{3z}{x+2z}+\frac{9z}{x+y+z}=$$

$$=\frac{(3x)^2}{3xy+6xz}+\frac{(3y)^2}{3y^2+6yz}+\frac{(3y)^2}{3xy+6yz}+\frac{(3z)^2}{3xz+6z^2}+\frac{(3x)^2}{3x^2+3xy+3xz}+$$

$$+\frac{(3z)^2}{3xz+3yz+3z^2}+\frac{(3z)^2}{3xz+3yz+3z^2}+\frac{(3z)^2}{3xz+3yz+3z^2}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(6x+6y+12z)^2}{3x^2+3y^2+15z^2+9xy+21xz+21yz};\ x,y,z>0$$

$$8\geq\frac{(6x+6y+12z)^2}{3x^2+3y^2+15z^2+9xy+21xz+21yz}\Leftrightarrow$$

$(z-y)^2+(z-x)^2\leq 0 \Leftrightarrow x=y=z$ and from $x^x+y^y+z^z=3$ we get:

$3x^x=3\Rightarrow x^x=1$. Therefore: $(x,y,z)=(1,1,1)$

A.067. Solution (Sudhir Jha) *We have:*

$$3(abc)^{\frac{1}{3}}\overset{GM\leq AM}{\leq}\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\overset{GM\leq AM}{\leq}\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$

$$\Rightarrow 3(abc)^{\frac{1}{3}} \le \sum \sqrt{ab} \le (a+b+c) \quad (1)$$

$$3(abc)^{\frac{1}{3}} \overset{GM\le AM}{\le} \sqrt[3]{a^2b} + \sqrt[3]{b^2c} + \sqrt[3]{c^2a} \overset{GM\le AM}{\le}$$

$$\le \frac{a+a+b}{3} + \frac{b+b+c}{3} + \frac{c+c+a}{3}$$

$$\Rightarrow 3(abc)^{\frac{1}{3}} \le \sum \sqrt[3]{a^2b} \le (a+b+c) \quad (2)$$

$$and\ 3(abc)^{\frac{1}{3}} \overset{GM\le AM}{\le} \sqrt[4]{a^3b} + \sqrt[4]{b^3c} + \sqrt[4]{c^3a}$$

$$\overset{GM\le AM}{\le} \frac{a+a+a+b}{4} + \frac{b+b+b+c}{4} + \frac{c+c+c+a}{4}$$

$$\Rightarrow 3(abc)^{\frac{1}{3}} \le \sum \sqrt[4]{a^3b} \le (a+b+c) \quad (3).\ Then\ (1)\times(2)\times(3)$$

$$\Rightarrow 27abc \le \left(\sum \sqrt{ab}\right)\left(\sum \sqrt[3]{a^2b}\right)\left(\sum \sqrt[4]{a^3b}\right) \le (a+b+c)^3$$

A.068. Solution (Ravi Prakash) *Given determinant*

$$= [\log(ex) - \log x][\log(e^2x) - \log x][\log(e^2x) - \log(ex)]$$

$$[\log(e^3x) - \log x][\log(e^xx) - \log(ex)]$$

$$[\log(e^3x) - \log(e^2x)]$$

$$\text{[Vandermode Determinant]} = (1)(2)(1)(3)(2)(1) = 12$$

$$\therefore 12 = 7 + 2^{x-10} + \log_{12} x \Rightarrow x = 12$$

A.069. Solution (Ravi Prakash) *Let* $z = x + iy$. *Rewrite the equation*

$$|z - (3+4i)| \le 1 \Longrightarrow \big||1| - |(3+4i)|\big| \le |z| \le 1 + |3+4i|$$

$$\Longrightarrow 4 \le |z| \le 6 \Longrightarrow 16 \le |z|^2 = x^2 + y^2 \le 36$$

A.070. Solution (Ravi Prakash)

$$Let\ a = log(1+sin^2x), b = log(1+cos^2x)$$

$$a + b = log(1+sin^2x)(1+cos^2x) = log\left(2 + \frac{1}{4}sin^2 2x\right)$$

$$\le log(2\cdot 25) < 1 \Longrightarrow a^2 + ab + b^2 \le (a+b)^2 < 1$$

$$(log(1+sin^2x))^2 + (log(1+cos^2x))^2 + (\log(1+sin^2x)(1+cos^2x))^2$$

$$= a^2 + b^2 + (a+b)^2 = 2(a^2+b^2+ab) < 2$$

A.071. Solution (Soumava Chakraborty)

$$\textit{Firstly, } \forall n \ge 1 \textit{ and } n \in \mathbb{N}, \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 \overset{(1)}{=} \binom{2n}{n}$$

$$\textit{Proof of (1): } (1+x)^n \overset{(i)}{=} \binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n}x^2 \textit{ and}$$

$$\therefore \left(1+\frac{1}{x}\right)^n \overset{(ii)}{=} \binom{n}{0} + \binom{n}{1}\frac{1}{x} + \dots + \binom{n}{n}\frac{1}{x^n}$$

$$\textit{Now, } \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2, = \textit{coefficient of term independent of } x \textit{ in}$$

$$\left[\binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n}x^n\right]\left[\binom{n}{0} + \binom{n}{1}\frac{1}{x} + \dots + \frac{\binom{n}{n}1}{x^2}\right]$$

$$\overset{by\ (i),(ii)}{=} \textit{coefficient of term independent of } x \textit{ in}$$

$$(1+x)^n\left(1+\frac{1}{x}\right)^n = \frac{(1+x)^{2n}}{x^n}$$

$$= \textit{coefficient of } x^n \textit{ in } (1+x)^{2n} = \binom{2n}{n} \Rightarrow \textit{(1) is true.}$$

$$\textit{Of course, putting } x = 1 \textit{ in (i), } \binom{n}{0} + \binom{n}{1} + \binom{n}{n} \overset{(2)}{=} 2^n$$

$$\textit{Now, } \frac{1}{\binom{2a}{a}^2} \cdot \sum_{k=0}^{a}\binom{a}{k}^3 \overset{by\ (1)}{=} \frac{\binom{a}{0}^3 + \binom{a}{1}^3 + \dots + \binom{a}{a}^3}{\left[\binom{a}{0}^2 + \binom{a}{1}^2 + \dots + \binom{a}{a}^2\right]^2}$$

$$= \frac{\left[\binom{a}{0}^3 + \binom{a}{1}^3 + \dots + \binom{a}{a}^3\right]\left[\binom{a}{0} + \binom{a}{1} + \dots + \binom{a}{a}\right]}{\left[\binom{a}{0}^2 + \binom{a}{1}^2 + \dots + \binom{a}{a}^2\right]^2 \cdot 2^a}$$

$$\overset{\substack{reverse\\CBS}}{\ge} \frac{\left[\binom{a}{0}^2 + \binom{a}{1}^2 + \dots + \binom{a}{a}^2\right]^2}{\left[\binom{a}{0}^2 + \binom{a}{1}^2 + \dots + \binom{a}{a}^2\right]^2 \cdot 2^a} \overset{(by\ (2))}{=} \frac{1}{2^a} \Rightarrow \frac{1}{\binom{2a}{a}^2} \cdot \sum_{k=0}^{a}\binom{a}{k}^3 \overset{(a)}{\ge} \frac{1}{2^a}$$

$$\textit{Similarly, } \frac{1}{\binom{2b}{b}^2} \cdot \sum_{k=0}^{b}\binom{b}{k}^3 \overset{(b)}{\ge} \frac{1}{2^b} \textit{ and } \frac{1}{\binom{2c}{c}^2} \cdot \sum_{k=0}^{c}\binom{c}{k}^3 \overset{(c)}{\ge} \frac{1}{2^c}$$

$$\textit{(a)+(b)+(c)} \Rightarrow \sum\left(\frac{1}{\binom{2a}{a}^2} \cdot \sum_{k=0}^{a}\binom{a}{k}^3\right) \ge \sum\frac{1}{2^a} \overset{Bergstrom}{\ge} \frac{9}{2^a+2^b+2^c} \quad \textit{(Proved)}$$

A.072. Solution (Tran Hong)

- $f(d) = (3+d)^4 - 256d \quad (d > 0) \Longrightarrow f'(d) = 4(3+d)^3 - 256$

$$\because f'(d)=0 \Leftrightarrow 4(3+d)^3=256 \Leftrightarrow (3+d)^3=64 \Leftrightarrow d=1>0$$
$$\Rightarrow f(d)\geq f_{min}(1)=(3+1)^4-256=0$$
$$\Rightarrow (3+d)^4-256d\geq 0 \overset{a+b+c=3}{\Longrightarrow} (a+b+c+d)^4\geq 256d \quad (*)$$

- $c(3-c)^2 \overset{0<a,b,c<3}{=} c(3-c)(3-c)$

$$=\frac{1}{2}(2c)(3-c)(3-c) \overset{AM-GM}{\leq} \frac{1}{2}\cdot\frac{1}{27}(2c+3+3-2c)^3$$
$$=\frac{1}{2}\cdot\frac{1}{27}\cdot 6^3=4 \Rightarrow c(a+b)^2\leq 4 \quad (**)$$
$$\overset{(*)+(**)}{\Longrightarrow} (a+b+c+d)^4+4\geq c(a+b)^2+256d$$

A.073. Solution (Michael Sterghiou)

$$256\sqrt{\frac{a^2+b^2}{2}}\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)^3$$
$$\leq 27\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)^4 ;(1)$$

Let: $t=\sqrt{\frac{a^2+b^2}{2}}$ and $u=\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}$

Then (1) becomes:

$$256tu^3-27(t+u)^4\leq 0 \Leftrightarrow -(u-3t)^2(27u^2+14ut+3t^2)\leq 0$$

Which is true for $u,t>0$.

A.074. Solution (Tran Hong)

$$27(a+b+c+d)^4-256d(a+b+c)^3 \overset{u=a+b+c\geq 0}{=} 27(u+d)^4-256du^3$$
$$=27u^4-148u^3d+162u^2d^2+108ud^3+27d^4$$
$$=(u-3d)^2(27u^2+14ud+3d^2)\geq 0 \quad \textit{(true for } u,d\geq 0)$$
$$4(a+b+c)^3-27c(a+b)^2 \overset{v=a+b\geq 0}{=} 4(v+c)^3-27cv^2$$
$$=4v^3-15v^2c+12cv^2+4c^3$$
$$=(v-2c)^2(4v+c)\geq 0 \textit{ (true for } v,c\geq 0)$$
$$LHS=27(a+b+c+d)^4+(4-256d)(a+b+c)^3$$
$$=27(a+b+c+d)^4-256d(a+b+c)^3+4(a+b+c)^3\geq 27c(a+b)^2$$
$$=RHS$$

A.075. Solution (Șerban George Florin)

$$3\sum\frac{1}{1+2x+4xy}=$$

$$=3\left(\frac{1}{1+2x+4xy}+\frac{2x}{2x+4xy+1}+\frac{4xy}{4xy+1+2x}\right)=$$

$$=3\frac{1+2x+4xy}{1+2x+4xy}=3=2(x+y+z)\Rightarrow x+y+z=\frac{3}{2}$$

$$x-y+z=\frac{1}{2}\Rightarrow 2y=1\Rightarrow y=\frac{1}{2}\Rightarrow x+z=1,8\ \cdot\frac{1}{2}xz=1$$

$$\Rightarrow xz=\frac{1}{4}:t^2-t+\frac{1}{4}=0,\left(t-\frac{1}{2}\right)^2=0\Rightarrow t_1=t_2=\frac{1}{2}$$

$$\Rightarrow x=z=\tfrac{1}{2}.\ So,\ x=y=z=\tfrac{1}{2}$$

A.076. Solution (Ravi Prakash)

As a,b,c are the sides of a triangle, $\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}$ are also sides of a triangle.

Let $x=\frac{1}{b+c},y=\frac{1}{c+a},z=\frac{1}{a+b}$. Then $x+y-z>0$, etc.

$$LHS=\prod_{cyc}(x+y-z)$$

$$=\sqrt{(x+y-z)(y+z-x)}\sqrt{(x+y-z)(z+x-y)}$$

$$\sqrt{(y+z-x)(z+x-y)}$$

$$\le\left[\frac{1}{2}(x+y-z+y+z-x)\right]\left[\frac{1}{2}(x+y-z+z+x-y)\right]$$

$$\times\left[\frac{1}{2}(y+z-x+z+x-y)\right]=xyz=\left(\frac{1}{b+c}\right)\left(\frac{1}{c+a}\right)\left(\frac{1}{a+b}\right)$$

Equality when triangle is equilateral. Let's assume $a\ge b\ge c,2s=a+b+c$

$$\Rightarrow 2s-a\le 2s-b\le 2s-c\Rightarrow\frac{1}{b+c}\ge\frac{1}{c+a}\ge\frac{1}{a+b}$$

It is sufficient to show that: $\frac{1}{b+c}<\frac{1}{c+a}+\frac{1}{a+b}$

$$\Leftrightarrow (c+a)(a+b)<(a+b)(b+c)+(b+c)(c+a)$$

$$\Leftrightarrow bc+a(a+b+c)<ac+b(a+b+c)+ab+c(a+b+c)$$

$$\Leftrightarrow bc<a(b+c)+(a+b+c)(b+c-a)$$

$$\Leftrightarrow bc<a(b+c)+(b+c)^2-a^2\Leftrightarrow 0<a(b+c-a)+b^2+c^2+bc$$

A.077. Solution (Serban George Florin)

$$0<a\le b.\ We\ denote\ x_1=\frac{2a+b}{3},x_2=\frac{3a+b}{4},x_3=\frac{4a+b}{5}$$

$$We\ prove\ that\ x_1,x_2,x_3\in(a,b),a<\frac{2a+b}{3}<b\Rightarrow 3a<2a+b\Rightarrow a<b$$

$$2a+b<3b\Rightarrow a<b\ (A)$$

$$a<\frac{3a+b}{4}<b\Rightarrow 4a<3a+b<4b\Rightarrow a<b$$

$$a<\frac{4a+b}{5}<b\Rightarrow 5a<4a+b<5b\Rightarrow a<b$$

$$R=\overline{1,3},x_k\in(a,b)\Rightarrow a<x_k<b\Rightarrow(x_k-a)(x_k-b)<0$$

$$x_k^2-(a+b)x_k+ab<0|:x_k\Rightarrow x_k-(a+b)+\frac{ab}{x_k}<0$$

$$\frac{ab}{x_k}<(a+b)-x_k,\frac{1}{x_k}<\frac{a+b-x_k}{ab},k\in\overline{1,3}$$

$$\left(\sum_{k=1}^{3}x_k\right)\left(\sum_{k=1}^{3}\frac{1}{x_k}\right)\le 2\left(\frac{b}{a}+\frac{a}{b}\right)+5=\frac{2a^2+2b^2+5ab}{ab}$$

$$S=\sum_{k=1}^{3}x_k,\left(\sum_{k=1}^{3}x_k\right)\left(\sum_{k=1}^{3}\frac{1}{x_k}\right)\le S\left[\frac{3(a+b)}{ab}-\frac{S}{ab}\right]\le\frac{2a^2+2b^2+5ab}{ab}$$

$$3(a+b)S-S^2\le 2a^2+2b^2+5ab,S^2-3(a+b)S+2a^2+2b^2+5ab\ge 0$$

$$\Delta=9a^2+9b^2+18ab-8a^2-8b^2-20ab=(a-b)^2$$

$$S^2-3(a+b)S+2a^2+2b^2+5ab=0,S_1=\frac{3(a+b)+a-b}{2}=2a+b$$

$$S_2=\frac{3(a+b)-a+b}{2}=a+2b\Rightarrow(S-S_1)(S-S_2)\ge 0$$

$$S-S_1=\frac{2a+b}{3}+\frac{3a+b}{4}+\frac{4a+b}{5}-2a-b$$

$$=\left(\frac{2a+b}{3}-a\right)+\left(\frac{3a+b}{4}-a\right)+$$

$$+\left(\frac{4a+b}{5}-b\right)\overset{b-a=t>0}{=}\frac{b-a}{3}+\frac{b-a}{4}+\frac{4(a-b)}{5}=\frac{t}{3}+\frac{t}{4}-\frac{4t}{5}$$

$$=\frac{20t+15t-48t}{60}=-\frac{13t}{60}<0\Rightarrow S-S_1<0\Rightarrow S<S_1$$

$$S - S_2 = \frac{2a+b}{3} + \frac{3a+b}{4} + \frac{4a+b}{5} - a - 2b$$

$$= \left(\frac{2a+b}{3} - a\right) + \left(\frac{3a+b}{4} - b\right) +$$

$$+\left(\frac{4a+b}{5} - b\right) \overset{b-a=t}{=} \frac{b-a}{3} + \frac{3(a-b)}{4} + \frac{4(a-b)}{5} = \frac{t}{3} - \frac{3t}{4} - \frac{4t}{5}$$

$$= \frac{20t - 43t - 48t}{60}$$

$$= -\frac{73t}{60} < 0 \Rightarrow S - S_2 < 0 \; but \; S - S_1 < 0 \Rightarrow (S - S_1)(S - S_2) > 0 \; true.$$

A.078. Solution (Remus Florin Stanca)

$$We\ know\ that\ x + y + z \ge 3\sqrt[3]{xyz} \Leftrightarrow x + y + z \ge \frac{3}{2}$$

$$We\ also\ know\ that: \frac{3}{1+2x+4xy} = \frac{3}{1+2x+\frac{1}{2z}} = \frac{3}{\frac{1}{1}+\frac{1}{\frac{1}{2x}}+\frac{1}{2z}} \le \frac{1+\frac{1}{2x}+2z}{3} \Rightarrow$$

$$\Rightarrow 3\left(\frac{1}{1+2x+4xy} + \frac{1}{1+2+4yz} + \frac{1}{1+2z+4xz}\right) \le$$

$$\le \frac{3+\frac{1}{2x}+\frac{1}{2y}+\frac{1}{2z}+2(x+y+z)}{3} = \frac{3+3+2(x+y+z)}{3} = \frac{6+2(x+y+z)}{3} \quad (1)$$

$$x + y + z \ge \frac{3}{2} \Rightarrow 4(x+y+z) \ge 6 \Rightarrow 6(x+y+z) \ge 6 + 2(x+y+z) \Rightarrow$$

$$\Rightarrow \frac{6+2(x+y+z)}{3} \le 2(x+y+z) \quad (2)$$

$$\overset{(1);(2)}{\Rightarrow} 3\left(\frac{1}{1+2x+4xy} + \frac{1}{1+2y+4yz} + \frac{1}{1+2z+4xz}\right) \le 2(x+y+z), \text{ but we know that:}$$

$$3\left(\frac{1}{1+2x+4xy} + \frac{1}{1+2y+4yz} + \frac{1}{1+2z+4xz}\right) = 2(x+y+z) \Rightarrow$$

$$\Rightarrow 1 = \frac{1}{2x} = 2z = \frac{1}{2y} = 2x = \frac{1}{2z} = 2y \Rightarrow x = y = z = \frac{1}{2}$$

A.079. Solution (Adrian Popa)

$$a, b, c > 1 \Rightarrow \sum a^{\frac{3}{a}}\left(b^{\frac{1}{b}} + c^{\frac{1}{c}} - a^{\frac{1}{a}}\right) \le a^{\frac{1}{a}} \cdot b^{\frac{1}{b}} \cdot c^{\frac{1}{c}}\left(a^{\frac{1}{a}} + b^{\frac{1}{b}} + c^{\frac{1}{c}}\right)$$

$$a > 1 \Rightarrow a^{\frac{1}{a}} > 1. \, Let \, a^{\frac{1}{a}} = x > 1$$

$$b > 1 \Rightarrow b^{\frac{1}{b}} > 1. \, Let \, b^{\frac{1}{b}} = y > 1$$

$$c > 1 \Rightarrow c^{\frac{1}{c}} > 1. \, Let \, c^{\frac{1}{c}} = z > 1$$

$$We \, must \, prove \, that \sum x^3 \,(y + z - x) \le xyz(x + y + z) \Leftrightarrow$$

$$\Leftrightarrow x^3y + x^3z - x^4 + y^3x + y^3z - y^4 + z^3x + z^3y - z^4$$

$$\le x^2yz + xy^2z + xyz^2 \Leftrightarrow$$

$$\Leftrightarrow x^4 + y^4 + z^4 - x^3y - x^3z - y^3x - y^3z - z^3x - z^3y + x^2yz + xy^2z$$

$$+ xyz^2 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow x^2(x^2 - xy - xz + yz) + y^2(y^2 - xy - yz + xz)$$

$$+ z^2(z^2 - zx - zy + xy) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow x^2(x - y)(x - z) + y^2(y - x)(y - z) + z^2(z - x)(z - y) \ge 0 \text{ – } True$$

Schur's inequality for $k = 2$.

A.080. Solution (Tran Hong)

$$LHS = \frac{y(8x + 5)}{48x^3 + 1} + \frac{z(8y + 5)}{48y^3 + 1} + \frac{x(8z + 5)}{48z^3 + 1}$$

$$\overset{BCS}{\le} \sqrt{(y^2 + z^2 + x^2)\left(\left(\frac{(8x + 5)}{48x^3 + 1}\right)^2 + \left(\frac{(8y + 5)}{48y^3 + 1}\right)^2 + \left(\frac{(8z + 5)}{48z^3 + 1}\right)^2\right)}$$

$$= \sqrt{(x^2 + y^2 + z^2)\left(\left(\frac{(8x + 5)}{48x^3 + 1}\right)^2 + \left(\frac{(8y + 5)}{48y^3 + 1}\right)^2 + \left(\frac{(8z + 5)}{48z^3 + 1}\right)^2\right)}$$

Because: $RHS = \sqrt{(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)}$

So, for $x > 0$, we need to prove:

$$\left(\frac{(8x + 5)}{48x^3 + 1}\right)^2 \le \frac{1}{x^2}; (analogs) \leftrightarrow \frac{(8x + 5)}{48x^3 + 1} \le \frac{1}{x};$$

$$\leftrightarrow 8x^2 + 5x \le 48x^3 + 1 \ \leftrightarrow 48x^3 - 8x^2 - 5x + 1 \ge 0;$$

$$\leftrightarrow (3x + 1)(4x - 1)^2 \ge 0;$$

Which is clearly true by: $x > 0 \to 3x + 1 > 0, (4x - 1)^2 \ge 0$;

Hence,

$$\sum\left(\frac{(8x+5)}{48x^3+1}\right)^2 \le \sum\frac{1}{x^2}$$

$$\to \frac{y(8x+5)}{48x^3+1}+\frac{z(8y+5)}{48y^3+1}+\frac{x(8z+5)}{48z^3+1} \le \sqrt{(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)}$$

Equality if and only if $x=y=z=\frac{1}{4}$

A.081. Solution (Adrian Popa)

$$(CA-CB)(A^2-B^2)=CA^3-CAB^2-CBA^2+CB^3$$

$$=CA^3-C(AB^2+BA^2)+CB^3=CA^3-CA^3+CB^3=O_4+CI_4=C$$

We must show that: $det(C)\neq 0$

We determine the last digit of $det(C)$:

$$U\left(28\cdot\begin{vmatrix}121 & 45 & 891\\ 27 & 151 & 210\\ 150 & 180 & 181\end{vmatrix}\right)=8, \qquad U\left(18\cdot\begin{vmatrix}120 & 45 & 891\\ 330 & 151 & 210\\ 450 & 180 & 181\end{vmatrix}\right)=0$$

$$U\left(36\cdot\begin{vmatrix}120 & 121 & 891\\ 330 & 27 & 210\\ 450 & 150 & 181\end{vmatrix}\right)=0, \qquad U\left(723\cdot\begin{vmatrix}120 & 121 & 45\\ 330 & 151 & 210\\ 450 & 150 & 180\end{vmatrix}\right)=0$$

$$U(det(C))=8\Rightarrow det(C)\neq 0$$

A.082. Solution (Abner Chinga Bazo)

$$(a-b)^2\ge 0\Leftrightarrow a^2+b^2\ge 2ab\Leftrightarrow 3(a^2+b^2)\ge 6ab$$

$$4(a^2+b^2)-4ab\ge a^2+b^2+2ab\Leftrightarrow 4(a^2-ab+b^2)\ge (a+b)^2$$

$$\Leftrightarrow \frac{a^2-ab+b^2}{(a+b)^2}\ge\frac{1}{4}\Leftrightarrow\frac{(a^2-ab+b^2)^6}{(a+b)^{12}}\ge\frac{1}{2^{12}}$$

$$\frac{(a^2-ab+b^2)^6}{(a+b)^{12}}+\frac{(b^2-bc+c^2)^6}{(b+c)^{12}}+\frac{(c^2-ca+a^2)^6}{(c+a)^{12}}\ge\frac{3}{4096}$$

A.083. Solution (Ravi Prakash)

$$\sum_{cyc}\frac{(x+1)(y+1)}{(x+2)(y+2)}=\frac{3}{4};\ \ (1)$$

Rewrite (1) as $\sum_{cyc}\left(\frac{(x+1)(y+1)}{(x+2)(y+2)}-\frac{1}{4}\right)=0;\quad (2)$

For $x,y\geq 0$ we have:

$4(x+1)(y+1)-(x+2)(y+2)=3xy+2x+2y\geq 0$, equality for $x=y=0$

$\frac{(x+1)(y+1)}{(x+2)(y+2)}-\frac{1}{4}\geq 0$ equality for $x=y=0$

As each terms of $LHS_{(2)}\geq 0$

(2) can hold iff $x=y=z=0\Rightarrow\sum_{cyc}\sqrt{(x+1)(y+1)}=3$

A.084. Solution (Florică Anastase)

$$log\left(\frac{sinb}{sina}\right)\geq\left(1+\sqrt{\frac{a}{b}}\right)log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

$$\Leftrightarrow log\left(\frac{sinb}{sina}\right)-log\left(\frac{sinb}{sin(\sqrt{ab})}\right)\geq\sqrt{\frac{a}{b}}log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

$$\Leftrightarrow log\left(\frac{sin\sqrt{ab}}{sina}\right)\geq\sqrt{\frac{a}{b}}log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

$$\Leftrightarrow\sqrt{b}log(sin\sqrt{ab})-\sqrt{b}log(sina)\geq\sqrt{a}log(sinb)-\sqrt{a}log(sin\sqrt{ab})$$

$$\Leftrightarrow(\sqrt{a}+\sqrt{b})log(sin\sqrt{ab})\geq\sqrt{a}log(sinb)+\sqrt{b}log(sina)$$

$$\Leftrightarrow log(sin\sqrt{ab})\geq\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}log(sinb)+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}log(sina);\quad (1)$$

We have:

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}log(sinb)+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}log(sina)\overset{logt-concave}{\leq}$$

$$\leq log\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}sinb+\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}sina\right)\overset{sint-concave\ (0,\pi)}{\leq}$$

$$\leq log\left(sin\left(\frac{b\sqrt{a}+a\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)=log\left(sin(\sqrt{ab})\right);(2)$$

From (1),(2) it follows that:

$$log\left(\frac{sinb}{sina}\right) \ge \left(1+\sqrt{\frac{a}{b}}\right) log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

A.085. Solution (Adrian Popa)

$$(a+b)\sqrt{c} \overset{Am-Gm}{\ge} 2\sqrt{ab}\cdot\sqrt{c} = 2\sqrt{abc} = 2 \to \frac{(a+b)\sqrt{c}}{2} \ge 1$$

Let: $\frac{(a+b)\sqrt{c}}{2} = t$

We must show that: $t(1-t) \le \frac{1}{t}\left(\frac{1}{t}-1\right); \forall t \ge 1$

$$t - t^2 \le \frac{1}{t^2} - \frac{1}{t} \leftrightarrow t + \frac{1}{t} \le t^2 + \frac{1}{t^2} \leftrightarrow t + \frac{1}{t} \le \left(t+\frac{1}{t}\right)^2 - 2$$

Let: $t + \frac{1}{t} = u; u \ge 2$. *We must show that:* $u \le u^2 - 2 \leftrightarrow u^2 - u - 2 \ge 0$

$$\Delta = 9, u_1 = -1; u_2 = 2$$

u	$-\infty \quad -1 \quad 2 \quad +\infty$
$u^2 - u - 2$	$++++0--0++++$

So, $u^2 - u - 2 \ge 0;\ \forall u \ge 2$ *then*

$$\frac{(a+b)\sqrt{c}}{2}\left(1-\frac{(a+b)\sqrt{c}}{2}\right) \le \frac{2}{(a+b)\sqrt{c}}\left(\frac{2}{(a+b)\sqrt{c}} - 1\right)$$

A.086. Solution (Khaled Abd Imouti)

Let us prove: $\left(M + G + \frac{G^2}{M}\right)^4 \overset{?}{\ge} M^4 + 15G^4 + 65\left(\frac{G^2}{M}\right)^4$

Where: $M = \frac{a+b}{2}, G = \sqrt{ab}$

Let us prove: $\frac{(M^2+MG+G^2)^4}{M^4} \ge \frac{M^8+15M^4G^4+65G^8}{M^4} \Leftrightarrow$

$$(M^2 + MG + G^2)^4 \ge M^8 + 15M^4G^4 + 65G^8$$

$$\left(G^2\left(\frac{M^2}{G^2} + \frac{M}{G} + 1\right)\right)^4 \ge G^8\left(\frac{M^8}{G^8} + 15\frac{M^4}{G^4} + 65\right)$$

$$\left(\frac{M^2}{G^2} + \frac{M}{G} + 1\right)^4 \ge \frac{M^8}{G^8} + 15\frac{M^4}{G^4} + 65$$

Suppose: $t = \frac{M}{G}; t \geq 1; (M \geq G)$

Let us prove: $(t^2 + t + 1)^4 \geq t^8 + 15t^4 + 65 \Leftrightarrow$

$$t^8 + 4t^7 + 6t^6 + 4t^5 + t^4 + 4(t^2 + t)^3 + 6(t^2 + t)^2 + 4(t^2 + t) + 1 \geq t^8 + 15t^4 + 65$$

$$l = 4t^7 + 10t^6 + 16t^5 + 10t^4 + 16t^3 + 4t^2 + 4t \geq 64$$

Because: $t \geq 1 \Rightarrow l \geq 64$ *(true)*

A.087. Solution (Jalil Hajimir)

From second equation $x + y + z > 0$*, since* $6x + 3y + 2z > 0$ *we conclude:*

$$x, y, z > 0$$

$$\left(\frac{x+y+z}{6}\right)^{x+y+z} = x\left(\frac{y}{2}\right)^2\left(\frac{z}{3}\right)^3 \text{ ;(1)}$$

$$x\left(\frac{y}{2}\right)^2\left(\frac{z}{3}\right)^3 \overset{Am-Gm}{\leq} \left(\frac{x+y+z}{6}\right)^6$$

$$\therefore \frac{x+y+z}{6} = 1;\ t^{t-1} \geq 1 \text{ ;(2)}$$

From (1),(2) we have:

$$\begin{cases} x = \frac{y}{2} = \frac{z}{3} \\ 6x + 3y + 2z = 18 \end{cases} \Rightarrow x = 1; y = 2; z = 3$$

A.088. Solution (Sudir Jha)

$$\frac{\left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{1}{ab}\right)}{\left(\sum_{cyc} \sqrt[3]{a}\right)\left(\sum_{cyc} \sqrt[3]{a^2}\right)} \geq \frac{\left(\sum_{cyc} \frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc} \frac{1}{\sqrt[3]{a^2}}\right)}{\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} \frac{1}{a^2b^2}\right)};\ \ (1)$$

Hence

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{1}{ab}\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} \frac{1}{a^2b^2}\right) \geq$$

$$\left(\sum_{cyc} \sqrt[3]{a}\right)\left(\sum_{cyc} \sqrt[3]{a^2}\right)\left(\sum_{cyc} \frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc} \frac{1}{\sqrt[3]{a^2}}\right)$$

Hence

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{c}{abc}\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc}\frac{c^2}{a^2b^2c^2}\right) \geq$$

$$\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)\left(\sum_{cyc}\frac{\sqrt[3]{bc}}{\sqrt[3]{abc}}\right)\left(\sum_{cyc}\frac{\sqrt[3]{b^2c^2}}{\sqrt[3]{a^2b^2c^2}}\right)$$

Hence

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \geq$$

$$a^2b^2c^2\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)\left(\sum_{cyc}\sqrt[3]{ab}\right)\left(\sum_{cyc}\sqrt[3]{a^2b^2}\right); \ (2)$$

By Chebyshev's inequality, we have:

$$\sum_{cyc} ab \geq \frac{\left(\sum_{cyc}\sqrt[3]{ab}\right)\left(\sum_{cyc}\sqrt[3]{a^2b^2}\right)}{3}; \ (3)$$

$$\left(\sum_{cyc} a\right) \geq \frac{\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)}{3}; \ (4)$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \overset{Am-Gm}{\geq} 3\cdot\sqrt[3]{\prod_{cyc} a^4}\cdot 3\cdot\sqrt[3]{\prod_{cyc} a^2} = 9a^2b^2c^2; \ \ (5)$$

Multiplying (3),(4),(5) we get (2) is true, then (1) is true.Proved.

A.089. Solution (Marian Ursărescu)

$$A + B = AB \Leftrightarrow A + B - AB = O_4 \Leftrightarrow AB - A - B + I_n = I_n$$

$$\Leftrightarrow A(B - I_n) - (B - I_n) = I_n \Leftrightarrow (A - I_n)(B - I_n) = I_n \text{ that mean}$$

$$XY = I_n \Leftrightarrow Y = X^{-1} \Rightarrow YX = I_n$$

$$(A - I_n)(B - I_n) = I_n \Rightarrow BA - B - A + I_n = I_n \Rightarrow BA = A + B \Rightarrow AB = BA$$

$$\text{and } (I_n - A)(I_n - B) = I_n; \quad (1)$$

$$I_n - A^3 - B^3 + (AB)^3 = I_n - A^3 - B^3 + A^3B^3 = I_n - A^3 - B^3(I_n - A^3)$$

$$= (I_n - A^3)(I_n - B^3) = (I_n - A)(I_n - B)(I_n + A + A^2)(I_n + B + B^2)$$

$$\overset{(1)}{=} (I_n + A + A^2)(I_n + B + B^2); \quad (2)$$

$$I_n - A^5 - B^5 + (AB)^5 = I_n - A^5 - B^5 + A^5B^5 = I_n - A^5 - B^5(I_n - A^5)$$

$$= (I_n - A^5)(I_n - B^5)$$

$$= (I_n - A)(I_n - B)(I_n + A + A^2 + A^3 + A^4)(I_n + B + B^2 + B^3 + B^4)$$

$$\overset{(1)}{\triangleq} (I_n + A + A^2 + A^3 + A^4)(I_n + B + B^2 + B^3 + B^4); \quad (3)$$

$$I_n - A^7 - B^7 + (AB)^7 = I_n - A^7 - B^7 + A^7B^7 = I_n - A^7 - B^7(I_n - A^7)$$

$$= (I_n - A^7)(I_n - B^7)$$

$$= (I_n - A)(I_n - B)(I_n + A + A^2 + A^3 + A^4 + A^5 + A^6)(I_n + B + B^2 + B^3 + B^4 + B^5 + +B^6)$$

$$\overset{(1)}{\triangleq} (I_n + A + A^2 + A^3 + A^4 + A^5 + A^6)(I_n + B + B^2 + B^3 + B^4 + B^5 + B^6); \quad (4)$$

From (2)+(3)+(4) we must show:

$$det(I_n + A + A^2)det(I_n + A + A^2 + A^3 + A^4)det(I_n + A + A^2 + A^3 + A^4 + A^5 + A^6)\cdot$$

$$\cdot det(I_n + B + B^2)det(I_n + B + B^2 + B^3 + B^4)det(I_n + B + B^2 + B^3 + B^4 + B^5 + B^6) \geq 0 \text{ true because}$$

$$det(I_n + X + X^2 + \cdots + X^{2n}) \geq 0 \quad \text{(R.M.M.-22)}$$

A.090. Solution (Adrian Popa)

$$([x] + \{x\})(y^2[x] + z^2\{x\}) = y^2[x]^2 + z^2[x]\{x\} + y^2[x]\{x\} + z^2\{x\}^2$$

$$= y^2[x]^2 + z^2\{x\}^2 + [x]\{x\}\underbrace{(z^2 + y^2)}_{2yz} \geq y^2[x]^2 + z^2\{x\}^2 + 2yz[x]\{x\}$$

$$= (y[x] + z\{x\})^2$$

A.091. Solution (Tran Hong)

For $a, b, c, d, e, f > 0$ we have:

$$a^a \cdot \left(\frac{d}{3}\right)^{\frac{d}{3}} \cdot \left(\frac{d}{3}\right)^{\frac{d}{3}} \cdot \left(\frac{d}{3}\right)^{\frac{d}{3}} \geq \left(\frac{a + \frac{d}{3} + \frac{d}{3} + \frac{d}{3}}{4}\right)^{a + \frac{d}{3} + \frac{d}{3} + \frac{d}{3}} = \frac{(a+d)^{a+d}}{4^{a+d}} \Rightarrow$$

$$(a + d)^{a+d} \leq 4^{a+d} \cdot a^a \cdot \frac{d^d}{3^d} \text{ and analogs}$$

$$(b + e)^{b+e} \leq 4^{b+e} \cdot b^b \cdot \frac{e^e}{3^e};$$

$$(c+f)^{c+f} \leq 4^{c+f} \cdot c^c \cdot \frac{f^f}{3^f}$$

$$(a+d)^{a+d} \cdot (b+e)^{b+e} \cdot \ (c+f)^{c+f}$$

$$\leq 4^{a+b+c+d+e+f} \cdot a^a \cdot b^b \cdot c^c \cdot d^d \cdot e^e \cdot f^f \cdot \frac{1}{3^{d+e+f}}$$

$$= 4^{12} \cdot a^a \cdot b^b \cdot c^c \cdot d^d \cdot e^e \cdot f^f \cdot \frac{1}{3^9} \Leftrightarrow$$

$$\frac{a^a \cdot b^b \cdot c^c \cdot d^d \cdot e^e \cdot f^f}{(a+d)^{a+d}(b+e)^{b+e}(c+f)^{c+f}} \geq \frac{3^9}{4^{12}}$$

$$\Leftrightarrow \frac{a^a \cdot b^b \cdot c^c \cdot d^d \cdot e^e \cdot f^f}{3(a+d)^{a+d}(b+e)^{b+e}(c+f)^{c+f}} \geq \left(\frac{3}{8}\right)^8$$

Proved.Equality for $a=b=c=1$ *and* $d=e=f=3$

A.092. *First we prove that if* $t>0$ *then:* $3t^6-3t^2 \geq 4t^3-4 \quad (1)$

$$3t^2(t^4-1)-4(t^3-1) \geq 0$$

$$\Leftrightarrow 3t^2(t-1)(t+1)(t^2+1)-4(t-1)(t^2+t+1) \geq 0$$

$$\Leftrightarrow (t-1)[3t^2(t^3+t+t^2+1)-4(t^2+t+1) \geq 0$$

$$\Leftrightarrow (t-1)[3t^4(t-1)+6t^3(t-1)+9t^2(t-1)+8t(t-1)+4(t-1)]$$

$$\geq 0 \ \Leftrightarrow (t-1)^2(3t^4+6t^3+9t^2+8t+4) \geq 0$$

Replace $x=t^6$ *in (1):* $3x-3\sqrt[3]{x} \geq 4\sqrt{x}-4 \quad (2)$

Analogous: $3y-3\sqrt[3]{y} \geq 4\sqrt{y}-4 \quad (3)$ *and* $3z-3\sqrt[3]{z} \geq 4\sqrt{z}-4 \quad (4)$

By adding (2),(3),(4) we have:

$$3(x+y+z)-3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right) \geq 4\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)-12=0$$

Then: $x+y+z \geq \sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}$

A.093. $\frac{a^2+ab+b^2}{a+b} \geq \frac{3}{4}(a+b) \ \ (1)$

$$4(a^2+ab+b^2) \geq 3(a+b)^2 \Leftrightarrow 4a^2+4ab+4b^2 \geq 3a^2+6ab+3b^2$$

$$\Leftrightarrow 4a^2+4ab+4b^2-3a^2-6ab-3b^2 \geq 0 \Leftrightarrow a^2-2ab+b^2 \geq 0 \Leftrightarrow$$

$(a-b)^2 \geq 0$. *By multypling (1) with c:*

$$\frac{a^2c+abc+b^2c}{a+b}\ge\frac{3}{4}(ac+bc)\Leftrightarrow\sum_{cyc}\frac{a^2c+abc+b^2c}{a+b}\ge\frac{3}{4}\sum_{cyc}(ac+bc)$$

$$\sum_{cyc}\frac{a^2c+abc+b^2c}{a+b}\ge\frac{3}{4}\cdot 2(ab+bc+ca)$$

$$\sum_{cyc}\frac{c(a^2+b^2)+1}{a+b}\ge\frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

A.094. Solution (Sanong Huayrerai)

For $x,y,z>0$ *we give* $x=a^2;y=b^3;z=c^6$

Hence $\frac{(1+\sqrt{x})(1+\sqrt[3]{y})(1+\sqrt[6]{z})}{\sqrt{1+x}\cdot\sqrt[3]{1+y}\cdot\sqrt[6]{1+z}}\ge 4$

Iff $(1+a)(1+b)(1+c)\le 4\sqrt{1+a^2}\cdot\sqrt[3]{1+b^3}\cdot\sqrt[6]{1+c^6}$

$$=4\sqrt[6]{(1+a^2)^3(1+b^3)^2(1+c^6)}$$

Iff $(1+a)^6(1+b)^6(1+c)^6\le 4^6\cdot(1+a^2)^3(1+b^3)^2(1+c^6)$ *true, because*

$$(1+a)^6\le 2^3(1+a^2)^3,\qquad (1+b)^6\le 2^4(1+b^3)^2$$

$$(1+c)^6\le 2^5(1+c^6)$$

A.095. Solution (Tran Hong)

$$\frac{1}{a+b}\overset{CBS}{\leq}\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)\Rightarrow\frac{ab}{12(a+b)}\le\frac{1}{48}(a+b)\quad(1)$$

$$\frac{1}{ab+bc+ca}\overset{CBS}{\leq}\frac{1}{9}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\Rightarrow\frac{abc}{8(ab+bc+ca)}\le\frac{1}{72}(a+b+c)\quad(2)$$

$$\frac{1}{abc+bcd+cda+dab}\le\frac{1}{16}\left(\frac{1}{abc}+\frac{1}{bcd}+\frac{1}{cda}+\frac{1}{dab}\right)\Rightarrow$$

$$\frac{abcd}{6(abc+bcd+cda+dab)}\le\frac{1}{96}(a+b+c+d)\quad(3)\xRightarrow{(1)+(2)+(3)}$$

$$LHS\le\frac{1}{48}(a+b)+\frac{1}{72}(a+b+c)+\frac{1}{96}(a+b+c+d)$$

$$=\frac{13}{288}a+\frac{13}{288}b+\frac{13}{288}c+\frac{1}{96}d<\frac{a+b+c+d}{12}=1$$

A.096. Solution (Jalil Hajimir)

Let: $f_1(x,y,z)=\frac{x}{y+z+1}+\frac{1}{3}xye^z,\ f_2(x,y,z)=\frac{y}{z+x+1}+\frac{1}{3}xye^z$

$$f_3(x,y,z)=\frac{z}{x+y+1}+\frac{1}{3}xye^z$$

$f(x,y,z)=f_1(x,y,z)+f_2(x,y,z)+f_3(x,y,z)$ *is convex function.*

$A=\{x,y,z\in\mathbb{R},0\le x,y,z\le 2\}$ *is a closed convex set*

$f(2,2,2)=\frac{6}{5}+4e^2$ *is the greatest value, then* $f(x,y,z)\le\frac{6}{5}+4e^2$ *is sum of there convex functions is convex.*

Let's prove $f_1(x,y,z)=\frac{x}{y+z+1}+\frac{1}{3}xye^z$ *is convex.*

$\nabla^2(f)$ *is a positive semi definite matrix.*

$$\begin{pmatrix} 0 & e^z-\frac{1}{(y+z+1)^2} & ye^z-\frac{1}{(y+z+1)^2} \\ e^z-\frac{1}{(y+z+1)^2} & \frac{2x}{(y+z+1)^3} & xe^z+\frac{2x}{(y+z+1)^3} \\ ye^z-\frac{1}{(y+z+1)^2} & xe^z+\frac{2x}{(y+z+1)^3} & xye^z+\frac{2x}{(y+z+1)^3} \end{pmatrix}$$

Therefore, $x=y=z=2$ *is only solution for the given equation.*

A.097. Solution (Tran Hong)

Let: $a=x-\frac{1}{2};b=y-\frac{1}{2};c=x+\frac{1}{2};d=y+\frac{1}{2}$

$$\Omega=\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2}+\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2}+\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2}+\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2}-8$$

$$=\frac{(a^2+d^2)(c^2+b^2)(c^2+d^2)+(a^2+b^2)(c^2+b^2)(c^2+d^2)+(a^2+b^2)(a^2+d^2)(c^2+d^2)}{(a^2+b^2)(a^2+d^2)(c^2+b^2)(c^2+d^2)}$$

$$+\frac{(a^2+b^2)(c^2+b^2)(c^2+d^2)-8(a^2+b^2)(c^2+b^2)(c^2+d^2)(a^2+d^2)}{(a^2+b^2)(a^2+d^2)(c^2+b^2)(c^2+d^2)}=$$

$$=\frac{-16[8(x^8+y^8)+32x^2y^2(x^4+y^4)+48x^4y^4+12x^2y^2(x^2+y^2)-28x^2y^2-4(x^6+y^6)+2(x^4+y^4)-(x^2+y^2)]}{(2x^2+2y^2-2x-2y+1)(2x^2+2y^2+2x-2y+1)(2x^2+2y^2-2x+2y+1)(2x^2+2y^2+2x+2y+1)}$$

$$\Leftrightarrow\ 8(x^8+y^8)+32x^2y^2(x^4+y^4)+48x^4y^4+12x^2y^2(x^2+y^2)+$$
$$2(x^4+y^4)\ \le 28x^2y^2+4(x^6+y^6)+(x^2+y^2)\ \ (*)$$

$$|x| < \frac{1}{2}, |y| < \frac{1}{2} \Rightarrow x^2 < \frac{1}{4};\ y^2 < \frac{1}{4}, x^8 + y^8 = x^2x^6 + y^2y^6 \le \frac{1}{4}x^6 + \frac{1}{4}y^6$$

$$8(x^8 + y^8) \le 2(x^6 + y^6) \le 4(x^6 + y^6) \quad (1)$$

$$x^4 + y^4 = x^2x^2 + y^2y^2 \le \frac{1}{4}x^2 + \frac{1}{4}y^2$$

$$\Rightarrow 2(x^4 + y^4) \le \frac{1}{2}(x^2 + y^2) \le x^2 + y^2 \quad (2)$$

$$x^4y^4 = x^2x^2 \cdot y^2y^2 \le \frac{1}{4}x^2 \cdot \frac{1}{4}y^2 = \frac{1}{16}x^2y^2 \Rightarrow 48x^4y^4 \le 3x^2y^2 \le 6x^2y^2 \quad (3)$$

$$x^2 + y^2 < \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \Rightarrow 12x^2y^2(x^2 + y^2) \le 6x^2y^2 \quad (4)$$

From (1)+(2)+(3)+(4)⇒(*) is true. Equality⇔ $x = y = 0$

So, $LHS \ge 8 > \frac{8}{\sqrt[4]{5}}$*.Proved.*

A.098. Solution (Lety Sauceda)

$$3x^6 - 9x^5 + 18x^4 - 21x^3 + 15x^2 - 6x + 1 = 0$$

$$x^6 - 3x^5 + 6x^4 - 7x^3 + 5x^2 - 2x + \frac{1}{3} = 0$$

For: $(x^2 - x + a)(x^2 - x + b)(x^2 - x + c) = 0$ *we have:*

$$x^6 - 3x^5 + (a + b + c + 3)x^4 - (2a + 2b + 2c + 1)x^3$$
$$- (a + b + c + ab + bc + ca)x^2 + (ab + bc + ca)x + abc = 0$$

$$\Rightarrow \begin{cases} 3abc = 1 \\ a + b + c = 3 \\ ab + bc + ca = 2 \end{cases}$$

$$3a^3 - 9a^2 + 6a - 1 = 0 \Leftrightarrow 3(a - 1)^3 - 3(a - 1) - 1 = 0$$

$$\xRightarrow{a-1=w} 3w^3 - 3w - 1 = 0 \Leftrightarrow w^3 - w + \frac{1}{3} = 0 \xRightarrow{w=s+r}$$

$$\begin{cases} s^3 + r^3 + (3sr + 1)w + \frac{1}{3} = 0 \\ r^3 = \frac{1}{27s^3} \\ s^3 + r^3 - \frac{1}{3} = 0 \end{cases}$$

$$27s^6 - 9s^3 + 1 = 0 \Rightarrow s^3 = \frac{3+\sqrt{-3}}{18} \overset{w=s+r}{\Longrightarrow}$$

$$\begin{cases} w_1 = \sqrt[3]{\frac{3+\sqrt{-3}}{18}} + +\sqrt[3]{\frac{3-\sqrt{-3}}{18}} \\ w_2 = \left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{3+\sqrt{-3}}{18}\right)^{\frac{1}{3}} - \left(\frac{1+\sqrt{-3}}{2}\right)\left(\frac{3-\sqrt{-3}}{18}\right)^{\frac{1}{3}} \\ w_3 = \left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{3-\sqrt{-3}}{18}\right)^{\frac{1}{3}} - \left(\frac{1+\sqrt{-3}}{2}\right)\left(\frac{3+\sqrt{-3}}{18}\right)^{\frac{1}{3}} \end{cases}$$

$$\begin{cases} a_1 = w_1 + 1 \\ a_2 = w_2 + 1 \\ a_3 = w_3 + 1 \end{cases}$$

Let: $a_1 = a;\ a_2 = b;\ a_3 = c \Rightarrow$

$$(x^2 - x + w_1 + 1)(x^2 - x + w_2 + 1)(x^2 - x + w_3 + 1) = 0$$

$$\begin{cases} a = 2.1371580426 \dots \\ b = .25777280103 \dots \\ c = .60506915636 \dots \end{cases}$$

$$(x^2 - x + 2.1371580426 \dots)(x^2 - x + .25777280103 \dots)(x^2 - x + c$$
$$= .60506915636 \dots) = 0$$

A.099. Solution (Tran Hong)

For $x > 0$ *we have:* $\varphi(x) = logx\ increasing\ on\ (1, \infty)$

$$\overset{a>2}{\Longrightarrow} 0 < log(a-1)loga \le log(a-1)log(a+1) \le logalog(a+1) \quad (1)$$

$$0 \le x \le y \le z; x + y + z = 3 \Rightarrow z \ge 1$$

Case1: $x \ge 1 \overset{y\ge x}{\Longrightarrow} y \ge 1$

$$\Rightarrow (x-1)logalog(a-1) + (y-1)log(a-1)log(a+1)$$
$$+(z-1)logalog(a+1) \ge 0$$

Case 2: $0 \le x \le 1 \overset{y\ge x}{\Longrightarrow} \begin{cases} y \ge 1 \ge x \\ 0 \le x \le y \le 1 \end{cases}$

If $0 \le x \le y \le 1$ *then:*

$$(x-1)logalog(a-1) + (y-1)log(a-1)log(a+1)$$

$$+(z-1)logalog(a+1)$$
$$\geq (x-1)logalog(a+1)+(y-1)logalog(a+1)$$
$$+(z-1)logalog(a+1)=(x+y+z-3)logalog(a+1)\overset{x+y+z=3}{=}0$$

If $0\leq x\leq 1\leq y$ then

$$(x-1)logalog(a-1)+(y-1)log(a-1)log(a+1)$$
$$+(z-1)log(a-1)log(a+1)$$
$$\geq (x-1)log(a-1)log(a+1)+(y-1)log(a$$
$$-1)log(a+1)$$
$$+(z-1)log(a-1)log(a+1)$$
$$=(x+y+z-3)log(a-1)log(a+1)\overset{x+y+z=3}{=}0$$

A.100. Solution (Tran Hong)

$$\left(\sqrt{x}+\sqrt{y}\right)^2\overset{CBS}{\leq}\left(\sqrt{1^2+1^2}+\sqrt{\sqrt{x}^2+\sqrt{y}^2}\right)^2=2(x+y)$$

$$\left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy}\overset{Am-Gm}{\leq}\left(\sqrt{x}+\sqrt{y}\right)\frac{\sqrt{x}+\sqrt{y}}{2}=\frac{\left(\sqrt{x}+\sqrt{y}\right)^2}{2}\leq\sqrt{x}^2+\sqrt{y}^2$$
$$=x+y$$

$$\overset{0\leq z\leq 1}{\Longrightarrow}\Omega=2(1-z)\left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy}+z\left(\sqrt{x}+\sqrt{y}\right)^2\leq(2-2z+2z)(x+y)$$
$$=2(x+y)$$

$$\left(\sqrt{x}+\sqrt{y}\right)^2\overset{Am-Gm}{\geq}4\sqrt{xy}\Rightarrow\sqrt{x}+\sqrt{y}\geq 2\sqrt[4]{xy}\Rightarrow\left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy}\geq 2\sqrt{xy}$$
$$\underset{0\leq z\leq 1}{\Longrightarrow}\Omega\geq\left(4z+4(1-z)\right)\sqrt{xy}=4\sqrt{xy}$$

A.101. Solution (Abdallah El Farissi)

Let $A=\frac{2\sqrt{ab}}{a+b};B=\sqrt{\frac{a^2+b^2}{2ab}};C=\frac{\sqrt{2(a^2+b^2)}}{a+b}$

We have: $1\leq A\leq\frac{a}{b};1\leq B\leq\frac{a}{b};1\leq C\leq\frac{a}{b}$

Let $f(x)=x+\frac{1}{x}, f$ –is positive and increasing function in $[1,\infty)$, it follows

that $f(A)f(B)f(C)\le f^3\left(\frac{a}{b}\right)\Leftrightarrow$

$$\left(\frac{2\sqrt{ab}}{a+b}+\frac{a+b}{2\sqrt{ab}}\right)\left(\sqrt{\frac{a^2+b^2}{2ab}}+\sqrt{\frac{2ab}{a^2+b^2}}\right)\left(\frac{\sqrt{2(a^2+b^2)}}{a+b}+\frac{a+b}{\sqrt{2(a^2+b^2)}}\right)$$
$$\le\left(\frac{a}{b}+\frac{b}{a}\right)^3$$

A.102. Solution (George Florin Şerban)

Denote: $a+1=x; b+1=y; c+1=z; x,y,z>0\Rightarrow$

$$\frac{(3x+2y+z)(x+3y+2z)(2x+y+3z)}{xyz}\ge 216$$

$$3x+2y+z=x+x+x+y+y+z\overset{Am-Gm}{\ge}6\cdot\sqrt[6]{x^3y^2z}$$

$$x+3y+2z=x+y+y+y+z+z\overset{Am-Gm}{\ge}6\cdot\sqrt[6]{xy^3z^2}$$

$$2x+y+3z=x+x+y+z+z+z\overset{Am-Gm}{\ge}6\cdot\sqrt[6]{x^2yz^3}$$

$$\prod_{cyc}(3x+2y+z)\ge 6^3\cdot\sqrt[6]{x^6y^6z^6}=216xyz,\qquad \frac{\prod(3x+2y+z)}{xyz}\ge 216$$

Equality for $a=b=c$

A.103. Solution (Adrian Popa) $a,b,c>0$

$$\frac{(a^2+a+1)^{\sqrt{3}}(b^2+b+1)^{\sqrt{3}}(a^2+a+1)^{\sqrt{3}}(c^2+c+1)^{\sqrt{3}}}{e^{2a}\cdot e^{2b}\cdot e^{2c}}\le 1\Leftrightarrow$$

$$\prod_{cyc}(a^2+a+1)^{\sqrt{3}}\le\prod_{cyc}e^{2a}\quad(*),\qquad \sqrt{3}\sum_{cyc}log(a^2+a+1)\le 2\sum_{cyc}a$$

Let: $f:(0,\infty)\to\mathbb{R}; f(x)=\sqrt{3}log(x^2+x+1)-2x$

$$f'(x)=\frac{-2x^2+(2\sqrt{3}-2)x+\sqrt{3}-2}{x^2+x+1},\ f'(x)<0,\forall x\ge 0 \textit{ then}$$

$$\begin{cases} f\searrow \\ f(0)=0\end{cases}\Rightarrow f(x)\le 0,\forall x\ge 0\Rightarrow\sqrt{3}log(x^2+x+1)\le 2x$$

$$\begin{cases}\sqrt{3}log(a^2+a+1)\le 2a\\ \sqrt{3}log(b^2+b+1)\le 2b\Rightarrow(*)\\ \sqrt{3}log(c^2+c+1)\le 2c\end{cases}$$

A.104. Solution (Adrian Popa)

$$(x+2)(y+3)=8\ldots(1)$$

$$\sqrt{[x]\cdot[y]}+\sqrt{(x-[x])(y-[y])}=\sqrt{xy}\ldots(2)$$

$$\begin{cases}x\ge 0\Rightarrow x+2\ge 2\\ y\ge 0\Rightarrow y+3\ge 3\\ (x+2)(y+3)=8\end{cases}\Rightarrow\begin{cases}x<1\Rightarrow[x]=0\\ y\le 1\Rightarrow[y]=0\ or\ [y]=1\end{cases}$$

$$i)\ [x]=0\ and\ [y]=0\overset{by\ (2)}{\Longrightarrow}\sqrt{xy}=\sqrt{xy}$$

$$ii) If\ x=0\ then\ y=1$$

$$iii) If\ x>0\ then\ y+3=\frac{8}{x+2}$$

$$y=\frac{2-3x}{x+2}>0\ then\ x<\frac{2}{3}\ldots(3)$$

$$\frac{2-3x}{x+2}<1\ \ then\ x>0\ldots(4)$$

From (3)+(4) we have:

$$x\in\left(0,\frac{2}{3}\right)\ and\ y=\frac{2-3x}{x+2}$$

A.105. Solution (George Florin Şerban)

$$a^2b^2+b^2c^2+c^2a^2=12abc\Rightarrow\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}=12$$

Let $x=\frac{ab}{c},y=\frac{bc}{a},z=\frac{ca}{b}\Rightarrow x+y+z=12$

$$\sum_{cyc}\sqrt[3]{\frac{a}{4a+bc}}=\sum_{cyc}\frac{1}{\sqrt[3]{4+\frac{bc}{a}}}=\sum_{cyc}\frac{1}{\sqrt[3]{4+x}}\overset{(*)}{\ge}\frac{3}{2}$$

Let be the function: $f:(0,\infty)\to\mathbb{R},f(x)=\frac{1}{\sqrt[3]{4+x}}=(x+4)^{-\frac{1}{3}}$ −convex.

$$f'(x)=-\frac{1}{3}(x+4)^{-\frac{4}{3}};f''(x)=\frac{4}{9}(x+4)^{-\frac{7}{3}}>0$$

From Jensen Inequality, we have: $f\left(\frac{x+y+z}{3}\right) \leq \frac{f(x)+f(y)+f(z)}{3} \Leftrightarrow$

$$\sum_{cyc} \frac{1}{\sqrt[3]{4+x}} \geq \frac{3}{\sqrt[3]{4+\frac{1}{3}\sum_{cyc} x}} \Leftrightarrow \sum_{cyc} \frac{1}{\sqrt[3]{4+x}} \geq \frac{3}{2}$$

A.106. Solution (Bedri Hajrizi)

$$5^{2x+1} + 20x^2 + 29x + 6 = 11 \cdot 5^x + x \cdot 5^{x+2}$$

$$5 \cdot 5^{2x} + 20x^2 + 29x + 6 = 11 \cdot 5^x + x \cdot 5^{x+2}$$

$$5 \cdot 5^{-2x} - (25x + 11) \cdot 5^x + 20x^2 + 29x + 6 = 0$$

$$5^x = \frac{25x + 11 \pm \sqrt{(25x+11)^2 - 20(20x^2 + 29x + 6)}}{10}$$

$$= \frac{25x + 11 \pm \sqrt{625x^2 + 550x + 121 - 400x^2 - 580x - 120}}{10}$$

$$= \frac{25x + 11 \pm (15x - 1)}{10}$$

$$5^x = 4x + 1 \ or \ 5^x = x + 2$$

$$i)\ 5^x = 4x + 1 \Leftrightarrow x \in \{0,1\}$$

$$ii)\ 5^x = x + 2 \Rightarrow x = -1 \ one\ solution$$

$$Let\ f(x) =)\ 5^x - x - 2$$

$$f(0) = 1 - 1 \cdot 2 < 0 \ and\ \ f(1) = 5 - 1 - 1 \cdot 2 > 0 \Rightarrow$$

$\exists \alpha \in (0,1)$ *such that* $f(\alpha) = 0.$ *So:* $x \in \{-1, 0, \alpha, 1\}$

A.107. Solution (Tran Hong)

Let: $f(a) = \sqrt[4]{a^3}; a > 0 \Rightarrow f'(a) = \frac{3}{4} a^{-\frac{1}{4}}$

$$f''(x) = -\frac{3}{16} a^{-\frac{5}{4}} < 0, \forall a > 0, \qquad \sum_{cyc} \sqrt[4]{a^3} \overset{Jensen}{\leq} 3 \cdot \sqrt[3]{\left(\frac{a+b+c}{3}\right)^3}$$

Similary

$$\sum_{cyc} \sqrt[5]{a^4} \overset{Jensen}{\leq} 3 \cdot \sqrt[5]{\left(\frac{a+b+c}{3}\right)^4}, \qquad \sum_{cyc} \sqrt[6]{a^5} \overset{Jensen}{\leq} 3 \cdot \sqrt[6]{\left(\frac{a+b+c}{3}\right)^5}$$

$$\left(\sum_{cyc}\sqrt[4]{a^3}\right)^4\left(\sum_{cyc}\sqrt[5]{a^4}\right)^5\left(\sum_{cyc}\sqrt[6]{a^5}\right)^6$$

$$\leq 3^4\left(\frac{a+b+c}{3}\right)^3\cdot 3^5\left(\frac{a+b+c}{3}\right)^4\cdot 3^6\left(\frac{a+b+c}{3}\right)^5$$

$$=27(a+b+c)^{12}$$

A.108. Solution (Tran Hong)

$$LHS=\sum_{cyc}\frac{(x^2+y^2+z^2+2xy+2zy)u^2}{xz}$$

$$=\sum_{cyc}\frac{(x^2+y^2+z^2+2xy+2zy+2xz-2xz)u^2}{xz}$$

$$=\sum_{cyc}\frac{((x+y+z)^2-2xz)u^2}{xz}=(x+y+z)^2\sum_{cyc}\frac{u^2}{xz}-2\sum_{cyc}u^2$$

$$\overset{C-B-S}{\geq}(x+y+z)^2\cdot\frac{(u+v+w)^2}{xy+yz+zx}-2\sum_{cyc}u^2$$

$$\overset{\sum x^2\geq 3\sum xy}{\geq}3(u+v+w)^2-2\sum_{cyc}u^2$$

$$=u^2+v^2+w^2+6(uv+vw+wu)=18+u^2+v^2+w^2$$

A.109. Solution (Tran Hong)

$$x,y,z\geq 1\Rightarrow x+3,y+3,z+3\geq 4\Rightarrow\prod_{cyc}(x+3)\geq 4\cdot 4\cdot 4$$

So, we must show that:

$$4\left(\prod_{cyc}(x+1)+8xyz\right)\geq\prod_{cyc}(3x+1)\Leftrightarrow$$

$$4(9xyz+xy+yz+zx+x+y+z+1)$$

$$\geq(27xyz+9(xy+yz+zx)+3(x+y+z)+1)$$

$$9xyz\geq 5(xy+yz+zx)-(x+y+z)-3\Leftrightarrow$$

$$9xyz+(x+y+z)+3\ge 5(xy+yz+zx)$$

Let $f(z)=\big(9xy+1-5(x+y)\big)z+x+y-5xy+3;(z\ge 1)$

$$9xy+1-5(x+y)=(9x-5)y+1-5x\ge(9x-5)\cdot 1+1-5x=$$

$$=4x+1-5\ge 4\cdot 1+1-5=0;(x,y\ge 1)$$

$$f(z)\nearrow[1,\infty)\Rightarrow f(z)\ge f(1)=4(x-1)(y-1)\ge 0,\forall x,y\ge 1$$

$$f(z)\ge 0\Rightarrow 9xyz+(x+y+z)+3\ge 5(xy+yz+zx),\forall x,y\ge 1$$

A.110. Solution (Tran Hong)

$$\left(\sum_{cyc}\sqrt{a+2b}\right)^2=3(a+b+c)+2\sum_{cyc}\left[\sqrt{a+2b}\cdot\sqrt{b+2c}\right]$$

$$=9+2\sum_{cyc}\left[\sqrt{a+2b}\sqrt{b+2c}\right]$$

$$\overset{Am-Gm}{\overbrace{\geq}}\ 9+2\cdot 3\cdot\sqrt[3]{\left(\sqrt{a+2b}\right)^2\left(\sqrt{b+2c}\right)^2\left(\sqrt{c+2a}\right)^2}$$

$$=9+6\cdot\sqrt[3]{(a+2b)(b+2c)(c+2a)}$$

$$=9+6\cdot\sqrt[3]{(3+a-b)(3+b-c)(3+c-a)}$$

A.111. Solution (Adrian Popa)

$$0<a\le b;1\le c\le 1$$

$$2c(a+b)\sqrt{ab}+(1-c)(a+b)^2\ge 4ab\ldots(1)\quad\therefore a+b\ \overset{Am-Gm}{\overbrace{\geq}}\ 2\sqrt{ab}\Rightarrow$$

$$2c(a+b)\sqrt{ab}+(1-c)(a+b)^2\ge 2c\cdot 2\sqrt{ab}+(1-c)\cdot 4ab=$$

$$4abc+4ab-4abc=4ab$$

$$2c(a+b)\sqrt{ab}+(1-c)(a+b)^2\ \overset{Am-Gm}{\overbrace{\leq}}\ c(a+b)(a+b)+(1-c)(a+b)^2$$

$$=(a+b)^2$$

We must show: $a+b\le\sqrt{2(a^2+b^2)}$ *true from Qm-Am.*

A.112. Solution (Petre Daniel Alexandru)

$$\frac{x^7}{y^{30}}+\frac{y^7}{z^{30}}+\frac{z^7}{x^{30}}=\frac{x^7}{(y^5)^6}+\frac{y^7}{(z^5)^6}+\frac{z^7}{(x^5)^6}\ \overset{Radon}{\overbrace{\geq}}\ \frac{(x+y+z)^7}{(x^5+y^5+z^5)^6}$$

Equality for

$$\frac{x}{y}=\frac{y}{z}=\frac{z}{x}\Leftrightarrow x=y=z$$

$$x\in\mathbb{Z}, x^4-3y^3-2z^2-3y+1=0\Leftrightarrow (x^2-4x+1)(x^2+x+1)=0$$

$$x^2-4x+1=0\Rightarrow x_{1,2}=2\pm\sqrt{3},\ \ x^2+x+1=0\Leftrightarrow x_{3,4}=\frac{-1\pm i\sqrt{3}}{2}$$

A.113. Solution (Ravi Prakash)

$$Let\ x=sina, y=sinb, z=sinc; 0\le a,b,c\le\tfrac{\pi}{2}$$

$$Consider\ \ 2\left(\sqrt{1-x^2}\sqrt{1-y^2}-xy\right)+2\alpha(x+y)=$$

$$=2(cosacosb-sinasinb)+2\alpha(sina+sinb)$$

$$=2cos(a+b)+2\alpha(sina+sinb)$$

$$=2\left[1-2sin^2\left(\frac{a+b}{2}\right)+2\alpha sin\left(\frac{a+b}{2}\right)cos\left(\frac{a-b}{2}\right)\right]$$

$$=2-4sin^2\left(\frac{a+b}{2}\right)+4\alpha sin\left(\frac{a+b}{2}\right)$$

$$=2-\left\{\left(2sin\left(\frac{a+b}{2}\right)-\alpha\right)^2-\alpha^2\right\}\le 2+\alpha^2\Rightarrow$$

$$2\left(\sqrt{1-x^2}\sqrt{1-y^2}-xy\right)\le 2+\alpha^2-2\alpha(x+y)$$

Similarly write the other two terms and add to obtain the desired inequality.

A.114. Solution (George Florin Şerban)

$$\frac{x^2z^2-2xyzt+y^2t^2+x^2zt+xyz^2+xyzt-xyt^2-y^2zt-y^2t^2}{xyzt}+$$

$$\frac{x^2t^2+y^2z^2+y^2t^2+2xyzt+2y^2zt+2xyt^2}{xyzt}\ge 9$$

$$\frac{x^2z^2+y^2z^2+y^2t^2+x^2t^2+x^2zt+xyz^2+xyzt+xyt^2+y^2zt}{xyzt}\ge 9$$

$$\left(\frac{xz}{yt}+\frac{yt}{xz}\right)+\left(\frac{yz}{xt}+\frac{xt}{yz}\right)+\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{z}{t}+\frac{t}{z}\right)\ge 8$$

$$\left(\frac{xz}{yt}+\frac{yt}{xz}\right)+\left(\frac{yz}{xt}+\frac{xt}{yz}\right)+\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{z}{t}+\frac{t}{z}\right)\overset{Am-Gm}{\geq}$$

$$2\sqrt{\frac{xz}{yt}\cdot\frac{yt}{xz}}+2\sqrt{\frac{yz}{xt}\cdot\frac{xt}{yz}}+2\sqrt{\frac{x}{y}\cdot\frac{y}{x}}+2\sqrt{\frac{z}{t}\cdot\frac{t}{z}}=8\ true.$$

A.115. Solution (Tran Hong)

$$x^4+y^4+z^4+u^4+v^4=(x^2+y^2+z^2+u^2)v^2\ ...(*)$$

$$\therefore x^4+\frac{1}{4}v^4\overset{Am-Gm}{\geq}2\sqrt{x^4\cdot\frac{1}{4}v^4}=x^2v^2$$

$$\therefore y^4+\tfrac{1}{4}v^4\overset{Am-Gm}{\geq}y^2v^2,\ \therefore z^4+\tfrac{1}{4}v^4\geq z^2v^2,\ \therefore u^4+\tfrac{1}{4}v^4\overset{Am-Gm}{\geq}u^2v^2$$

$$x^4+y^4+z^4+u^4+\frac{4}{4}v^4=(x^2+y^2+z^2+u^2)v^2\Leftrightarrow$$

$$x^4+y^4+z^4+u^4+v^4=(x^2+y^2+z^2+u^2)v^2\Rightarrow(*)\text{true.}$$

$$\text{Equality}\Leftrightarrow x^2=y^2=z^2=u^2=\frac{1}{2}v^2=\alpha^2\ \ (\alpha>0)$$

$$\Rightarrow |x|=|y|=|z|=|u|=\alpha;\ |v|=\alpha\sqrt{2}$$

• $If\ x;y;z;u\geq 0\Rightarrow x+y+z+u=4\alpha=4\Rightarrow\alpha=1\Rightarrow x=y=z=u$
$=1;v=\pm\sqrt{2}\ \Rightarrow(x;y;z;u)=(1;1;1;1),v=\pm\sqrt{2}$

• $If\ x;y;z;u<0\Rightarrow x+y+z+u<0<4$

No solution.

• $If\ \begin{cases}xy<0\\zu<0\end{cases}$ or $\begin{cases}xz<0\\yu<0\end{cases}$ or $\begin{cases}xu<0\\yz<0\end{cases}$ then $x+y+z+u=0<4\Rightarrow$

No solution.

• $If\ x\geq 0;y=z=u\leq 0$ (and cyclic)then $x+y+z+u=\alpha-\alpha-\alpha-\alpha=4\Rightarrow\alpha=-2\Rightarrow$no solution.

• $If\ x<0;y=z=u=2$(and cyclic)then $x+y+z+u=-\alpha+\alpha+\alpha+\alpha=2\alpha=4\Rightarrow\alpha=2\Rightarrow x=-2;y=z=u=2;v=\pm 2\sqrt{2}$

$$\Rightarrow(x;y;z;u)=(-2;2;2;2)\text{(and cyclic)},v=\pm 2\sqrt{2}$$

A.116. Solution (Sanong Huayrerai)

$$For\ a,b,c>0\ and\ abc=1, we\ have$$

$$\left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)\left(\frac{b+c}{\sqrt{b}+\sqrt{c}}\right)\left(\frac{c+a}{\sqrt{c}+\sqrt{a}}\right)\ge\frac{(\sqrt{a}+\sqrt{b})^2(\sqrt{b}+\sqrt{c})^2(\sqrt{c}+\sqrt{a})^2}{8(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})}$$

$$=\frac{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})}{8}\ge\frac{\left(\sqrt[6]{abc}+\sqrt[6]{abc}\right)^3}{8}=1$$

$$\left(\frac{a+b}{\sqrt[4]{a}+\sqrt[4]{b}}\right)\left(\frac{b+c}{\sqrt[4]{b}+\sqrt[4]{c}}\right)\left(\frac{c+a}{\sqrt[4]{c}+\sqrt[4]{a}}\right)$$

$$\ge\frac{(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{b}+\sqrt[4]{c})(\sqrt[4]{c}+\sqrt[4]{a})(\sqrt[4]{a^3}+\sqrt[4]{b^3})(\sqrt[4]{b^3}+\sqrt[4]{c^3})(\sqrt[4]{c^3}+\sqrt[4]{a^3})}{8(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{b}+\sqrt[4]{c})(\sqrt[4]{c}+\sqrt[4]{a})}$$

$$=\frac{(\sqrt[4]{a^3}+\sqrt[4]{b^3})(\sqrt[4]{b^3}+\sqrt[4]{c^3})(\sqrt[4]{c^3}+\sqrt[4]{a^3})}{8}\ge\frac{\left(\sqrt[12]{(abc)^3}+\sqrt[12]{(abc)^3}\right)^3}{8}=1$$

$$\left(\frac{a+b}{\sqrt[8]{a}+\sqrt[8]{b}}\right)\left(\frac{b+c}{\sqrt[8]{b}+\sqrt[8]{c}}\right)\left(\frac{c+a}{\sqrt[8]{c}+\sqrt[8]{a}}\right)$$

$$\ge\frac{(\sqrt[8]{a}+\sqrt[8]{b})(\sqrt[8]{b}+\sqrt[8]{c})(\sqrt[8]{c}+\sqrt[8]{a})(\sqrt[8]{a^7}+\sqrt[8]{b^7})(\sqrt[8]{b^7}+\sqrt[8]{c^7})(\sqrt[8]{c^7}+\sqrt[8]{a^7})}{8(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{b}+\sqrt[4]{c})(\sqrt[4]{c}+\sqrt[4]{a})}$$

$$=\frac{(\sqrt[8]{a^7}+\sqrt[8]{b^7})(\sqrt[8]{b^7}+\sqrt[8]{c^7})(\sqrt[8]{c^7}+\sqrt[8]{a^7})}{8}\ge\frac{\left(\sqrt[24]{(abc)^7}+\sqrt[24]{(abc)^7}\right)^3}{8}=1$$

A.117. Solution (Orlando Irahola Ortega)

$$32x^6-48x^4+36x^2-2-\sqrt{3}=0\Leftrightarrow$$

$$(2x)^6-6(2x)^4+36(2x)^2-4-2\sqrt{3}=0$$

$$Let\ 2x=\sqrt{t+2}\Rightarrow x=\frac{1}{2}\sqrt{t+2}\Rightarrow t^3+\underbrace{6}_{p}t+\underbrace{(16-2\sqrt{3})}_{q}=0$$

$$\Delta=\frac{q^2}{4}+\frac{p^3}{27}=75-16\sqrt{3}>0\Rightarrow\exists t_1\in\mathbb{R},t_2,t_3\in\mathbb{C}$$

Applying Cardano Theorem:

$$t=\sqrt[3]{\sqrt{3}-8+\sqrt{75-16\sqrt{3}}}+\sqrt[3]{\sqrt{3}-8-\sqrt{75-16\sqrt{3}}}$$

$$\text{How } t=\pm\frac{1}{2}\sqrt{t+2}\Rightarrow x=$$

$$\pm\sqrt[3]{\sqrt[3]{\sqrt{3}-8+\sqrt{75-16\sqrt{3}}}+\sqrt[3]{\sqrt{3}-8-\sqrt{75-16\sqrt{3}}}+2}$$

A.118. Solution (George Florin Şerban)

$$\frac{a^2}{\sqrt[3]{4}}+\frac{b^2}{\sqrt[3]{2}}+\frac{c^2}{\sqrt[3]{2}}+\frac{d^2}{\sqrt[3]{2}}+\frac{e^2}{1}\overset{Bergstrom}{\geq}\frac{(a+b+c+d+e)^2}{\sqrt[3]{4}+3\sqrt[3]{2}+1}$$

$$=\frac{41^2}{\sqrt[3]{4}+3\sqrt[3]{2}+1}\overset{(1)}{=}41\left(8\sqrt[3]{4}-\sqrt[3]{2}-5\right)$$

$$(1)\Leftrightarrow\left(\sqrt[3]{4}+3\sqrt[3]{2}+1\right)\left(8\sqrt[3]{4}-\sqrt[3]{2}-5\right)=41\ true.$$

Equality holds if and only if:

$$\frac{a}{\sqrt[6]{4}}=\frac{b}{\sqrt[6]{2}}=\frac{c}{\sqrt[6]{2}}=\frac{d}{\sqrt[6]{2}}=\frac{e}{1}=\frac{a+b+c+d+e}{\sqrt[6]{4}+3\sqrt[6]{2}+1}=\frac{41}{\sqrt[6]{4}+3\sqrt[6]{2}+1}$$

$$\begin{cases} a=\frac{41\sqrt[6]{4}}{\sqrt[6]{4}+3\sqrt[6]{2}+1}\\ b=c=d=\frac{41\sqrt[6]{2}}{\sqrt[6]{4}+3\sqrt[6]{2}+1}\\ e=\frac{41}{\sqrt[6]{4}+3\sqrt[6]{2}+1}\end{cases}$$

A.119. Solution (Tran Hong)

$$\therefore (y+z)^3+2=(y+z)^3+1+1\overset{AM-GM}{\geqq}3\sqrt[3]{(y+z)^3\cdot 1\cdot 1}=3(y+z)$$

Similary: $(z+t)^3+2\geq 3(z+t)and\ (t+y)^3+2\geq 3(t+z)$

$$\Omega=\frac{6}{(y+z)^3+2}+\frac{6}{(z+t)^3+2}+\frac{6}{(t+y)^3+2}$$

$$\leq\frac{2}{y+z}+\frac{2}{z+t}+\frac{2}{t+y}\leq\frac{1}{2}\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{1}{2}\left(\frac{1}{z}+\frac{1}{t}\right)+\frac{1}{2}\left(\frac{1}{t}+\frac{1}{y}\right)$$

$$=\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\Psi\ \left(\therefore We\ using: \frac{4}{\alpha+\beta}\leq\frac{1}{\alpha}+\frac{1}{\beta},\forall\alpha,\beta>0\right)$$

$$\Omega=\Psi\Leftrightarrow\begin{cases} y=z=t\\ y+z=z+t=t+y=1\end{cases}\Leftrightarrow y=z=t=\frac{1}{2}$$

$$[x](x-[x])+y+t=x^2+2z \overset{y=z=t}{\Leftrightarrow} [x](x-[x])$$
$$=x^2 \ldots (1); (x \geq 0 \Rightarrow [x] \geq 0)$$
$$(1) \Leftrightarrow [x]\{x\}=(\{x\}+[x])^2 \Leftrightarrow [x]\{x\}=([x])^2+2[x]\{x\}+(\{x\})^2$$
$$\Leftrightarrow ([x])^2+[x]\{x\}+(\{x\})^2=0$$

But: $[x] \geq 0;\ \{x\} \geq 0, equality\ for\ [x]=\{x\}=0 \Rightarrow x=0$

So, $(x,y,z,t)=\left(0;\frac{1}{2};\frac{1}{2};\frac{1}{2}\right)$

A.120. Solution (Tran Hong) *Let* $f(x)=\frac{x}{4x^2+2x+1}, x>0, f'(x)=\frac{1-4x^2}{(4x^2+2x+1)^2}$

$$\therefore f'(x)=0 \Leftrightarrow 1-4x^2=0 \Leftrightarrow \begin{cases} x=\frac{1}{2} \in [0,\infty) \\ x=-\frac{1}{2} \notin [0,\infty) \end{cases} \Rightarrow$$

$$f'(x)<0 \Leftrightarrow x \in \left(\frac{1}{2},\infty\right);\ f'(x)>0 \Leftrightarrow x \in \left(0,\frac{1}{2}\right) \Rightarrow f(x) \leq f_{max}\left(\frac{1}{2}\right)=\frac{1}{6}$$

$$LHS=\sum_{cyc}\frac{a}{4a^2+2a+1}=f(a)+f(b)+f(c) \leq \frac{1}{2} \ldots \ldots (true) for\ all$$
$$a,b,c>0$$

$$Equality\ for\ a=b=c=\frac{1}{2}$$

A.121. Solution (George Florin Şerban)

$$\frac{x}{x+1}+\frac{y}{(x+1)(y+1)}+\frac{z}{(x+1)(y+1)(z+1)}+\frac{1}{8\sqrt{xyz}}=1 \Leftrightarrow$$

$$\frac{x}{x+1}+\frac{y+1-1}{(x+1)(y+1)}+\frac{z+1-1}{(x+1)(y+1)(z+1)}+\frac{1}{8\sqrt{xyz}}=1 \Leftrightarrow$$

$$\frac{x}{x+1}+\frac{y+1}{(x+1)(y+1)}-\frac{1}{(x+1)(y+1)}+\frac{z}{(x+1)(y+1)(z+1)}-$$
$$-\frac{1}{(x+1)(y+1)(z+1)}+\frac{1}{8\sqrt{xyz}}=1$$

$$\frac{x}{x+1}+\frac{1}{x+1}-\frac{1}{(x+1)(y+1)}+\frac{1}{(x+1)(y+1)}-\frac{1}{(x+1)(y+1)(z+1)}+\frac{1}{8\sqrt{xyz}}=1$$

$$\frac{x+1}{x+1}-\frac{1}{(x+1)(y+1)(z+1)}+\frac{1}{8\sqrt{xyz}}=1 \Leftrightarrow$$

$$1-\frac{1}{(x+1)(y+1)(z+1)}+\frac{1}{8\sqrt{xyz}}=1 \Leftrightarrow$$

$$\frac{1}{(x+1)(y+1)(z+1)}=\frac{1}{8\sqrt{xyz}}=0 \Leftrightarrow$$

$(x+1)(y+1)(z+1)=8\sqrt{xyz}$ true from AM-GM we have

$x+1\geq 2\sqrt{x};\ \ y+1\geq 2\sqrt{y}; z+1\geq 2\sqrt{z}$. Proved.

A.122. Solution (George Florin Şerban)

If $b=0\Rightarrow \frac{a}{2}\geq\frac{a}{2}$ *true. If* $b\neq 0, \frac{a}{b}=t>0\Rightarrow$

$$\sqrt[7]{\left(\frac{2t}{t+1}\right)^7-\left(\sqrt{t}\right)^7+\left(\frac{t+1}{2}\right)^7}\geq\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\Leftrightarrow$$

$$\left(\frac{2t}{t+1}\right)^7-\left(\sqrt{t}\right)^7+\left(\frac{t+1}{2}\right)^7\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^7\Leftrightarrow$$

$$\left(\frac{t+1}{2}\right)^7-\left(\sqrt{t}\right)^7\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^7-\left(\frac{2t}{t+1}\right)^7\Leftrightarrow$$

$$\left(\frac{t+1}{2}-\sqrt{t}\right)\left[\left(\frac{t+1}{2}\right)^6+\left(\frac{t+1}{2}\right)^5\sqrt{t}+\cdots+\left(\sqrt{t}\right)^6\right]\geq$$

$$\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}-\frac{2t}{t+1}\right)\left[\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^6\right.$$

$$\left.+\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^5\frac{2t}{t+1}+\cdots+\left(\frac{2t}{t+1}\right)^6\right]$$

$$\frac{t+1}{2}\geq\sqrt{t}\Rightarrow\frac{t+1}{2}-\sqrt{t}\geq 0$$

$$\frac{t+1}{2}-\sqrt{t}\geq\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}-\frac{2t}{t+1}=\frac{t+1}{2}-\sqrt{t}\geq 0$$

$$\frac{2t}{t+1}\leq\sqrt{t}\leq\frac{t+1}{2}$$

$$\left(\frac{t+1}{2}\right)^6 \geq \left(\frac{2t}{t+1} - \sqrt{t} + \frac{t+1}{2}\right)^6 \Rightarrow \frac{t+1}{2} \geq \frac{2t}{t+1} - \sqrt{t} + \frac{t+1}{2}$$

$\Rightarrow \sqrt{t} \geq \frac{2t}{t+1}$ *true by Gm-Hm.*

$$\left(\frac{t+1}{2}\right)^5 \geq \left(\frac{2t}{t+1} - \sqrt{t} + \frac{t+1}{2}\right)^5 \Rightarrow \frac{t+1}{2} \geq \frac{2t}{t+1} - \sqrt{t} + \frac{t+1}{2}$$

$\Rightarrow \sqrt{t} \geq \frac{2t}{t+1}$ *true by Gm-Hm.*

$\left(\sqrt{t}\right)^6 \geq \left(\frac{2t}{t+1}\right)^6 \Rightarrow \sqrt{t} \geq \frac{2t}{t+1}$ *true by Gm-Hm.*

A.123. Solution (George Florin Serban)

$$\prod_{cyc}(x+1) + \prod_{cyc}(2x+1) \overset{AM-GM}{\geq} 2\sqrt{\prod_{cyc}(x+1)(2x+1)} \overset{(1)}{\geq} \frac{2\prod_{cyc}(3x+2)}{\prod_{cyc}(x+2)}$$

$$(1) \Leftrightarrow \prod_{cyc}(x+1)(2x+1) \geq \left(\frac{\prod_{cyc}(3x+2)}{\prod_{cyc}(x+2)}\right)^2 \Leftrightarrow$$

$$\prod_{cyc}(x+1)(2x+1)(x+2)^2 \geq \prod_{cyc}(3x+2)^2 \Leftrightarrow$$

$$(2x^2+3x+1)(x^2+4x+4) \geq 9x^2+12x+4 \Leftrightarrow$$

$$2x^4+11x^3+12x^2+4x \geq 0, \forall x \geq 0\ (true).$$

A.124. Solution (Khanh Hung Vu)

$$\frac{1}{\sqrt{(x+y)(y+z)}} + \frac{1}{\sqrt{(y+z)(z+x)}} + \frac{1}{\sqrt{(z+x)(x+y)}} \overset{BCS}{\leq}$$

$$\leq \sqrt{3\left(\frac{1}{(x+y)(y+z)} + \frac{1}{(y+z)(z+x)} + \frac{1}{(z+x)(x+y)}\right)}$$

$$\frac{1}{\sqrt{(x+y)(y+z)}} + \frac{1}{\sqrt{(y+z)(z+x)}} + \frac{1}{\sqrt{(z+x)(x+y)}}$$

$$\leq \sqrt{\frac{6(x+y+z)}{(x+y)(y+z)(z+x)}}; (1)$$

On the other hand, we have:

$$9(x+y)(y+z)(z+x)=9(x+y+z)(xy+yz+zx)-9xyz$$

And by Cauchy inequality, we have:

$$9xyz=3\sqrt[3]{xyz}\cdot 3\sqrt[3]{x^2y^2z^2}\le (x+y+z)(xy+yz+zx)$$

$$9(x+y)(y+z)(z+x)$$
$$\ge 9(x+y+z)(xy+yz+zx)-(x+y+z)(xy+yz+zx)$$

Or we have:

$$9(x+y)(y+z)(z+x)\ge 8(x+y+z)(xy+yz+zx)\Rightarrow$$

$$\frac{6(x+y+z)}{(x+y)(y+z)(z+x)}\le\frac{27}{4(xy+yz+zx)}\Leftrightarrow$$

$$\sqrt{\frac{6(x+y+z)}{(x+y)(y+z)(z+x)}}\le\frac{3}{2}\sqrt{\frac{3}{xy+yz+zx}};\ \ (2)$$

From (11),(2) we have the thing to prove:

$$\frac{1}{\sqrt{(x+y)(y+z)}}+\frac{1}{\sqrt{(y+z)(z+x)}}+\frac{1}{\sqrt{(z+x)(x+y)}}\le\frac{3}{2}\sqrt{\frac{3}{xy+yz+zx}}$$

A.125. Solution (Ravi Prakash)

Let $f(x)=x^7-15x^5+49x^3-36x=x(x^6-15x^4+49x^2-36)=$

$$\overset{t=x^2}{=}x(t^3-14t^2+49t-36)=x(t^3-t^2-13t^2+13t+36t-36)=$$

$$x(t-1)(t^2-13t+36)=x(t-1)(t-4)(t-9)=$$

$$=x(x^2-1)(x^2-4)(x^2-9)$$

$$=(x+3)(x+2)(x+1)x(x-1)(x-2)(x-3)$$

Note that $f(x)=0$ *for* $x\in\{0,\pm1,\pm2\pm3\}\Rightarrow\left[\frac{f(x)}{56}\right]=0$ *for*

$$x\in\{0,\pm1,\pm2\pm3\}\Rightarrow A=\{0,\pm1,\pm2\pm3\}$$

For $x\in\mathbb{Z};\ x\ge 4$

$$f(x)=P_7^{x+4}>\binom{x+4}{7}(7!)\Rightarrow f(x)>7!=5040,\forall x\ge 4\Rightarrow$$

$$\left[\frac{f(x)}{56}\right]>1,\forall x\ge 4$$

$$for\ x \le -4, x \in \mathbb{Z}\ f(x) = (-1)(P_7^{-x-4}) < -7! = -5040 \Rightarrow$$

$$\left[\frac{f(x)}{56}\right] < -1, \forall x \le -4$$

$$\Omega = \sum_{x \in A} x = 0$$

A.126. Solution (Remus Florin Stanca)

$$a^2b^2(2-a-b)^2(2+a+b)^2 \le (1-a^2)(1-b^2)(a+b)^4 \Leftrightarrow$$

$$\frac{a}{1-a^2} \cdot \frac{b}{1-b^2} \cdot \frac{ab}{(a+b)^2} \le \left(\frac{a+b}{4-(a+b)^2}\right)^2 \Leftrightarrow$$

$$\frac{1}{4} \cdot \frac{2a}{1-a^2} \cdot \frac{2b}{1-b^2} \cdot \frac{ab}{(a+b)^2} \le \left(\frac{\frac{a+b}{4}}{1-\left(\frac{a+b}{2}\right)^2}\right)^2 \Leftrightarrow$$

$$\frac{2a}{1-a^2} \cdot \frac{2b}{1-b^2} \cdot \frac{ab}{(a+b)^2} \le \left(\frac{2 \cdot \frac{a+b}{2}}{1-\left(\frac{a+b}{2}\right)^2}\right)^2 \cdot \frac{1}{4} \Leftrightarrow$$

Let: $tan\frac{x}{2} = a, tan\frac{y}{2} = b, \frac{tan\frac{x}{2}+tan\frac{y}{2}}{2} = tan\alpha \Leftrightarrow$

$$tanxtany \cdot \frac{tan\frac{x}{2}tan\frac{y}{2}}{tan^2\alpha} \le tan^2(2\alpha) \Leftrightarrow$$

$$tanxtany \cdot tan\frac{x}{2}tan\frac{y}{2} \le tan^2\alpha tan^2(2\alpha); (1)$$

Let's prove that: $tan\frac{x}{2}tan\frac{y}{2} \le tan^2\alpha \Leftrightarrow tan\frac{x}{2}tan\frac{y}{2} \le \left(\frac{tan\frac{x}{2}+tan\frac{y}{2}}{2}\right)^2$

$$(true\ by\ \frac{a+b}{2} \ge \sqrt{ab}, \forall a, b \ge 0; (2)$$

Let's prove that: $tanxtany \le tan^2(2\alpha) \Leftrightarrow$

$$\frac{1}{2}log\left(\frac{a}{1-a^2}\right) + \frac{1}{2}log\left(\frac{b}{1-b^2}\right) \le log\left(\frac{\frac{a+b}{2}}{1-\left(\frac{a+b}{2}\right)^2}\right)$$

Let: $f(x)=log\left(\frac{x}{1-x^2}\right)\Rightarrow\frac{\partial f}{\partial x}=\frac{1}{2}\cdot\frac{1-x^2}{x}\cdot\left(\frac{1}{1-x}-\frac{1}{1+x}\right)'=\frac{1}{2}\cdot\frac{1-x^2}{x}\cdot$

$$\left(\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}\right)=\frac{x^2+1}{x-x^3}$$

$$\Rightarrow\frac{\partial^2 f}{\partial x^2}=\frac{(3x^2-1)(x^2+1)}{(x-x^3)^2}\leq 0\left(a\leq\frac{1}{\sqrt{3}}\right)\Rightarrow f-concave\Rightarrow$$

$$\frac{1}{2}log\left(\frac{a}{1-a^2}\right)+\frac{1}{2}log\left(\frac{b}{1-b^2}\right)\leq log\left(\frac{\frac{a+b}{2}}{1-\left(\frac{a+b}{2}\right)^2}\right)\Rightarrow true(3)$$

$$(2);(3)\Rightarrow(1)true\Rightarrow a^2b^2(2-a-b)^2(2+a+b)^2$$
$$\leq(1-a^2)(1-b^2)(a+b)^4$$

A.127. Solution (Ravi Prakash)

$$sin^4xsin^2y+sin^2xsin^4y+cos^4xcos^2y+cos^2xcos^4y-sin^2x+sin^4x$$
$$-cos^2y+cos^4y=0\Rightarrow$$
$$sin^4x(1+sin^2y)+(sin^4y-1)sin^2x+cos^4y(1+cos^2x)$$
$$+(cos^4x-1)cos^2y=0$$
$$sin^4x(1+sin^2y)-sin^2xcos^2y(1+sin^2y)+cos^4y(1+cos^2x)$$
$$-cos^2ysin^2x(1+cos^2x)=0\Rightarrow$$
$$sin^2x(1+sin^2y)(sin^2x-cos^2y)+cos^2y(1+cos^2x)(cos^2y-sin^2x)$$
$$=0(sin^2x-cos^2y)[sin^2x(2-cos^2y)-cos^2y(2-sin^2x)]$$
$$=0$$

$$2(sin^2x-cos^2y)^2=0\Rightarrow sin^2x=cos^2y\ \Rightarrow sinx=cosy\left(\therefore 0<x,y<\frac{\pi}{2}\right)$$

$$\Rightarrow x=\frac{\pi}{2}-y\Rightarrow x+y=\frac{\pi}{2}\text{ but }x+y=\frac{5\pi}{6}$$

So the system has no solution.

A.128. Solution (Abner Chinga Bazo)

$$64x^5(x-1)+32x^2(x^2+x+1)-64x+19=0$$
$$64x^6-64x^5+32x^4+32x^3+32x^2-64x+19=0\,;(Horner)\Leftrightarrow$$
$$(2x-1)^2(16x^4+4x^2+12x+19)=0$$

$$16x^4 + (4x^2 + 12x + 9) + 10 = 0$$

$$16x^4 + (2x + 3)^2 + 10 > 0, \forall x \in \mathbb{R} \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

So, $S = \left\{\frac{1}{2}\right\}$

A.129. Solution (Adrian Popa) $\sqrt[7]{2+x} + \sqrt[7]{5-x} = \sqrt[7]{7}$

Let: $\sqrt[7]{2+x} = a, \sqrt[7]{5-x} = b \Rightarrow \begin{cases} a + b = \sqrt[7]{7} \\ a^7 + b^7 = 7 \end{cases} \Rightarrow \begin{cases} (a+b)^7 = 7 \\ a^7 + b^7 = 7 \end{cases}; (1) \Rightarrow$

$$7a^6b + 7ab^6 + 21a^5b^2 + 21a^2b^5 + 35a^3b^4 + 35a^4b^3 = 0$$

$$7ab(a^5 + b^5) + 21a^2b^2(a^3 + b^3) + 35a^3b^3(a + b) = 0$$

$$ab(a+b)(7(a^4 - a^3b + a^2b^2 - ab^3 + b^4) + 21ab(a^2 - ab + b^2) + 35a^2b^2) = 0$$

If $ab = 0$ then $a = 0 \Rightarrow b = \sqrt[7]{7}$ and $b = 0 \Rightarrow a = \sqrt[7]{7}$

$$a + b = \sqrt[7]{7} \neq 0$$

$$7(a^4 - a^3b + a^2b^2 - ab^3 + b^4) + 21ab(a^2 - ab + b^2) + 35a^2b^2 = 0$$

$$a^4 + b^4 - a^3b - ab^3 + a^2b^2 + 3a^3b - 3a^2b^2 + 3ab^3 + 5a^2b^2 = 0$$

$$a^4 + b^4 + 2a^3b + 2ab^3 + 3a^2b^2 = 0$$

$$a^4 + b^4 + 2a^2b^2 + 2ab(a^2 + b^2) + a^2b^2 = 0$$

$(a^2 + ab + b^2)^2 = 0$ impossible, because $\Delta < 0$.

So, $\begin{cases} \sqrt[7]{2+x} = 0 \\ \sqrt[7]{2+x} = \sqrt[7]{7} \end{cases} \Rightarrow x \in \{-2, -5\}$

$$\sqrt[9]{3+x} + \sqrt[9]{6-x} = \sqrt[9]{9}$$

Let: $\sqrt[9]{3+x} = a; \sqrt[9]{6-x} = b \Rightarrow \begin{cases} a + b = \sqrt[9]{9} \\ a^9 + b^9 = 9 \end{cases} \Rightarrow (a+b)^9 = 9 \Rightarrow$

$$9a^8b + 9ab^8 + 36a^7b^2 + 36a^2b^7 + 84a^6b^3 + 84a^3b^6 + 126a^5b^4 + 126a^4b^5 = 0$$

$$ab\left(3a^7 + 3b^7 + 12ab(a^5 + b^5) + 28a^2b^2(a^3 + b^3) + 42a^3b^3(a + b)\right) = 0$$

$$ab(a+b)(3(a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6)$$
$$+12ab(a^4-a^3b+a^2b^2-ab^3+b^4)$$
$$+28a^2b^2(a^2-ab+b^2)+42a^3b^3)=0$$

If $ab=0$ then $a=0\Rightarrow b=\sqrt[9]{9}$ and $b=0\Rightarrow a=\sqrt[9]{9}$

$$a+b=\sqrt[9]{9}\neq 0$$

$$3a^6+3b^6+19a^4b^2+23a^3b^3+19a^2b^4+9ab^5$$
$$=3\underbrace{(a^2+ab+b^2)^2}_{\geq 0}+\underbrace{a^2b^2(a+b)^2}_{\geq 0}=0\Leftrightarrow a=b=0$$

So, $\sqrt[9]{3+x}=0\Leftrightarrow x=-3$ or $\sqrt[9]{3+x}=\sqrt[9]{9}\Leftrightarrow x=6$ then $x\in\{0,6\}$

$$A=\{-2,5\},B=\{-3,6\}$$

$$A\Delta\Omega_1=B\Rightarrow(\{-2,5\}\,/\,\Omega_1)\cup(\Omega_1/\{-2,5\})=\{-3,6\}\Rightarrow\Omega_1=\{-2,5;-3,6\}$$

$$\Omega_2\Delta B=A\Rightarrow(\{-3,6\}/\Omega_2)\cup(\Omega_2/\{-3,6\})=\{-2,5\}\Rightarrow\Omega_2=\{-2,5;3,6\}$$

A.131. Solution (Remus Florin Stanca)

Let's prove that: $\sum\frac{(1+x^2)(1+y^2)}{(1+x)(1+y)}=36-24\sqrt{2}$, let

$$x=tan\frac{A}{2},y=tan\frac{B}{2},z=tan\frac{C}{2},A,B,C\in\left[0,\frac{\pi}{2}\right],$$

$$sinA=\frac{2x}{x^2+1},sinB=\frac{2y}{y^2+1},sinC=\frac{2z}{z^2+1}$$

$$sinA+\frac{2}{x^2+1}=2\cdot\frac{x+1}{x^2+1}\Rightarrow sinA+2cos^2\frac{A}{2}-1+1=2\cdot\frac{x+1}{x^2+1}\Rightarrow$$

$$sinA+cosA+1=2\cdot\frac{x+1}{x^2+1}\Rightarrow\frac{x^2+1}{x+1}=\frac{2}{sinA+cosA+1}\Rightarrow$$

$$\sum_{cyc}\frac{(1+x^2)(1+y^2)}{(1+x)(1+y)}=\sum_{cyc}\frac{4}{(sinA+cosA+1)(sinB+cosB+1)}$$
$$\geq 36-24\sqrt{2}\Leftrightarrow$$

$$\sum_{cyc}\frac{1}{(sinA+cosA+1)(sinB+cosB+1)}\geq 9-6\sqrt{2};(1)$$

$$\sum_{cyc}\frac{1}{(sinA+cosA+1)(sinB+cosB+1)}\overset{Bergstrom}{\geq}$$

$$\geq\frac{9}{\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1)};(2)$$

Let's prove that:

$$\frac{9}{\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1)}\geq 9-6\sqrt{2}$$

We need to prove that:

$$\frac{3}{\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1)}\geq 3-2\sqrt{2}=\frac{1}{\left(\sqrt{2}+1\right)^2}\Leftrightarrow$$

$$\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1)\leq 9+6\sqrt{2}\Leftrightarrow$$

$$\sum_{cyc}(sinAsinB+sinAcosB+sinA+sinAsinB+cosAcosB+cosA+sinB$$

$$+cosB+1)\leq 9+6\sqrt{2}\Leftrightarrow$$

$$\sum_{cyc}\big(cos(A-B)+sin(A+B)\big)+2(sinA+sinB+sinC)$$

$$+2(cosA+cosB+cosC)\leq 6+6\sqrt{2}\Leftrightarrow$$

$$\sum_{cyc}\left(cos(A-B)+cos\left(\frac{\pi}{2}-A-B\right)\right)+2(sinA+sinB+sinC)$$

$$+2(cosA+cosB+cosC)\leq 6+6\sqrt{2}\Leftrightarrow$$

$$2\sum_{cyc}(cos\left(\frac{\pi}{4}-B\right)cos\left(\frac{\pi}{4}-A\right)+2\sqrt{2}\sum_{cyc}cos\left(\frac{\pi}{4}-A\right)\leq 6+6\sqrt{2}$$

Let: $cos\left(\frac{\pi}{4}-A\right)=x_1;\ cos\left(\frac{\pi}{4}-B\right)=x_2;\ cos\left(\frac{\pi}{4}-C\right)=x_3\Leftrightarrow$

$$(x_1x_2+x_2x_3+x_3x_1)+\sqrt{2}(x_1+x_2+x_3)\leq 3+3\sqrt{2};(3)$$

$$\frac{(x_1+x_2+x_3)^2}{3}\geq x_1x_2+x_2x_3+x_3x_1\Rightarrow$$

$$(x_1x_2+x_2x_3+x_3x_1)+\sqrt{2}(x_1+x_2+x_3)$$
$$\leq \frac{(x_1+x_2+x_3)^2}{3}+\sqrt{2}(x_1+x_2+x_3)\overset{(3)}{\Rightarrow}$$

We need to prove that:

$\frac{(x_1+x_2+x_3)^2}{3}+\sqrt{2}(x_1+x_2+x_3)\leq 3+3\sqrt{2}$ *let* $x_1+x_2+x_3=s$ *and we know that* $0\leq s\leq 3\Rightarrow s^2+3s\sqrt{2}-9-9\sqrt{2}\leq 0\xRightarrow{(1),(2)}$ *we peove that (1),(2) are true.*

But we have in the equality case $s^2+3s\sqrt{2}-9-9\sqrt{2}=0;\ \Delta=54+36\sqrt{2}\Rightarrow$ $s=3$*, because* $s\geq 0\Rightarrow$

$$\sum_{cyc} cos\left(\frac{\pi}{4}-A\right)=3\Rightarrow 2\sum_{cyc} sin^2\left(\frac{\frac{\pi}{4}-A}{2}\right)=0\Rightarrow A=B=C=\frac{\pi}{4}\Rightarrow$$

$$x=y=z=tan\frac{\pi}{8}$$

$$tan\frac{\pi}{8}=\frac{sin\frac{\pi}{8}}{cos\frac{\pi}{8}}=\frac{\sqrt{\frac{1-cos\frac{\pi}{4}}{2}}}{\sqrt{\frac{1+cos\frac{\pi}{4}}{2}}}=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}=\sqrt{2}-1\Rightarrow x=y=z=\sqrt{2}-1.$$

A.132. Solution (Remus Florin Stanca) *We know that:*

$$\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\leq\frac{x_1+\cdots+x_n}{n}\Rightarrow\frac{128}{42z+43(x+y)}=$$

$$=\frac{128}{z+z+\cdots+z+x+x+\cdots+x+y+y+\cdots+y}\leq\frac{\frac{42}{z}+\frac{43}{x}+\frac{43}{y}}{128}$$

$$(z-42\ times, x-43\ times, y-53-times)\Rightarrow$$

$$\frac{1}{42z+43(x+y)}\leq\frac{\frac{42}{z}+43\left(\frac{1}{x}+\frac{1}{y}\right)}{128^2}\Rightarrow$$

$$\sum_{cyc}\frac{1}{42z+43(x+y)}\le\frac{128\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}{128^2}=\frac{1}{128}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

But $\sum_{cyc}\frac{1}{42z+43(x+y)}=\frac{1}{128}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\Rightarrow x=y=z\Rightarrow$

$$x^4+2x^3-6x^2+1=0\Rightarrow$$

$$x^4-x^3+3x^3-3x^2+3x-x+1=0\Rightarrow$$

$$x^3(x-1)+3x(x^2-1)-3x(x-1)-(x-1)=0\Rightarrow$$

$$(x-1)(x^2+3x^2-3x-1)=0\Rightarrow(x-1)^2(x^2+4x+1)=0\Rightarrow x=1$$

$$\Rightarrow(x;y;z)\in\{(1;1;1)\}$$

A.133. Solution (Adrian Popa)

Denote: $\begin{cases}f(x)+g(x)+h(x)=S_1\\ f(x)\cdot g(x)+g(x)\cdot h(x)+h(x)\cdot f(x)=S_2\\ f(x)\cdot g(x)\cdot h(x)=S_3\end{cases}$

$$S_1=3x+3;\ S_1^2-2S_2=3x^2+6x+5\Rightarrow 9x^2+18x+9-2S_2$$

$$=3x^2+6x+5\Rightarrow 2S_2=6x^2+12x+4\Rightarrow S_2$$

$$=3x^2+6x+2$$

$$S_1^3-3S_1S_2+3S_3=3x^3+9x^2+15x+9$$

$$27(x^3+3x^2+3x+1)-9(x+1)(3x^2+6x+2)+3S_3$$

$$=3x^3+9x^2+15x+9$$

$$3S_3=3x^3+9x^2+6x\Rightarrow S_3=x^3+3x^2+2x$$

$$f(x)\cdot g(x)\cdot h(x)=x^3+3x^2+2x=0\Leftrightarrow$$

$$x(x^2+3x+2)=0\Rightarrow x\in\{-2;-1;0\}$$

A.134. Solution (Ravi Prakash)

Let $\frac{1}{2}(a+b)=A;\sqrt{ab}=G;\frac{2ab}{a+b}=H$ then

$$A+H=\frac{1}{2}(a+b)+\frac{2ab}{a+b}\ge 2\sqrt{\frac{a+b}{2}\cdot\frac{2ab}{a+b}}=2G\Rightarrow A+H-2G\ge 0$$

Also, $AH=G^2$

$$\frac{(a+b)^3}{8}+\frac{8a^3b^3}{(a+b)^3}-ab\sqrt{ab}-\left(\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}+\frac{2ab}{a+b}\right)^3=$$

$$=A^3+H^3-G^3-(A+H-G)^3=$$

$$=A^3+H^3-G^3-[(A+H)^3-3(A+H)^2G+3(A+H)G^2-G^3]=$$

$$=A^3+H^3-G^3-$$

$$-[A^3+H^3-G^3+3A^2H+3AG^2--3(A+H)^2G+3(A+H)G^2]=$$

$$=3(A+H)^2G-3(A+H)G^2-3AH(A+H)=$$

$$=3(A+H)G[A+H-G-G]\geq 0$$

$$\frac{(a+b)^3}{8}+\frac{8a^3b^3}{(a+b)^3}\geq ab\sqrt{ab}+\left(\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}+\frac{2ab}{a+b}\right)^3$$

A.135. Solution (Tran Hong)

$$\frac{\left(a+b+\sqrt{ab}\right)^3}{(a+b)^2}+\frac{\left(b+c+\sqrt{bc}\right)^3}{(b+c)^2}+\frac{\left(c+a+\sqrt{ca}\right)^3}{(c+a)^2}\overset{Radon}{\geq}$$

$$\geq\frac{\left(2a+2b+2c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}{(2a+2b+2c)^2}=\frac{(2a+2b+2c+12)^3}{(2a+2b+2c)^2}=$$

$$=\frac{8(a+b+c+6)^3}{4(a+b+c)^2}=\frac{2(a+b+c+6)^3}{(a+b+c)^2}$$

Let: $t=a+b+c\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ca}=12$

We need to prove:

$$\frac{2(a+b+c+6)^3}{(a+b+c)^2}\geq 81\Leftrightarrow\frac{2(t+6)^3}{t^2}\geq 81\Leftrightarrow 2(t+6)^3\geq 81t^2\Leftrightarrow$$

$$2(t+6)^3-81t^2\geq 0\Leftrightarrow 2t^3-45t^2+216t+432\geq 0$$

$$\Leftrightarrow(t-12)^2(2t+3)\geq 0$$

Which is true because: $t\geq 12$.

A.136. Solution (Tran Hong)

$$\sum_{cyc}\frac{c+\sqrt{ab}}{\sqrt{ab}(a+b+2c)}=$$

$$=\frac{c+\sqrt{ab}}{\sqrt{ab}(a+b+2c)}+\frac{b+\sqrt{ac}}{\sqrt{ac}(a+c+2b)}+\frac{a+\sqrt{bc}}{\sqrt{bc}(b+c+2a)}$$

$$=\frac{1}{\sqrt{ab}(a+b+2c)}+\frac{1}{\sqrt{ac}(a+c+2b)}+\frac{1}{\sqrt{bc}(b+c+2a)}+$$

$$+\frac{c}{\sqrt{ab}(a+b+2c)}+\frac{b}{\sqrt{ac}(a+c+2b)}+\frac{a}{\sqrt{bc}(b+c+2a)}=\Omega$$

$$\Rightarrow \Omega \overset{Am-Gm}{\geq} \left(\frac{1}{a+b+2c}+\frac{1}{b+c+2a}+\frac{1}{a+c+2b}\right)$$

$$+\left(\frac{\frac{2c}{a+b}}{a+b+2c}+\frac{\frac{2b}{a+c}}{a+c+2b}+\frac{\frac{2a}{b+c}}{b+c+2a}\right)$$

$$=\frac{1+\frac{2c}{a+b}}{a+b+2c}+\frac{1+\frac{2b}{a+c}}{a+c+2b}+\frac{1+\frac{2a}{b+c}}{b+c+2a}$$

$$=\frac{\frac{a+b+2c}{a+b}}{a+b+2c}+\frac{\frac{a+c+2b}{a+c}}{a+c+2b}+\frac{\frac{b+c+2a}{b+c}}{b+c+2a}$$

$$=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

A.137. Solution (Marian Ursărescu)

We must show: $\sqrt[5]{F_m^2F_n^3L_n^2L_m^3}+\sqrt[5]{F_m^3F_n^2L_n^3L_m^2}<2F_{m+n}$; (1)

$$\sqrt[5]{F_m^2F_n^3L_n^2L_m^3}=\sqrt[5]{(F_m\cdot L_n)^2\cdot(F_n\cdot L_m)^3}\overset{Am-Gm}{\leq}\frac{2F_m\cdot L_n+3F_n\cdot L_m}{5};\ (2)$$

$$\sqrt[5]{F_m^3F_n^2L_n^3L_m^2}=\sqrt[5]{(F_m\cdot L_n)^3\cdot(F_n\cdot L_m)^2}\overset{Am-Gm}{\leq}\frac{3F_m\cdot L_n+2F_n\cdot L_m}{5};\ (3)$$

From (2),(3) we have: $\sqrt[5]{F_m^2F_n^3L_n^2L_m^3}+\sqrt[5]{F_m^3F_n^2L_n^3L_m^2}<F_m\cdot L_n+F_n\cdot L_m$; (4)

But: $F_m\cdot L_n+F_n\cdot L_m=2F_{m+n}$ *(Ferns identity); (5)*

From (4),(5) we get (1) is true.

A.138.Solution (Marian Ursărescu)

$$det\left(\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n\right)=$$

$$=det\left(A^2+\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n\right)$$

$$= det\left[\left(A+\frac{\sqrt{3}(a+b)+i(a-b)}{2}I_n\right)\left(A+\frac{\sqrt{3}(a+b)-i(a-b)}{2}I_n\right)\right]$$

$$= det\left[\left(A+\left(\frac{\sqrt{3}(a+b)}{2}+i\frac{a-b}{2}\right)I_n\right)\overline{\left(A+\left(\frac{\sqrt{3}(a+b)}{2}+\frac{\imath(a-b)}{2}\right)I_n\right)}\right] \geq 0. \text{ Because}$$

$$det(X\bar{X}) \geq 0, \forall X \in M_n(\mathbb{R})$$

A.139. Solution (Tran Hong)

For $m \in \mathbb{N} \Rightarrow$

$$3\sqrt{3}\left(\frac{m^3}{(m+3)!}\right) \overset{(*)}{\leq} \frac{\sqrt{3}}{3}\cdot m! \Leftrightarrow 9\cdot\frac{m^3}{(m+3)!} < m! \Leftrightarrow 9m^3 < m!\,(m+3)!$$

Which is true because:

If $m = 0$ *then:* $0!\,(0+3)! = 1 > 0 = 0\cdot 9^3$

If $m \geq 1, m \in \mathbb{N}$ *then:* $m! \geq m$ *and*

$$(m+3)! = (m+3)(m+2)(m+1)m! \geq 4\cdot 3(m+1)m! \geq 12(m+1)m! \geq$$

$$9(m+1)m > 9m^2 \Rightarrow m!\,(m+3)! > m\cdot 9m^2 = 9m^3 \Rightarrow (*) \text{ true.}$$

Similary:

$$3\sqrt{3}\left(\frac{n^5}{(n+5)!}\right) < \frac{\sqrt{3}}{3!}\cdot n!, \forall n \in \mathbb{N} \text{ and } 3\sqrt{3}\left(\frac{p^7}{(p+7)!}\right) < \frac{\sqrt{3}}{3!}\cdot p!, \forall p \in \mathbb{N}$$

So,

$$LHS < \frac{\sqrt{3}}{3}(m!+n!+p!) \overset{BCS}{\leq} \frac{\sqrt{3}}{3}\cdot\sqrt{3}\cdot\sqrt{(m!)^2+(n!)^2+(p!)^2} =$$

$$= \sqrt{(m!)^2+(n!)^2+(p!)^2}$$

A.140.Solution (Ravi Prakash)

Let z_0 *−be any complex number such that* $|z_0| = 1$. *We have:*

$$|z| = |(z-z_0)+z_0| \leq |z-z_0|+|z_0| \to |z-z_0| \geq |z|-|z_0| \to$$

$$|z-z_0| \geq |z|-1; (|z_0| = 1)$$

Let: $z_A = cosA + isinA, z_B = cosB + isinB, z_C = cosC + isinC$ *then*

$$|z_A| = |z_B| = |z_C| = 1 \to |z-z_A|, |z-z_B|, |z-z_C| \geq |z|-1$$

Adding, we get: $|z-z_A|+|z-z_B|+|z-z_C| \geq 3(|z|-1)$

A.141. Solution (Rahim Shahbazov)

$$\textit{It is clear that: } x > 0, y > 0$$

$$\textit{Let: } x = a^{10}, y = b^{10} \to 5(a^2 + b^2) - 2(a^5 + b^5) = 6 \to$$

$$2a^5 + 3 + 2b^5 + 3 = 5a^2 + 5b^2$$

$$\begin{cases} 2a^5 + 3 = a^5 + a^5 + 1 + 1 + 1 \geq 5\sqrt[5]{a^{10}} \\ 2b^5 + 3 \geq 5b^2 \end{cases} \to a = b = 1 \to x = y = 1$$

$$\textit{Then } 2(z^4 + 10z^2 + 5) = 5z^4 + 10z^2 + 1 \leftrightarrow$$

$$z^5 + 5z^4 + 10z^3 - 10z^2 + 5z - 1 = 0 \leftrightarrow (z - 1)^5 = 0 \leftrightarrow z = 1.$$

$$\textit{So, } (x, y, z) = (1,1,1)$$

A.142. Solution (Rahim Shahbazov)

$$abcd = e^4 \to loga + logb + logc + logd = 4$$

$$a, b, c, d > 1 \to loga, logb, ogc, logd \geq$$

$$\text{Let: } x = loga, y = logb, z = logc, t = logd \to$$

$$\frac{(2 - x)(2 - y)(2 - z)(2 - t)}{(x + y)(y + z)(z + t)(t + x)} \leq \frac{1}{16}$$

$$\to (x + y)(y + z)(z + t)(t + x) \geq (4 - 2x)(4 - 2y)(4 - 2z)(4 - 2t) \to$$

$$(x + y)(y + z)(z + t)(t + x) \geq$$

$$\geq (z + y + t - x)(z + x + t - y)(z + x + y - t)(x + y + t - z)$$

$$\begin{cases} y + z + t - x = A > 0 \\ z + x + t - y = B > 0 \\ z + x + y - t = C > 0 \\ x + y + t - z = D > 0 \end{cases} \to \begin{cases} 2(z + t) = A + B \\ 2(x + y) = C + D \\ 2(x + t) = B + D \\ 2(y + z) = A + C \end{cases}$$

$$\to (A + B)(C + D)(B + D)(A + C) \geq 16ABCD \textit{ true from Am-Gm.}$$

A.143. Solution (Do Chinh)

$$\frac{3}{\sqrt[3]{1 + x}} + \frac{x}{\sqrt[3]{1 + x^3}} = 2\sqrt[3]{4}, x \neq 1; (1) \Leftrightarrow$$

$$\frac{3}{\sqrt[3]{1 + x}} + \frac{x}{\sqrt[3]{(1 + x)(1 - x + x^2)}} = 2\sqrt[3]{4}$$

$$3+\frac{3}{\sqrt[3]{1+x}}+\frac{x}{\sqrt[3]{1-x+x^2}}=2\sqrt[3]{4(x+1)}$$

Let: $f(x)=\frac{x}{\sqrt[3]{1-x+x^2}}, x\in\mathbb{R},\ f'(x)=\frac{-x^2-1}{3\sqrt[3]{(1-x+x^2)^2}}, f(x)<0, \forall x\in\mathbb{R}$

$f(x)$ *–is decreasing function (2)*

Let: $g(x)=2\sqrt[3]{4(x+1)}-3, x\in\mathbb{R},\ g'(x)=\frac{2\sqrt[3]{4}}{3\sqrt[3]{(x+1)^2}}, g(x)>0, \forall x\in\mathbb{R}$

$g(x)$ *–is increasing function (3)*

We have, $f(1)=g(1)$ *and from* $(2),(3)\Rightarrow x=1$ *is only solution.*

A.144. Solution (Bedri Hajrizi)

It's clear that trivial solution is $(0,0,0)$. *Suppose that* $(x,y,z)\neq(0,0,0)$
Produce all we get: $\prod((4xy-1)^2+16)=16^3$

$$16^3=\prod((4xy-1)^2+16)\geq 16^3\Rightarrow\begin{cases}xy=\frac{1}{4}\\ yz=\frac{1}{4}\\ zx=\frac{1}{4}\end{cases}$$

Produce all $x^2y^2z^2=\left(\frac{1}{4}\right)^3$ *and* $xy=\frac{1}{4}\Rightarrow x^2y^2z^2=\frac{1}{16}z^2\Rightarrow$
$\frac{1}{16}z^2=\frac{1}{64}\Rightarrow z=\pm\frac{1}{2}$. *Similarly:* $x=y=\pm\frac{1}{2}$.
Finally solutions are: $(0,0,0),\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

A.145. Solution (Sanong Huayrerai)

For $x,y,z>0, 3(xy+yz+zx)=1$ we get:

$$27(x^3y+y^3z+z^3x)+36(x^2y+y^2z+z^2x)+6(x+y+z)\geq$$

$$\geq\frac{27(x^2+y^2+z^2)(xy+yz+zx)}{3}+\frac{36(x+y+z)(xy+yz+zx)}{3}+6(x+y+z)$$

$$=\frac{3(x^2+y^2+z^2)}{3}+\frac{12(x+y+z)}{3}+6(x+y+z)\geq 3\cdot\frac{1}{3}+4+6=11$$

Because: $3(xy+yz+zx)=1\Rightarrow xy+yz+zx=\frac{1}{3}$

$$\Rightarrow(x+y+z)^2\geq 1\Rightarrow x+y+z\geq 1\Rightarrow x^2+y^2+z^2\geq\frac{1}{3}$$

A.146. Solution (Sanong Huayrerai)

$$\sum_{cyc}\frac{(a^{10}+b^{10})(a^9+b^9)}{(a^4+b^4)(a^3+b^3)}\geq$$

$$\geq\sum_{cyc}\frac{(a^4+b^4)(a^3+b^3)(a^6+b^6)(a^6+b^6)}{(a^4+b^4)(a^3+b^3)}\geq\sum_{cyc}\frac{(a^6+b^6)^2}{4}\geq$$

$$\geq\frac{\left(\sum\frac{a^6+b^6}{2}\right)^2}{3}=\frac{(a^6+b^6+c^6)^2}{3}\geq 3$$

If $abc=1\Rightarrow a+b+c\geq 3\Rightarrow a^6+b^6+c^6\geq 3$

A.147. Solution (George Florin Şerban)

Applying Lagrange identity: $(ad-bc)^2+(ac+bd)^2=(a^2+b^2)(c^2+d^2)$

$$(ad-bc)^6+(ac+bd)^6=\frac{[(ad-bc)^2]^3}{1}+\frac{[(ac+bd)^2]^3}{1}\overset{Holder}{\geq}$$

$$\geq\frac{[(ad-bc)^2+(ac+bd)^2]^3}{(1+1)\cdot 2^{3-2}}=\frac{(a^2+b^2)^3(c^2+d^2)^3}{4}$$

$$4(ad-bc)^6+4(ac+bd)^6\geq(a^2+b^2)^3(c^2+d^2)^3$$

A.148. Solution (Santos Martins Junior)

Observe that $x\neq 1, y\neq 1, z\neq 1$.

Indeed, if one of the three equals to 1 then from the three first equations of the system we have $x=y=z=1$ but that does not satisfy the last equation of the system.

Let $a=\left(\frac{x+y}{2}\right)^{\frac{1}{xy}}, b=\left(\frac{z+x}{2}\right)^{\frac{1}{xy}}\ c=\left(\frac{y+z}{2}\right)^{\frac{1}{yz}}$ *and* $p=x^{\frac{1}{x(x+y)}}, q=y^{\frac{1}{y(y+z)}}$,

$r=z^{\frac{1}{x(z+x)}}$. *Where x,y,z naturals, $a,b,c,p,q,r>0$ real numbers.*

System of the first equations, becomes: $\begin{cases}ab=pq; & (1)\\ ca=qr; & (2)\\ bc=rp; & (3)\end{cases}$

Doing $\frac{(2)}{(3)}+(1)$: $a^2=q^2\Leftrightarrow a=q;\ \ (4)$

(4) implies from (1): $b = p; (5)$

(5) implies from (3): $c = r;\ \ (6)$

Considering (5): $\left(\frac{z+x}{2}\right)^{\frac{1}{xy}} = x^{\frac{1}{x(x+y)}} \Leftrightarrow \left(\frac{z+x}{2}\right)^{\frac{1}{xy}} = x^{\frac{1}{x(x+y)}}; (7)$

Since $\frac{y}{x+y} < 1$*, we have* $x^{\frac{y}{x+y}} < x \Rightarrow \frac{z+x}{2} = x^{\frac{y}{x+y}} < x \Rightarrow z < x; (8)$

Considering (6): $\left(\frac{y+z}{2}\right)^{\frac{1}{yz}} = z^{\frac{1}{z(z+x)}} \Leftrightarrow y + z = 2z^{\frac{y}{z+x}};\ \ (9)$

Assume $z \geq y$ *then* $\frac{y}{z+x} \leq \frac{z}{z+x}$ *and from (8):* $\frac{y}{z+x} \leq \frac{z}{z+x} \leq \frac{z}{z+z} = \frac{1}{2}$*, since* $z < x$.

Hence: $y + z < 2\sqrt{z}$ *from AM-GM* $2\sqrt{y}\sqrt{y} < y + z \Rightarrow$

$$2\sqrt{y}\sqrt{y} \leq y + z < 2\sqrt{z}$$

$\Rightarrow y < 1$ *impossible, since we have* y *natural number.*

Therefore, we must have $y > z; (10)$

We must have that $y > z + x;\ \ (11)$

Indeed if $y < z + x$*, then from (9):* $z^{\frac{y}{z+x}} \leq z$ *implying* $\frac{y+z}{2} = z^{\frac{y}{z+x}} \leq z$

$\Leftrightarrow y \leq z$ *impossible by (10)*

Hence, we have $z < x;\ \ \ (8)$*and* $z < y;\ \ \ (9)$

$(11): z + x < y \Leftrightarrow 2z < y;\ \ (12)$

For the 4^{th} *equation:* $(x + 1)(y + 1)(z + 1) = 336$*; (13) we get using (8) and (12):*

$$(z + 1)(2z + 2)(z + 3) < 336$$

$$(z + 1)^2(z + 3) < 168 \Leftrightarrow 1 \leq z \leq 3.$$

i) $z = 1 \Rightarrow x = y = 1$ *using the 3 first equations but does not satisfy (13).*

ii) $z = 2$ *is not possible because from (13) that implies* $5\ /\ 336$ *impossible.*

iii) $z = 3$ *then from (13):* $(x + 1)(y + 2) = 56;\ \ (14)$

Also from (12): $y > 2 \Rightarrow z \geq 7.$

Also from (8): $x > z \Rightarrow x \geq 4$ *but couples* $(x, y) = (3,12), (1,26)$ *that satisfy (14) don't satisfy* $y \geq 7$ *and* $x \geq 4$. *No solution for natural numbers.*

A.149. Solution (Tran Hong)

$$(a^2+b^2)(c^2+d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$= (ac+bd)^2 + (ad-bc)^2 \Rightarrow$$

$$(ad-bc)^8(a^2+b^2)(c^2+d^2) + (ac+bd)^{10} =$$

$$= (ad-bc)^8[(ac+bd)^2 + (ad-bc)^2] + (ac+bd)^{10} =$$

$$= (ac+bd)^{10} + (ad-bc)^{10} + (ad-bc)^8(ac+bd)^2 =$$

$$= [(ad-bc)^2]^5 + [(ac+bd)^2]^5 + [(ad-bc)^2]^4(ac+bd)^2$$

Let: $x = (ad-bc)^2, y = (ac+bd)^2$. *We need to prove:*

$$x^5 + y^5 + x^4y \leq (x+y)^5 \Leftrightarrow$$

$$x^5 + y^5 + x^4y \leq x^5 + y^5 + 10xy(x+y) + 5x^4y + 5xy^4$$

$$xy[4x^3 + 10xy(x+y) + 5y^3] \geq 0$$

Which is true, because:

$$x, y \geq 0 \Rightarrow xy \geq 0, 4x^3 + 10xy(x+y) + 5y^3 \geq 0.$$

A.150. Solution (George Florin Şerban)

$$\sum_{cyc} \frac{1}{\sqrt{x^2-xy+y^2}} = \sum_{cyc} \frac{\sqrt{x}}{\sqrt{xyz}} = \sum_{cyc} \frac{1}{\sqrt{yz}} = \sum_{cyc} \frac{1}{\sqrt{xy}}$$

$$\sqrt{x^2-xy+y^2} \geq \sqrt{xy} \Leftrightarrow (x-y)^2 \geq 0 \text{ true.}$$

$\sum_{cyc} \frac{1}{\sqrt{x^2-xy+y^2}} \leq \sum_{cyc} \frac{1}{\sqrt{xy}}$ *equality holds if* $x = y = z \Rightarrow [x] \cdot \{x\} + 1 = x \Leftrightarrow$

$$[x] - \{x\} - 1 - [x] - \{x\} = 0 \Leftrightarrow ([x]-1)(\{x\}-1) = 0$$

If $[x] - 1 = 0 \Rightarrow x \in [1,2) \Rightarrow x, y, z \in [1,2)$

If $\{x\} - 1 = 0 \Rightarrow \{x\} = 1$ *contradiction.*

A.151. Solution (Bedri Hajrizi)

$$\frac{(x-2)!!\,(x-3)!!}{(x-4)!!\,(x-5)!!} + \frac{(x-3)!!\,(x-4)!!}{(x-5)!!\,(x-6)!!} + \frac{(x-4)!!\,(x-5)!!}{(x-6)!!\,(x-7)!!} = 38 \Leftrightarrow$$

$$(x-2)(x-3) + (x-3)(x-4) + (x-4)(x-5) = 38 \Leftrightarrow$$

$$3x^2-21x=0\Leftrightarrow 3x(x-7)=0, x\in\mathbb{N}\Leftrightarrow x=7.$$

A.152. Solution (Tran Hong)

We have: $576=24^2=(x^3+y^3+z^3)^2$. *Inequality becomes as:*

$$(x^5y+y^5z+z^5x)(x^2y+y^2z+z^2x)\geq(x^3+y^3+z^3)^2\cdot xyz;\ \ (1)$$

$$x^5y+y^5z+z^5x=\frac{x^6}{\frac{x}{y}}+\frac{y^6}{\frac{y}{z}}+\frac{z^6}{\frac{z}{x}}\overset{Bergstrom}{\geq}\frac{(x^3+y^3+z^3)^2}{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}$$

$$=\frac{(x^3+y^3+z^3)^2\cdot xyz}{x^2z+y^2x+z^2y}$$

WLOG, suppose: $x\geq y\geq z$. *We need to prove:*

$$x^2y+y^2z+z^2x\geq x^2z+y^2x+z^2y\Leftrightarrow(x-y)(y-z)(x-z)\geq 0$$

true by $x\geq y\geq z\Rightarrow x-y\geq 0; x-z\geq 0; y-z\geq 0\Rightarrow(1)$ *is true.*

A.153. Solution (Michael Sterghiou)

$$\begin{cases}0\leq x,y,z\leq 1\\(x^2+1)(y^2+1)(z^2+1)=8+(x^2-1)(y^2-1)(z^2-1);\ \ (1)\end{cases}$$

Expanding (1) we get

$$2\left(\sum_{cyc}x^2y^2+1\right)=8\ or\ \sum_{cyc}x^2y^2=3$$

But $x,y,z\in[0,1]$ *so* $\sum_{cyc}x^2y^2\leq 3$ *with equality when* $x^2=y^2=z^2=1$.

Given $x,y,z\geq 0$ *the only solution is the set* $\{1,1,1\}$.

A.154. Solution (Sanong Huayrerai)

For $a,b>0$ *we have:*

$$\sqrt[3]{\frac{a^3+b^3}{2}}\cdot\sqrt[4]{\frac{a^4+b^4}{2}}\cdot\sqrt[5]{\frac{a^5+b^5}{2}}=\left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}}\cdot\left(\frac{a^4+b^4}{2}\right)^{\frac{1}{4}}\cdot\left(\frac{a^5+b^5}{2}\right)^{\frac{1}{5}}=$$

$$=\left(\frac{a^3+b^3}{2}\right)^{\frac{20}{60}}\cdot\left(\frac{a^4+b^4}{2}\right)^{\frac{15}{60}}\cdot\left(\frac{a^5+b^5}{2}\right)^{\frac{12}{60}}\leq\frac{a^5+b^5}{a^2+b^2}\Leftrightarrow$$

$$(a^2+b^2)^{60}\cdot(a^3+b^3)^{20}\cdot(a^4+b^4)^{15}\cdot(a^5+b^5)^{12}=2^{47}\cdot(a^5+b^5)^{60}\Leftrightarrow$$

$$(a^2+b^2)^{60}\cdot(a^3+b^3)^{20}\cdot(a^4+b^4)^{15}\le 2^{47}\cdot(a^5+b^5)^{48}\Leftrightarrow$$

$$(a^2+b^2)^{60}\cdot(a^3+b^3)^{20}\cdot(a^4+b^4)^{15}$$

$$\le 2^{47}\left[\frac{(a^2+b^2)(a^3+b^3)}{2}\right]^{20}(a^5+b^5)^{28}\Leftrightarrow$$

$$(a^2+b^2)^{40}\cdot(a^4+b^4)^{15}\le 2^{27}\cdot(a^5+b^5)^{28}\Leftrightarrow$$

$$((a^2+b^2)^4)^{10}\cdot(a^4+b^4)^{15}\le 2^{27}\cdot(a^5+b^5)^{28}\Leftrightarrow$$

$$(2^2(a^4+b^4)^2)^{10}\cdot(a^4+b^4)^{15}\le 2^{27}\cdot(a^5+b^5)^{28}\Leftrightarrow$$

$$(a^4+b^4)^{35}\le 2^7\cdot(a^5+b^5)^{28}\Leftrightarrow$$

$$(a^4+b^4)^5\le 2\cdot(a^5+b^5)^4\Leftrightarrow 2\cdot(a^5+b^5)^4\le 2\cdot(a^5+b^5)^4\ (true).$$

A.155. Solution (Rovsen Pirguliyev)

Denote: $z_1=z^2, z_2=z, z_3=cost+isint$ *and* $z_1+z_2+z_3=z_k$ *then we have:*

$$LHS=|z_k-2z_1|+|z_k-2z_2|+|z_k-2z_3|;\ \ (1)$$

Now, applying Cauchy-Schwartz inequality, we get:

$$(|z_k-2z_1|+|z_k-2z_2|+|z_k-2z_3|)^2$$

$$\le 3(|z_k-2z_1|^2+|z_k-2z_2|^2+|z_k-2z_3|^2);(2)$$

$$\sum_{i=1}^{3}|z_k-2z_i|^2=3|z_k|^2-2z_k\overline{z_k}-2z_k\overline{z_k}+4(|z_1|^2+|z_2|^2+|z_3|^2)=$$

$-|z_k|^2+12$. *Hence:*

$$(|z_k-2z_1|+|z_k-2z_2|+|z_k-2z_3|)^2\le 3(-|z_k|^2+12)\le 36\Leftrightarrow$$

$$|z^2+z-cost+isint|+|cost+isint-z^2+z|+|cost+isint+z^2-z|$$

$$\le 6$$

A.156. Solution (Ravi Prakash)

Let $f(x)=x^{\frac{1}{x}}, x\ge e; logf(x)=\frac{1}{x}logx$

$\frac{f'(x)}{f(x)}=\frac{1}{x^2}(1-logx)<0, \forall x\ge e$ the $f-$ strictly decreasing on $[e,\infty)$.

If $x\ge 4$, then $4^{\frac{1}{4}}\ge x^{\frac{1}{x}}\Rightarrow 4^x\ge x^4, \forall x\ge 4\Rightarrow 4^{2x}+4^x\ge x^8+x^4$

Therefore,

$$4^n(4^n+1)+4^m(4^m+1)+4^p(4^p+1)+4^q(4^q+1)\geq$$
$$\geq n^8+m^8+p^8+q^8+n^4+m^4+p^4+q^4\geq$$
$$\geq 4\sqrt[4]{n^8m^8p^8q^8}+4\sqrt[4]{n^4m^4p^4q^4}=4mnpq(mnpq+1)$$

A.157. Solution (Tran Hong)

For $a,b,c>0$ *and* $\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}=\frac{3}{4}$. *Inequality becomes as:*

$$16\sum\frac{\sqrt{ab}}{a+b}+\sum\frac{(a+b)^2}{ab}\geq 16\sum\frac{ab}{(a+b)^2}+4\sum\frac{a+b}{\sqrt{ab}};\ (1)$$

Hence, we must that:

$$16.\frac{\sqrt{ab}}{a+b}+\frac{(a+b)^2}{ab}\geq 16.\frac{ab}{(a+b)^2}+4.\frac{a+b}{\sqrt{ab}};$$
$$\Leftrightarrow\frac{16v}{u}+\frac{u^2}{v^2}\geq\frac{16v^2}{u^2}+\frac{4u}{v};(\because u=a+b;v=\sqrt{ab};\ u\geq 2v>0)$$
$$\Leftrightarrow 16uv^3+u^4\geq 16v^4+4vu^3\Leftrightarrow u^4-16v^4-4uv(u^2-4v^2)\geq 0;$$
$$\Leftrightarrow(u^2+4v^2)(u^2-4v^2)-4uv(u^2-4v^2)\geq 0;$$
$$\Leftrightarrow(u^2-4v^2)(u^2+4v^2-4uv)\geq 0;$$
$$\Leftrightarrow(u^2-4v^2)(u-2v)^2\geq 0\Leftrightarrow(u-2v)^3(u+2v)\geq 0;$$

Which is true by: $u\geq 2v>0$

$\Rightarrow$ (1) *true. Proved . Equality* $\Leftrightarrow u=2v\Leftrightarrow a=b=c=1.$

A.158. Solution (George Florin Şerban)

$$\frac{x^6}{yzt}+\frac{y^6}{ztx}+\frac{z^6}{txy}+\frac{t^6}{xyz}=1\Rightarrow\frac{1}{xyzt}\sum_{cyc}x^7=\sum_{cyc}xyz\Rightarrow$$
$$\sum_{cyc}x^7=xyzt\sum_{cyc}xyz\Rightarrow$$
$$\sum_{cyc}x^7=(xyz)^2t+(xyt)^2z+(xzt)^2y+(yzt)^2x$$
$$2x^7+2y^7+2z^7+t^7\overset{AM-GM}{\geq}7\sqrt[7]{(xyzt)^{14}t^7}=7(xyz)^2t$$

$$2x^7+2y^7+z^7+2t^7 \overset{AM-GM}{\geq} 7(xyt)^2z$$
$$2x^7+y^7+2z^7+2t^7 \overset{AM-GM}{\geq} 7(xzt)^2y$$
$$x^7+2y^7+2z^7+t^7 \overset{AM-GM}{\geq} 7(yzt)^2x$$

Therefore,

$$7\sum_{cyc} x^7 \geq 7(xyz)^2t+7(xyt)^2z+7(xzt)^2y+7(yzt)^2x$$
$$\sum_{cyc} x^7 \geq (xyz)^2t+(xyt)^2z+(xzt)^2y+(yzt)^2x$$

Equality holds if and only if $x=y=z=t \Rightarrow 4x^3=1 \Rightarrow$
$$x=y=z=t=\frac{1}{\sqrt[3]{4}}$$

A.159. Solution (Tran Hong)

By Cauchy-Schwartz Inequality, we have:

$$(x-a)^2+2x^2=\left(\frac{a-x}{2}\right)^2+\left(\frac{a-x}{2}\right)^2+\left(\frac{a-x}{2}\right)^2+\left(\frac{a-x}{2}\right)^2$$
$$=\frac{\left(4\cdot\frac{a-x}{2}+x+x\right)^2}{6}=\frac{(2a-2x+2x)^2}{6}=\frac{2a^2}{3}$$

Similarly: $(y-b)^2+2y^2 \geq \frac{2b^2}{3}$, $(z-c)^2+2z^2 \geq \frac{2c^2}{3}$

$$\Rightarrow \Omega_1=((x-a)^2+y^2+z^2)^2+(x^2+(y-b)^2+z^2)^2$$
$$+(x^2+y^2+(z-c)^2)^2$$
$$\geq \frac{(x-a)^2+2x^2+(y-b)^2+2y^2+(y-c)^2+2z^2}{3}$$
$$\geq \frac{\left(\frac{2a^2}{3}+\frac{2b^2}{3}+\frac{2c^2}{3}\right)^2}{3}=\frac{4(a^2+b^2+c^2)^2}{27} \overset{(*)}{\geq} \frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{2(a^2+b^2+c^2)}=\Omega_2$$
$$(*) \Leftrightarrow (2a^2+2b^2+2c^2)^3 \geq 27(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

Which is true by AM-GM Inequality:

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) \leq \frac{(2a^2+2b^2+2c^2)^3}{3^3}$$

Equality holds if and only if: $\begin{cases}\frac{a-x}{2}=x\\ \frac{b-y}{2}=y\\ \frac{c-z}{2}=z\end{cases} \Leftrightarrow x=y=z=\frac{a}{3}.$

A.160. Solution (George Florin Şerban)

$$2\sum_{cyc}\frac{log_y x}{x+y}=\frac{16}{x+y+z+t}\Rightarrow\sum_{cyc}\frac{log_y x}{x+y}=\frac{8}{x+y+z+t}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}4\sqrt[4]{\prod_{cyc}\frac{log_y x}{x+y}}=\frac{4}{\sqrt[4]{\prod_{cyc}(x+y)}}\overset{AM-GM}{\geq}\frac{4}{\frac{\sum_{cyc}(x+y)}{4}}=\frac{8}{\sum_{cyc}x}$$

Equality holds if $x+y=y+z=z+t=t+x\Leftrightarrow x=y=z=t.$

$$(2x+3x+4x+100)^{10}=10^{12}x^2x^3x^4$$

$$\Leftrightarrow(9x+100)^{10}=10^{12}x^9\Rightarrow 9x+100=\sqrt[10]{10^{12}x^9}$$

$$\Rightarrow 9x+100=10\sqrt[10]{100x^9}\Rightarrow$$

$$\frac{9x+100}{10}=\sqrt[10]{100\cdot\underbrace{x\cdot x\cdot\ldots\cdot x}_{9-times}}\overset{AM-GM}{\leq}\frac{9x+100}{10}$$

Therefore: $x=y=z=t=100.$

A.161. Solution (Rovsen Pirguliyev)

Let's prove that: $log_{10}^2 9+log_{10}^2 11>log_{10}98$

$$log_{10}^2 9+log_{10}^2 11=\left(1+log_{10}\frac{9}{10}\right)^2+\left(1+log_{10}\frac{11}{10}\right)^2>$$

$$>2+2\left(log_{10}\frac{9}{10}+log_{10}\frac{11}{10}\right)=2+2log_{10}\frac{99}{10}=2log_{10}99>log_{10}98$$

$$\left(\frac{1}{a}+\frac{1}{c}\right)log_{10}^4 9+\left(\frac{1}{b}+\frac{1}{d}\right)log_{10}^4 11\overset{Bergstrom}{\geq}\frac{(1+1)^2}{a+c}log_{10}^4 9$$

$$+\frac{(1+1)^2}{b+c}log_{10}^4 11=$$

$$=\frac{(2log_{10}^2 9)^2}{a+c}+\frac{(2log_{10}^2 11)^2}{b+d}\overset{Bergstrom}{\geq}\frac{4(log_{10}^2 9+log_{10}^2 11)^2}{a+b+c+d}>$$

$$\frac{4log_{10}^2 98}{log_{10}^2 98}=4$$

SOLUTIONS

GEOMETRY

G.001. Solution (Tran Hong)

The nine-point center N satisfies:

$$AN^2 + BN^2 + CN^2 = 3R^2 - ON^2$$

Other, $ON = \frac{OH}{2}$ (N *−midpoint* OH) $\Rightarrow$

$$AN^2 + BN^2 + CN^2 = 3R^2 - \frac{1}{4}OH^2 = 3R^2 - \frac{1}{4}\left(9R^2 - (a^2 + b^2 + c^2)\right)$$

$$= \frac{a^2 + b^2 + c^2 + 3R^2}{4}$$

$$Lhs = \sum_{cyc}\left(\frac{a^2 + R^2}{NB}\right)^2 = \sum_{cyc}\frac{(a^2 + R^2)^2}{NB^2} \overset{Bergstrom}{\geq} \frac{(a^2 + b^2 + c^2 + 3R^2)^2}{AN^2 + BN^2 + CN^2} =$$

$$= \frac{(a^2 + b^2 + c^2 + 3R^2)^2}{\frac{a^2 + b^2 + c^2 + 3R^2}{4}} = 4(a^2 + b^2 + c^2 + 3R^2) \geq 4\left(4\sqrt{3}S + 3R^2\right) =$$

$$= 16\sqrt{3} \cdot sr + 12R^2 \overset{\substack{s \geq 3\sqrt{3}r \\ R \geq 2r}}{\geq} 16 \cdot 9r^2 + 12 \cdot 4r^2 = 192r^2$$

G.002. Solution (Ravi Prakash)

$$|cosx| + |cosy| = \sqrt{(2 + sinx + siny)(2 - sinx - siny)}; \quad (1)$$
$$\Rightarrow (|cosx| + |cosy|)^2 = 4 - (sinx + siny)^2 \Rightarrow$$
$$cos^2x + cos^2y + 2|cosx||cosy| = 4 - 2sinxsiny - sin^2x - sin^2y \Leftrightarrow$$
$$|cosxcosy| + sinxsiny = 1$$

If $cosxcosy \geq 0$ we get:

$$cos(x - y) = 1 \Leftrightarrow x - y = 2r\pi \Leftrightarrow x = 2r\pi + y, r \in \mathbb{Z} \Rightarrow$$

$y = 2n\pi + x, n \in \mathbb{Z}$. If $cosxcosy \leq 0$ we get:

$$cosxcosy + sinxsiny = 1 \Leftrightarrow cos(x + y) = -1 \Leftrightarrow x + y = (2r + 1)\pi \Leftrightarrow$$
$$y = (2r + 1)\pi - x, r \in \mathbb{Z}$$

Thus, $y = 2n\pi + x, n \in \mathbb{Z}$ *or* $y = (2r + 1)\pi - x, r \in \mathbb{Z}$

G.003. Solution (Avishek Mitra)

$$\sum sin^2\alpha \overset{AM-GM}{\geq} 4\cdot\sqrt[4]{\prod sin^2\alpha} = 4\cdot\sqrt{\prod sin\alpha} = 4\cdot\sqrt{\frac{1}{16}} = 1$$

$$\sum(1-cos^2\alpha) \geq 1 \Rightarrow \sum cos^2\alpha \leq 4-1 = 3$$

$$\left(\therefore \left(\frac{\sum cos\alpha}{4}\right)^2 \overset{power\ mean}{\leq} \frac{\sum cos^2\alpha}{4} \Rightarrow \frac{(\sum cos\alpha)^2}{16} \leq \frac{3}{4} \Rightarrow \sum cos\alpha \leq 2\sqrt{3}\right)$$

$$\sum \frac{sin^2\alpha}{cos\alpha} \overset{Bergstrom}{\geq} \frac{(\sum sin\alpha)^2}{\sum cos\alpha} \overset{AM-GM}{\geq} \frac{\left(4\sqrt[4]{\prod sin\alpha}\right)^2}{2\sqrt{3}} = \frac{16\cdot\frac{1}{\sqrt{16}}}{2\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

G.004. Solution (George Florin Şerban)

$$X = a(a-3b-3c)^2 + b(3a-b-c)^2 + c(3a-b-c)^2 =$$
$$= a^3 + 9ab^2 + 9ac^2 - 6a^2b + 18abc - 6a^2c + 9a^2b + b^3 + bc^2 - 6ab^2 + 2b^2c -$$
$$-6abc + 9a^2c + b^2c + c^3 - 6abc + 2bc^2 - 6ac^2 =$$
$$= a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3ac^2 + 3a^2c + 3b^2c + 3bc^2 =$$
$$= (a+b+c)^3 = 8s^3$$

Applying Mitrinovic Inequality: $3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}R}{2}$ *we get:*

$$\left(3\sqrt{3}r\right)^3\cdot 8 \leq 8s^3 \leq 8\cdot\left(\frac{3\sqrt{3}R}{2}\right)^3 \Leftrightarrow 27\cdot 3\sqrt{3}\cdot 8r^3 \leq X \leq 8\cdot\frac{27\cdot 3\sqrt{3}R^3}{8}$$

$$648\sqrt{3}r^3 \leq a(a-3b-3c)^2 + b(3a-b-c)^2 + c(3a-b-c)^2 \leq 81\sqrt{3}R^3$$

G.005.Solution (Soumava Chakraborty)

$$\pi = A + 2A + 4A \Rightarrow A = \frac{\pi}{7}, B = \frac{2\pi}{7} \text{ and } C = \frac{4\pi}{7}$$

$$\text{Now, } \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{2\sin\frac{\pi}{7}\cos\frac{\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{3\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{5\pi}{7}}{2\sin\frac{\pi}{7}}$$

$$= \frac{\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{2\pi}{7} + \sin\frac{6\pi}{7} - \sin\frac{4\pi}{7}}{2\sin\frac{\pi}{7}}$$

$$= \frac{\sin\left(\pi - \frac{\pi}{7}\right)}{2\sin\frac{\pi}{7}} = \frac{\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}} = \frac{1}{2} \therefore \boxed{\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} \overset{(1)}{=} \frac{1}{2}}$$

$$\therefore \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\left(\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\right)\overset{\text{by (1)}}{\cong}-\frac{1}{2}$$

$$\therefore \boxed{\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\overset{(2)}{\cong}-\frac{1}{2}}$$

$$\text{Now, }\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)^2+\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2$$

$$=3+2\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+2\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}+2\cos\frac{8\pi}{7}\cos\frac{2\pi}{7}$$

$$+2\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}+2\sin\frac{4\pi}{7}\sin\frac{8\pi}{7}+2\sin\frac{8\pi}{7}\sin\frac{2\pi}{7}$$

$$=3+2\left(\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}-\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)$$

$$+2\left(\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}-\sin\frac{4\pi}{7}\sin\frac{8\pi}{7}\right)$$

$$+2\left(\cos\frac{8\pi}{7}\cos\frac{2\pi}{7}-\sin\frac{8\pi}{7}\sin\frac{2\pi}{7}\right)=3+2\left(\cos\frac{6\pi}{7}+\cos\frac{12\pi}{7}+\cos\frac{10\pi}{7}\right)$$

$$=3-2\left(\cos\frac{\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}\right)\overset{\text{by (1)}}{\cong}3-1=2$$

$$\therefore\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)^2+\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2$$

$$=2\overset{\text{by (2)}}{\Rightarrow}\frac{1}{4}+\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2=2$$

$$\Rightarrow\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2=\frac{7}{4}\Rightarrow\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{\sqrt{7}}{2}$$

$$\Rightarrow\boxed{\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{\pi}{7}\overset{(3)}{\cong}\frac{\sqrt{7}}{2}}$$

$$\textit{Again},\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)\left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)$$

$$=\frac{\left(2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\left(2\sin\frac{3\pi}{7}\cos\frac{3\pi}{7}\right)}{8}$$

$$=\frac{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\sin\frac{6\pi}{7}}{8}$$

$$=\frac{\left(\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{\pi}{7}\right)}{8}\Rightarrow\boxed{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\overset{(4)}{\cong}\frac{1}{8}}$$

$$\text{Also, }\left(2\sin^2\frac{\pi}{7}\right)\left(2\sin^2\frac{2\pi}{7}\right)\left(2\sin^2\frac{3\pi}{7}\right)=\left(1-\cos\frac{2\pi}{7}\right)\left(1-\cos\frac{4\pi}{7}\right)\left(1-\cos\frac{6\pi}{7}\right)$$

$$=1+\frac{1}{2}\left(2\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+2\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+2\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}\right)-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}$$

$$-\cos\frac{6\pi}{7}-\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$$

$$= 1 + \frac{1}{2}\left(\cos\frac{6\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{4\pi}{7}\right) - \cos\frac{2\pi}{7} - \cos\frac{4\pi}{7} - \cos\frac{6\pi}{7} - \cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\cos\frac{\pi}{7}$$

$$\overset{\text{by (4)}}{=} 1 + \frac{1}{2}\left(-\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7}\right) - \cos\frac{2\pi}{7} - \cos\frac{4\pi}{7} - \cos\frac{6\pi}{7} - \frac{1}{8}$$

$$= \frac{7}{8} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{\pi}{7} = \frac{7}{8} \Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sqrt{\frac{7}{64}} \therefore \boxed{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} \overset{(5)}{=} \frac{\sqrt{7}}{8}}$$

$$Moreover, \frac{1}{\sin\frac{2\pi}{7}} + \frac{1}{\sin\frac{3\pi}{7}} = \frac{\sin\frac{3\pi}{7} + \sin\frac{5\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}} = \frac{2\sin\frac{4\pi}{7}\cos\frac{\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\sin\frac{4\pi}{7}} = \frac{1}{\sin\frac{\pi}{7}}$$

$$\Rightarrow \left(\frac{1}{\sin\frac{2\pi}{7}} + \frac{1}{\sin\frac{3\pi}{7}} - \frac{1}{\sin\frac{\pi}{7}}\right)^2 = 0$$

$$\Rightarrow \frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{3\pi}{7}} + \frac{1}{\sin^2\frac{\pi}{7}} + 2\left(\frac{1}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}} - \frac{1}{\sin\frac{3\pi}{7}\sin\frac{\pi}{7}} - \frac{1}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}}\right) = 0$$

$$\Rightarrow \frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{3\pi}{7}} + \frac{1}{\sin^2\frac{\pi}{7}} - \left(\frac{2}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}}\right)\left(\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7}\right) = 0$$

$$\overset{\text{by (3) and (5)}}{\Longrightarrow} \frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{3\pi}{7}} + \frac{1}{\sin^2\frac{\pi}{7}} - \left(\frac{16}{\sqrt{7}}\right)\left(\frac{\sqrt{7}}{2}\right) = 0$$

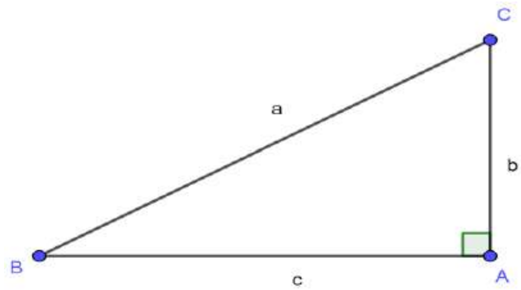
$$\Rightarrow \boxed{\frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{3\pi}{7}} + \frac{1}{\sin^2\frac{\pi}{7}} \overset{(6)}{=} 8}$$

$$\text{Now}, \sum h_a^2 = \sum\left(\frac{bc}{2R}\right)^2 = \sum\left(\frac{4R^2\sin B\sin C}{2R}\right)^2 = 4R^2\sum\sin^2 B\sin^2 C$$

$$= 4R^2\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2\left(\frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{3\pi}{7}} + \frac{1}{\sin^2\frac{\pi}{7}}\right)$$

$$\overset{\text{by (5) and (6)}}{=} 4R^2\left(\frac{56}{64}\right) = \frac{7R^2}{2} \therefore \boxed{\text{LHS} = \frac{7R^2}{2}} > \frac{7\sqrt{21}R^2}{10}\ (\textit{Proved})$$

G.006. Solution (Ravi Prakash)



$$Let: c = acos\theta; b = asin\theta$$

$$LHS = \frac{bc}{a(b+c-a)} + \frac{2bc+(a+b+c)^2}{a(a+b+c)} =$$

$$= \frac{sin\theta cos\theta}{sin\theta + cos\theta - 1} + \frac{2sin\theta cos\theta}{sin\theta + cos\theta + 1} + sin\theta + cos\theta + 1 =$$

$$= \frac{sin\theta cos\theta(sin\theta + cos\theta + 1 + 2sin\theta + 2cos\theta - 2)}{(sin\theta + cos\theta)^2 - 1} + sin\theta + cos\theta + 1 =$$

$$= \frac{3sin\theta + 3cos\theta - 1}{2} + sin\theta + cos\theta + 1 =$$

$$= \frac{1}{2}(5sin\theta + 5cos\theta + 1) \le \frac{1}{2}(5\sqrt{2} + 1)$$

It is sufficient to show that: $5\sqrt{2} < 3\sqrt{3} + \sqrt{2} + 1 \Leftrightarrow 4 < 6\sqrt{3}\ (true)$.

G.007. Solution (Sudhir Jha)

$$Lhs = cos^2x + cos^2y + cos^2z + 1 + (sin^2x + sin^2y + sin^2z) +$$

$$+(sin^2x \cdot sin^2y + sin^2y \cdot sin^2z + sin^2z \cdot sin^2x) + sin^2x \cdot sin^2y \cdot sin^2z =$$

$$= (sin^2x + cos^2x) + (sin^2y + cos^2y) + (sin^2z + cos^2z) + 1 +$$

$$+(sin^2x \cdot sin^2y + sin^2y \cdot sin^2z + sin^2z \cdot sin^2x) + sin^2x \cdot sin^2y \cdot sin^2z =$$

$$= 4 + (sin^2x \cdot sin^2y + sin^2y \cdot sin^2z + sin^2z \cdot sin^2x) + sin^2x \cdot sin^2y \cdot$$

$sin^2z \le 8$ true from:

$$(\therefore sin^2x \cdot sin^2y \le 1; sin^2y \cdot sin^2z \le 1;\ sin^2z \cdot sin^2x$$

$$\le 1; sin^2x \cdot sin^2y \cdot sin^2z \le 1\ )$$

$$cos^2x + cos^2y + cos^2z + (1 + sin^2x)(1 + sin^2y)(1 + sin^2z) \le 8$$

Equality holds for $x = y = z = \frac{\pi}{2}$

G.008. Solution (Khaled Abd Imouti)

$$\begin{cases} sinx + siny = 1; \ (1) \\ cosx + cosy = \sqrt{3}; \quad (3) \\ \sqrt[4]{z + sin^{-6}x} + \sqrt[4]{z + sin^{-6}y} = 4\sqrt{2}; \quad (3) \end{cases}$$

From (1): $sin^2x + sin^2y + 2sinxsiny = 1$

From (2): $cos^2x + cos^2y + 2cosxcosy = 3$

Adding (1) and (2) we get: $1 + 2(sinxsiny + cosxcosy) + 1 = 4 \Leftrightarrow$

$$cos(x - y) = 1 \Leftrightarrow x - y = 2k\pi, k \in \mathbb{Z} \Rightarrow x = y + 2k\pi, k \in \mathbb{Z}$$

$$cos(y + 2k\pi) + cosy = \sqrt{3} \Rightarrow 2cosy = \sqrt{3} \Rightarrow cosy = \frac{\sqrt{3}}{2} \Rightarrow cosy = cos\frac{\pi}{6}$$

$\Rightarrow \begin{cases} y = \pi + 2\pi \\ y = \frac{5\pi}{6} + 2k\pi \end{cases}$. *Similarly,* $cosx = \frac{\sqrt{3}}{2}, sinx = \frac{1}{2}$

$$\sqrt[4]{z + 2^6} + \sqrt[4]{z + 2^6} = 4\sqrt{2} \Rightarrow 2\sqrt[4]{z + 2^6} = 4\sqrt{2} \Rightarrow \sqrt[4]{z + 2^6} = 2\sqrt{2} \Rightarrow$$

$$z + 2^6 = 2^4 \cdot 2 \Rightarrow z = 0.$$

$$S = \left\{x = y = \frac{\pi}{6} + 2k\pi, z = 0 / k \in \mathbb{Z}\right\}$$

G.009. Solution (Bedri Hajrizi)

$x + y = \pi \Rightarrow y = \pi - x$. *The system is equivalent with:*

$$\begin{cases} tan^2x(1 - sin^8x) + cot^2x(1 - cos^8x) = \frac{30}{16}; \ (1) \\ tan^2x(1 - sin^{10}x) + cot^2x(1 - cos^{10}x) = \frac{31}{16}; \ (2) \end{cases}$$

From $(2) - (1)$ *we get:*

$$tan^2x(sin^8x - sin^{10}x) + cot^2x(cos^8x - cos^{10}x) = \frac{1}{16} \Leftrightarrow$$

$$\frac{sin^2x}{cos^2x} \cdot sin^8x \cdot cos^2x + \frac{cos^2x}{sin^2x} \cdot cos^8x \cdot sin^2x = \frac{1}{16} \Leftrightarrow$$

$$sin^{10}x + cos^{10}x - \frac{1}{16} = 0$$

$$\text{Let be the function } f(x) := \sin^{10}x + \cos^{10}x - \frac{1}{16}$$

$$f'(x) = 10\sin x\cos x(\sin^8 x - \cos^8 x)$$

$$f'(x) = 0 \Leftrightarrow x = m\pi; x = \frac{\pi}{2} + n\pi; x = \frac{\pi}{4} + \frac{k\pi}{2}; m, n, k \in \mathbb{Z}$$

$$f(m\pi) = f\left(\frac{\pi}{2} + n\pi\right) = \frac{15}{16}, \qquad f\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) = 0$$

$$\text{So, } f(x) \ge 0, \forall x \in \mathbb{R}, (x, y) \in \left\{\left(\frac{\pi}{4} + \frac{k\pi}{2}, \frac{3\pi}{4} - \frac{k\pi}{2}\right) / k \in \mathbb{Z}\right\}$$

G.010. Solution (Soumava Chakraborty)

$$x^2 + y^2 \ge 2xy \Rightarrow x^2 + y^2 + 1 + x^2y^2 \ge 2xy + 1 + x^2y^2$$
$$\Rightarrow (1 + x^2)(1 + y^2) \ge (1 + xy)^2$$
$$\Rightarrow \sqrt{(1 + x^2)(1 + y^2)} \overset{(1)}{\ge} 1 + xy \ \forall\, x, y > 0$$
$$\prod(\tan A + \cot A)(\cos A + \sec A) \ge \prod(\tan A + \cot B)(\cos A + \sec B)$$
$$\Leftrightarrow \prod\left(\tan A + \frac{1}{\tan A}\right)\left(\cos A + \frac{1}{\cos A}\right) \ge \prod\left(\tan A + \frac{1}{\tan B}\right)\left(\cos A + \frac{1}{\cos B}\right)$$
$$\Leftrightarrow \left(\prod\left(\frac{1 + \tan^2 A}{\tan A}\right)\right)\left(\prod\left(\frac{1 + \cos^2 A}{\cos A}\right)\right)$$
$$\ge \left(\prod\left(\frac{1 + \tan A\tan B}{\tan B}\right)\right)\left(\prod\left(\frac{1 + \cos A\cos B}{\cos B}\right)\right)$$
$$\Leftrightarrow \left(\prod\sqrt{(1 + \tan^2 B)(1 + \tan^2 C)}\right)\left(\prod\sqrt{(1 + \cos^2 B)(1 + \cos^2 C)}\right) \overset{(i)}{\ge} \left(\prod(1 + \tan B\tan C)\right)\left(\prod(1 + \cos B\cos C)\right)$$
$$\text{Now, } (1) \Rightarrow \sqrt{(1 + \tan^2 B)(1 + \tan^2 C)}$$
$$\ge 1 + \tan B\tan C \text{ and analogs} \overset{\text{multiplying together}}{\Rightarrow}$$
$$\prod\sqrt{(1 + \tan^2 B)(1 + \tan^2 C)} \overset{(a)}{\ge} \prod(1 + \tan B\tan C)$$
$$\text{Also, } (1) \Rightarrow \sqrt{(1 + \cos^2 B)(1 + \cos^2 C)}$$
$$\ge 1 + \cos B\cos C \text{ and analogs} \overset{\text{multiplying together}}{\Rightarrow}$$
$$\prod\sqrt{(1 + \cos^2 B)(1 + \cos^2 C)} \overset{(b)}{\ge} \prod(1 + \cos B\cos C) \therefore (a).(b)$$
$$\Rightarrow (i) \text{ is true (Proved)}$$

G.011. Solution (Khaled Abd Imouti)

We know that: $cosx=\frac{e^{ix}+e^{-ix}}{2}$; $cos^5x=\frac{1}{32}\left(e^{ix}+e^{-ix}\right)^5$

$$cos^5x=\frac{1}{32}\left[\sum_{k=1}^{5}\binom{5}{k}e^{ikx}\cdot e^{-(5-k)x}\right]=$$

$$=\frac{1}{32}\left[2\cdot\frac{e^{i5x}+e^{-i5x}}{2}+5\cdot 2\cdot\frac{e^{i3x}+e^{-i3x}}{2}+10\cdot 2\cdot\frac{e^{ix}+e^{-ix}}{2}\right]=$$

$$=\frac{1}{32}[2cos(5x)+10cos(3x)+20cosx]$$

$$32cos^5x=2cos(5x)+10cos(3x)+20cosx$$

$$16cos^5x=cos(5x)+5cos(3x)+10cosx$$

$$cos(5x)=16cos^5x-5cos(3x)-10cosx$$

$$cos(5x)=16cos^5x-5(4cos^3x-3cosx)-10cosx$$

$$cos(5x)=16cos^5x-20cos^3x+5cosx;(cosx\neq 0)$$

$$\frac{cos(5x)}{cosx}=16cos^4x-20cos^2x+5=16\left(cos^2x-\frac{5}{8}\right)^2-\frac{5}{4}$$

So, we get:

$$\frac{cos(5x)}{cosx}+\frac{cos(5y)}{cosy}+\frac{cos(5z)}{cosz}=\frac{15}{4}$$

$$16\left(cos^2x-\frac{5}{8}\right)^2-\frac{5}{4}+16\left(cos^2y-\frac{5}{8}\right)^2-\frac{5}{4}+16\left(cos^2z-\frac{5}{8}\right)^2-\frac{5}{4}=\frac{15}{4}$$

$$16\left[\left(cos^2x-\frac{5}{8}\right)^2+\left(cos^2y-\frac{5}{8}\right)^2+\left(cos^2x-\frac{5}{8}\right)^2\right]-\frac{15}{4}=\frac{15}{4}$$

$$16\left[\left(cos^2x-\frac{5}{8}\right)^2+\left(cos^2y-\frac{5}{8}\right)^2+\left(cos^2x-\frac{5}{8}\right)^2\right]=\frac{15}{2}$$

If $cosx=cosy=cosz$ then:

$$3\cdot 16\left(cos^2x-\frac{5}{8}\right)^2=\frac{15}{2}\Leftrightarrow\left(cos^2x-\frac{5}{8}\right)^2=\frac{5}{32}$$

$$cos^2x=\frac{5}{8}\pm\frac{\sqrt{5}}{4\sqrt{2}}$$

$$\text{Case I. } cosx=-\sqrt{\frac{5}{8}\mp\frac{\sqrt{5}}{4\sqrt{2}}}\Rightarrow cosx=-\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{4\sqrt{2}}}<-1\ (impossible)$$

$$\text{Case II. } cosx=\sqrt{\frac{5}{8}\mp\frac{\sqrt{5}}{4\sqrt{2}}}\Rightarrow cosx=\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{4\sqrt{2}}}\in(0,1)$$

$$\text{So, } S=\{x,y,z/x=y=z=cos^{-1}\left(\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{4\sqrt{2}}}\right)$$

G.012. Solution (Khaled Abd Imouti)

Let denote $y=sinxcosx=\frac{1}{2}sin2x,0\le 2x\le\frac{\pi}{2}$

We must show that: $tanh(2y)\le tanhy+\frac{y}{cosh^2y}$

Let us prove that: $tanh(2y)-tanhy\le\frac{y}{cosh^2y}$. *By using M.V.T. we have:*

$$tanh(2y)-tanhy=y\cdot tanhc;y<c<2y$$

$$tanh(2y)-tanhy=\frac{y}{cosh^2c};y<c<2y$$

$$tanh(2y)-tanhy\le\frac{y}{cosh^2y}$$

G.013. Solution (Soumava Chakraborty)

$$:\frac{h_ar_b^2}{m_a}+\frac{h_br_c^2}{m_b}+\frac{h_cr_a^2}{m_c}$$

$$=\frac{2rsr_b^2}{am_a}+\frac{2rsr_c^2}{bm_b}$$

$$+\frac{2rsr_a^2}{cm_c}\overset{Bergstrom}{\geq}(2rs)\frac{(r_a+r_b+r_c)^2}{am_a+bm_b+cm_c}\overset{CBS}{\geq}\frac{2rs(4R+r)^2}{\sqrt{3}\sqrt{\sum a^2m_a^2}}$$

$$=\frac{4rs(4R+r)^2}{\sqrt{3}\sqrt{\sum a^2(2b^2+2c^2-a^2)}}=\frac{4rs(4R+r)^2}{\sqrt{3}\sqrt{4\sum a^2b^2-\sum a^4}}$$

$$=\frac{4rs(4R+r)^2}{\sqrt{3}\sqrt{2\sum a^2b^2+16r^2s^2}}\overset{Goldstone}{\geq}\frac{4rs(4R+r)^2}{\sqrt{3}\sqrt{8R^2s^2+16r^2s^2}}$$

$$=\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\frac{r(4R+r)^2}{\sqrt{R^2+2r^2}}$$

$$\overset{?}{\gtreqless}\frac{\sqrt{6}}{3}\cdot\frac{r(4R+r)^2}{R+(\sqrt{6}-2)r}$$

$$\Leftrightarrow R-2r+\sqrt{6}r\overset{?}{\gtreqless}\sqrt{R^2+2r^2}\overset{squaring}{\Leftrightarrow}(R-2r)^2+6r^2$$

$$+2\sqrt{6}r(R-2r)\overset{?}{\gtreqless}R^2+2r^2$$

$$\Leftrightarrow 8r^2-4Rr+2\sqrt{6}r(R-2r)\overset{?}{\gtreqless}0\Leftrightarrow -4r(R-2r)+2\sqrt{6}r(R-2r)\overset{?}{\gtreqless}0$$

$$\Leftrightarrow 2(\sqrt{6}-2)r(R-2r)\overset{?}{\gtreqless}0\to true \because R\overset{Euler}{\gtreqless}2r\ and\ \sqrt{6}>2$$

$$\therefore\frac{h_ar_b^2}{m_a}+\frac{h_br_c^2}{m_b}+\frac{h_cr_a^2}{m_c}\geq\frac{\sqrt{6}}{3}\cdot\frac{r(4R+r)^2}{R+(\sqrt{6}-2)r}$$

G.014. Solution (Adrian Popa)

$$sin^2x\cdot(1-sin^2t)+sin^2y\cdot cos^2x+sin^2z\cdot cos^2y+sin^2t\cdot cos^2z=2\Leftrightarrow$$

$$sin^2x\cdot cos^2t+(1-cos^2y)\cdot cos^2x+sin^2z\cdot(1-sin^2y)+(1-cos^2t)\cdot cos^2z=2$$

$$\Leftrightarrow sin^2x\cdot sin^2t+cos^2y\cdot cos^2x+sin^2z\cdot sin^2y+cos^2t\cdot cos^2z$$

$$\Leftrightarrow sinx\cdot sint=0, cosy\cdot cosx=0, sinz\cdot siny=0,$$

$$cost\cdot coxz=0$$

I) $sinx=cosy=sinz=cost=0\Leftrightarrow$

$$(x,y,z,t)=\left(m\pi,n\pi+\frac{\pi}{2},p\pi,q\pi+\frac{\pi}{2}\right),m,n,p,q\in\mathbb{Z}$$

ii) $sint=cosz=siny=cosx=0$

$$(x,y,z,t)=\left(m'\pi+\frac{\pi}{2},n'\pi,p'\pi+\frac{\pi}{2},q'\pi\right),m',n',p',q'\in\mathbb{Z}$$

G.015. Solution (Abdul Hannan)

$$det\begin{pmatrix}x & y\\ y & x\end{pmatrix}=x^2-y^2=(x-y)(x+y);\ \ (1)$$

For square matrices A,B of the same size, we have:

$$\begin{pmatrix}I & -I\\ 0 & I\end{pmatrix}\begin{pmatrix}A & B\\ B & A\end{pmatrix}\begin{pmatrix}I & I\\ 0 & I\end{pmatrix}=\begin{pmatrix}A-B & 0\\ B & A+B\end{pmatrix}$$

$$det\begin{pmatrix}I & -I\\ 0 & I\end{pmatrix}det\begin{pmatrix}A & B\\ B & A\end{pmatrix}det\begin{pmatrix}I & I\\ 0 & I\end{pmatrix}=det\begin{pmatrix}A-B & 0\\ B & A+B\end{pmatrix}\Rightarrow$$

$$det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = det(A-B)det(A+B)$$

Take $A = \begin{pmatrix} 9R^2 & a^2 \\ a^2 & 9R^2 \end{pmatrix}, B = \begin{pmatrix} b^2 & c^2 \\ c^2 & b^2 \end{pmatrix}$

$$A+B = \begin{pmatrix} 9R^2+b^2 & a^2+c^2 \\ a^2+c^2 & 9R^2+b^2 \end{pmatrix} \text{ and } A-B = \begin{pmatrix} 9R^2-b^2 & a^2-c^2 \\ a^2-c^2 & 9R^2-b^2 \end{pmatrix}$$

$$\Rightarrow det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = det\begin{pmatrix} 9R^2+b^2 & a^2+c^2 \\ a^2+c^2 & 9R^2+b^2 \end{pmatrix} det\begin{pmatrix} 9R^2-b^2 & a^2-c^2 \\ a^2-c^2 & 9R^2-b^2 \end{pmatrix} \overset{(1)}{=}$$

$$= (9R^2-b^2+a^2-c^2)(9R^2-b^2-a^2+c^2)(9R^2+b^2+a^2+c^2)(9R^2 + b^2-a^2-c^2)$$

$$9R^2-b^2+a^2-c^2 \overset{Leibniz}{\geq} a^2+b^2+c^2-b^2+a^2-c^2 = 2a^2 > 0$$

$$9R^2-b^2-a^2+c^2 \overset{Leibniz}{\geq} a^2+b^2+c^2-b^2-a^2+c^2 = 2c^2 > 0$$

$$9R^2+b^2+a^2+c^2 > 0$$

$$9R^2+b^2-a^2-c^2 \overset{Leibniz}{\geq} a^2+b^2+c^2+b^2-a^2-c^2 = 2b^2 > 0$$

Therefore,

$$det\begin{pmatrix} A & B \\ B & A \end{pmatrix} > 0$$

G.016. Solution (Adrian Popa)

Let's denote: $tanA = x, tanB = y, tanC = z$ *then*

$$\sum_{cyc}\frac{x}{\sqrt[3]{x+y+z}}\left(1+\frac{y}{\sqrt[3]{x+y+z}}\right) = \sum_{cyc}\frac{x}{\sqrt[3]{x+y+z}} + \sum_{cyc}\frac{xy}{\sqrt[3]{x+y+z}} =$$

$$= \frac{x+y+z}{\sqrt[3]{x+y+z}} + \frac{xy+yz+zx}{\sqrt[3]{(x+y+z)^2}} = \sqrt[3]{(x+y+z)^2} + \frac{xy+yz+zx}{\sqrt[3]{(x+y+z)^2}} \overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq} 2\sqrt{xy+yz+zx} \overset{(1)}{\geq} 6, \qquad (1) \Leftrightarrow xy+yz+zx \geq 9$$

$$xy+yz+zx \overset{AM-GM}{\geq} 3\sqrt[3]{x^2y^2z^2} \geq 9 \Leftrightarrow x^2y^2z^2 \geq 27 \Leftrightarrow$$

$$xyz \geq 3\sqrt{3} \Leftrightarrow tanAtanBtanC \geq 3\sqrt{3}; \ \ (2)$$

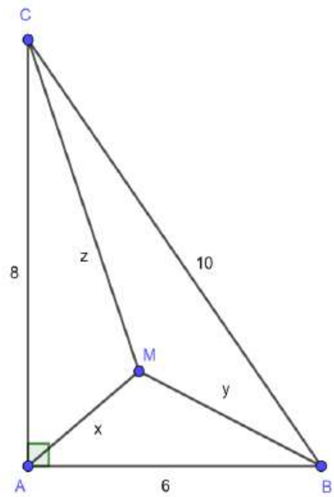
$$(2) \Leftrightarrow \frac{sinAsinBsinC}{cosAcosBcosC} = \frac{\frac{sr}{2R^2}}{\frac{s^2-4R^2-4Rr-r^2}{4R^2}} = \frac{2rs}{s^2-4R^2-4Rr-r^2}$$

From $s^2 \leq 4R^2 + 4Rr + 3r^2$ $(Gerretsen)$ *it follows that:*

$$s^2 - 4R^2 - 4Rr - r^2 \leq 2r^2 \Rightarrow \frac{2rs}{s^2-4R^2-4Rr-r^2} \geq \frac{s}{r} \geq 3\sqrt{3} \text{ true from}$$

$$s \geq 3\sqrt{3}\ (Mitrinovic) \Rightarrow (1) \text{ is true.}$$

G.017. Solution (George Florin Şerban)



Let be $\Delta ABC, \mu(\sphericalangle A) = 90^{\circ}, M \in Int(ABC), AB = 6$

$$, BC = 10, CA = 8$$

$$\mu(\sphericalangle AMB) = \mu(\sphericalangle BMC) = \mu(\sphericalangle CMA) = 120^{\circ}$$

$$AM = x, BM = y, CM = z$$

In ΔAMB *from cosines law:*

$$AB^2 = MA^2 + MB^2 - 2MA \cdot MBcos(\sphericalangle AMB) \Rightarrow$$

$$x^2 + y^2 + xy = 36$$

In ΔBMC *from cosines law:*

$$BC^2 = MB^2 + MC^2 - 2MB \cdot MCcos(\sphericalangle BMC) \Rightarrow$$

$$y^2 + z^2 + yz = 60$$

In ΔAMC *from cosines law:*

$$AC^2 = AM^2 + MC^2 - 2MA \cdot MCcos(\sphericalangle AMC) \Rightarrow$$

$$x^2 + z^2 + xz = 64$$

$$\sigma_{ABC} = \sigma_{AMB} + \sigma_{BMC} + \sigma_{CMA} \Rightarrow \frac{8 \cdot 6}{2} = \frac{xysin120^{\circ}}{2} + \frac{yxsin120^{\circ}}{2} + \frac{zxsin120^{\circ}}{2}$$

$$= xy + yz + zx = 32\sqrt{3}$$

$$a^2x + b^2y + c^2z = \frac{a^2}{\frac{1}{x}} + \frac{b^2}{\frac{1}{y}} + \frac{c^2}{\frac{1}{z}} \overset{Bergstrom}{\geq} \frac{(a+b+c)^2xyz}{xy+yz+zx} = \frac{64\sqrt{3}xyz}{32\sqrt{3}}$$

$$= 2xyz$$

G.018. Solution (George Florin Şerban)

$$tanx = a, tany = b, tanz = c; a, b, c > 0, abc = 1, a^2 + 1 = \frac{1}{cos^2x}$$

$$8 + (1 + a^3)(1 + b^3)(1 + c^3) \geq 2(a^2 + 1)(b^2 + 1)(c^2 + 1)$$

$$9+\sum_{cyc}a^3+\sum_{cyc}(ab)^3+1\ge 2+2\sum_{cyc}(ab)^2+2\sum_{cyc}a^2+2$$

$$\sum_{cyc}a^3+\sum_{cyc}\frac{1}{a^3}+6\ge 2\sum_{cyc}a^2+2\sum_{cyc}\frac{1}{a^2}$$

$$\sum_{cyc}\left(a^3+\frac{1}{a^3}+2-\frac{2}{a^2}-2a^2\right)\ge 0;(1)$$

$$a^3+\frac{1}{a^3}+2-\frac{2}{a^2}-2a^2\ge 0,\forall a>0$$

$$a^6-2a^5+2a^3-2a+1\ge 0\Leftrightarrow (a-1)^2\left[\left(a^2-\frac{1}{2}\right)^2+\frac{3}{4}\right]\ge 0$$

True from $(a-1)^2\ge 0$ and $\left(a^2-\frac{1}{2}\right)^2+\frac{3}{4}\ge\frac{3}{4}>0$

Hence,

$$\sum_{cyc}\left(a^3+\frac{1}{a^3}+2-\frac{2}{a^2}-2a^2\right)\ge 0$$

Therefore,

$$8+(1+tan^3x)(1+tan^3y)(1+tan^3z)\ge\frac{2}{cos^2xcos^2ycos^2z}$$

G.019. Solution (Carlos Eduardo Aguiar Paiva)

$$4sin\frac{\pi}{26}+4xsin\frac{3\pi}{26}+4sin\frac{9\pi}{26}=x+\sqrt{13}$$

$$\Leftrightarrow cos\frac{2\pi}{13}+cos\frac{6\pi}{13}-xcos\frac{8\pi}{13}=\frac{x+\sqrt{13}}{4}$$

If $m=\frac{\pi}{13}$ and $k=cos(2m)+cos(6m)+cos(8m),k>0$ then:

$$k^2=cos^2(2m)+cos^2(6m)+cos^2(8m)+$$

$$+2[cos(2m)cos(6m)+cos(2m)cos(8m)+cos(6m)cos(8m)]$$

If $n=2[cos(2m)cos(6m)+cos(2m)cos(8m)+cos(6m)cos(8m)]$, then:

$$n=cos(2m)+cos(4m)+cos(6m)+cos(8m)+cos(10m)+cos(14m)$$

But $cos(14m)=cos(12m)$, hence

$$n = cos(2m) + cos(4m) + cos(6m) + cos(8m) + cos(10m) + cos(12m)$$

$$n = \frac{cos\left(\frac{2m + 12m}{2}\right) sin\left(\frac{6(2m)}{2}\right)}{sin\left(\frac{2m}{2}\right)}$$

$$n = \frac{cos(7m)sin(6m)}{sin(m)} = \frac{2cos(7m)sin(6m)}{2sin(m)}$$

$$n = \frac{sin(13m) - sin(m)}{2sin(m)}; n = \frac{sin\pi - sin\frac{\pi}{13}}{2sin\frac{\pi}{13}}; n = -\frac{1}{2}$$

$$k^2 = \frac{cos(4m) + 1 + cos(12m) + 1 + cos(16m) + 1}{2} - \frac{1}{2}$$

$$2k^2 = cos(4m) + cos(12m) + cos(16m) + 2$$

But, $cos(16m) = cos(10m)$. Then:

$2k^2 = cos(4m) + cos(10m) + cos(12m)$, Adding $2k^2$ and k, we have:

$$2k^2 + k = cos(2m) + cos(4m) + cos(6m) + cos(8m) + cos(10m) + cos(12m) + 2$$

$$2k^2 + k = n + 2$$

$$2k^2k = -\frac{1}{2} + 2; 4k^2 + 2k - 3 = 0; k = \frac{-1 + \sqrt{13}}{4}$$

Therefore: $cos\frac{2\pi}{13} + cos\frac{6\pi}{13} - xcos\frac{8\pi}{13} = \frac{-1+\sqrt{13}}{4}$

$$x = -1$$

G.020. Solution (Abdul Hannan)

The function $f(t) \coloneqq t^x$ *is concave in the interval* $(0, \infty)$ *since*

$f''(t) = x(x-1)t^{x-2} \le 0$. *Therefore,*

$$a^x + b^x + c^x = f(a) + f(b) + f(c) \le 3f\left(\frac{a+b+c}{3}\right) = 3\left(\frac{a+b+c}{3}\right)^x$$

Similarly,

$$a^y + b^y + c^y \le 3\left(\frac{a+b+c}{3}\right)^y$$

$$a^z+b^z+c^z\le\ 3\left(\frac{a+b+c}{3}\right)^z$$

Hence,

$$(a^x+b^x+c^x)(a^y+b^y+c^y)(a^z+b^z+c^z)\le 3\left(\frac{a+b+c}{3}\right)^{x+y+z}$$

Let $x+y+z=3p$. *Since* $0\le x,y,z\le 1$, *we have,* $1\le p\le 1$.

So, it is enough to prove that:

$$27\left(\frac{a+b+c}{3}\right)^{3p}\le\frac{(a+b+c)^3}{\sqrt[3]{(abc)^{3-3p}}}$$

$\Leftrightarrow (27abc)^{1-p}\le(a+b+c)^{3(1-p)}$ *which is true by AM-GM.*

G.021. Solution (Abdul Hannan)

$$\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a'}+\sqrt{b'}+\sqrt{c'})}{\sqrt[6]{aa'bb'cc'}}\overset{BCS}{\le}\frac{\sqrt{3(a+b+c)}\sqrt{3(a'+b'+c')}}{\sqrt[6]{aa'bb'cc'}}=$$

$$=\frac{6\sqrt{ss'}}{\sqrt[6]{4RF\cdot 4R'F'}}$$

It is enough to prove that:

$$\frac{6\sqrt{ss'}}{\sqrt[6]{4RF\cdot 4R'F'}}\le\frac{2ss'}{\sqrt[3]{2RR'FF'}}\Leftrightarrow\frac{3}{\sqrt[6]{4RF\cdot 4R'F'}}\le\frac{\sqrt{ss'}}{\sqrt[3]{2RR'FF'}}$$

$$\Leftrightarrow\frac{3^6}{4RF\cdot 4R'F'}\le\frac{s^3s'^3}{4R^2R'^2F^2F'^2}\Leftrightarrow\frac{3^6}{4}\le\frac{s^3s'^3}{RR'FF'}$$

So, it is enough to prove that for any triangle: $\frac{3^3}{2}\le\frac{s^3}{RF}$

Indeed, we have

$$\frac{s^3}{RF}=\frac{s^3}{Rrs}=\frac{s^2}{Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2}{Rr}=\frac{16R-5r}{R}\overset{Euler}{\ge}\frac{16R-\frac{5R}{2}}{R}=\frac{27}{2}$$

G.022. Solution (Adrian Popa)

$$S=\sum_{0\le i<j\le 5}\frac{(-1)^{j-i}}{2}\binom{5}{i}\binom{5}{j}cos(2j-2i)x+63=$$

$$=-\frac{cos2x}{2}\cdot\left[\binom{0}{5}\binom{1}{5}+\binom{1}{5}\binom{2}{5}+\binom{2}{5}\binom{3}{5}+\binom{3}{5}\binom{4}{5}+\binom{4}{5}\binom{5}{5}\right]$$

$$+\frac{cos4x}{2}\cdot\left[\binom{0}{5}\binom{2}{5}+\binom{1}{5}\binom{3}{5}+\binom{2}{5}\binom{4}{5}+\binom{3}{5}\binom{5}{5}\right]$$

$$+\frac{cos6x}{2}\cdot\left[\binom{0}{5}\binom{4}{5}+\binom{1}{5}\binom{4}{5}+\binom{2}{5}\binom{5}{5}\right]$$

$$-\frac{cos8x}{2}\cdot\left[\binom{0}{5}\binom{4}{5}+\binom{1}{5}\binom{5}{5}\right]$$

$$-\frac{cos10x}{2}\cdot\binom{0}{5}\binom{5}{5}+63=$$

$$=-105cos2x+60cos4x-\frac{45}{2}cos6x+5cos8x-\frac{1}{2}cos10x+63=0$$

We observe that sum of coefficients – $105+60-\frac{45}{2}+5-\frac{1}{2}+63=0$*, hence*

$$cos2x=cos4x=cos6x=cos8x=cos10x=1$$

Therefore: $x\in\{2k\pi/k\in\mathbb{Z}\}$

G.023. Solution (George Florin Şerban)

$$\sum_{cyc}tanA=\prod_{cyc}tanA\overset{AGM}{\geq}3\sqrt[3]{\prod_{cyc}tanA}$$

$$\mu(\sphericalangle A)\geq 45^{\circ},\mu(\sphericalangle B)\geq 45^{\circ},\mu(\sphericalangle C)\geq 45^{\circ}\Rightarrow tanA,tanB,tanC\geq 1$$

$$\left(\prod_{cyc}tanA\right)^3\geq 27\prod_{cyc}tanA\Rightarrow\left(\prod_{cyc}tanA\right)^2\geq 27\Rightarrow\prod_{cyc}tanA\geq 3\sqrt{3}$$

Hence,

$$(tanA+tanB+tanC)^2\left(\frac{1}{cos^2A}+\frac{1}{cos^2B}+\frac{1}{cos^2C}-3\right)=$$

$$=\left(\prod_{cyc}tanA\right)^2\left(\sum_{cyc}\left(\frac{1}{cos^2A}-1\right)\right)=\left(\prod_{cyc}tanA\right)^2\left(\sum_{cyc}\frac{1-cos^2A}{cos^2A}\right)=$$

$$=\left(\prod_{cyc}tanA\right)^2\left(\sum_{cyc}\frac{sin^2A}{cos^2A}\right)=\left(\prod_{cyc}tanA\right)^2\left(\sum_{cyc}tan^2A\right)\overset{AGM}{\geq}$$

$$\overset{AGM}{\geq}\left(\prod_{cyc} tanA\right)^2 \cdot 3\sqrt[3]{\prod_{cyc} tan^2 A} \geq 27 \cdot 3\sqrt[3]{27} = 243$$

G.024. Solution (Khaled Abd Imouti)

$$4 - 2(cotx + cot^3 x) \geq 0 \Leftrightarrow 2 - (cotx + cot^3 x) \geq 0$$

$$\Leftrightarrow cotx + cot^3 x \leq 2 \Leftrightarrow \frac{cosx}{sin^3 x} \leq 2$$

Let $f: D - \{k\pi \mid k \in \mathbb{Z}\} \to \mathbb{R}, f(x) = cotx(1 + cot^2 x)$

$$f(-x) = -f(x), x \in [0, \pi]$$

$$\lim_{x\to 0} f(x) = +\infty;\ \lim_{x\to \pi} f(x) = -\infty$$

$$f'(x) = \frac{-sin^4 x - 3sin^2 x cos^2 x}{sin^6 x} < 0 \Rightarrow f \searrow [0, \pi]$$

Let $g(x) = 4 - 2(cotx + cot^3 x) \geq 0$

$$\left|\frac{tan^2 x - 1}{tan^4 x}\right| = -(1 + cot^2 x) + 2cot^3 x + 4 - 2cotx(1 + cot^2 x)$$

I. $cot^2 x - cot^4 x =$

$$-1 - cot^2 x + 2cot^3 x + 4 - 2cotx - 2cot^3 x \Leftrightarrow$$

$$-cot^4 x + 2cot^2 x + 2cotx - 3 = 0$$

$$cotx = y \Rightarrow -y^4 - 2y^2 - 2y + 3 = 0$$

$$\Leftrightarrow (y - 1)(y^3 + y^2 - y - 3) = 0$$

$$y - 1 = 0 \Rightarrow cotx = 1 \Rightarrow x = \frac{\pi}{4}$$

$$y^3 + y^2 - y - 3 = 0$$

Let $h(y) = y^3 + y^2 - y - 3$

$$h'(y) = 3y^2 + 2y - 1; h'(y) = 0 \Leftrightarrow y_1 = -1, y_2 = \frac{1}{3}$$

$$\Rightarrow y^3 + y^2 - y - 3 < 0, \forall x \leq 1$$

II. $cot^4 x - cot^2 x = -1 - cot^2 x + 2cot^3 x + 4 - 2cotx -$

$2cot^3 x \Leftrightarrow$

$$cot^4x + 2cotx - 3 = 0; y = cotx \Rightarrow$$

$$y^4 + 2y - 3 = 0 \Leftrightarrow (y-1)(y^3 + y^2 + y + 3) = 0$$

$$y = 1 \Rightarrow cotx = 1 \Rightarrow x = \frac{\pi}{4}, \qquad S = \left\{\frac{\pi}{4} + k\pi \mid k \in \mathbb{Z}\right\}$$

G.025. Solution (Santos Martins Junior)

We know that: $\frac{1}{cos^2m} = 1 + tan^2m$

Hence system becomes:

$$\begin{cases}(tan^2x + tan^2y)(1 + tan^2y) + tan^2y(1 + tan^2z) + tan^2z(1 + tan^2x) = 29\\(tan^2y + tan^2z)(1 + tan^2z) + tan^2z(1 + tan^2x) + tan^2x(1 + tan^2y) = 19\\(tan^2z + tan^2x)(1 + tan^2x) + tan^2x(1 + tan^2y) + tan^2y(1 + tan^2z) = 23\end{cases}$$

Let: $tan^2x = a; tan^2y = b; tan^2z = c$ *where* $a, b, c > 0$

$$\begin{cases}(a+b)(b+1) + b(c+1) + c(a+1) = 29; \ (1)\\(b+c)(c+1) + c(a+1) + a(b+1) = 19; \ (2)\\(c+a)(a+1) + a(b+1) + b(c+1) = 23; \ (3)\end{cases}$$

We can rewrite (1) as $(a+b)(b+1) + c(a+b) + (b+c) = 29 \Leftrightarrow$

$$(a+b+1)(b+c+1) = 30; \ (4)$$

We can rewrite (2) as $(b+c)(c+1) + a(b+c) + (c+a) = 19 \Leftrightarrow$

$$(b+c+1)(c+a+1) = 20$$

We can rewrite (3) as $(c+a+1)(a+b+1) = 24; \ (6)$

Doing $(6) \cdot \frac{(4)}{(5)}: (a+b+1)^2 = 36 \Leftrightarrow a+b+c = 6; \ (7)$

(7) in (6): $c+a+1 = 4; \ (8)$ *and (7) in (4):* $b+c+1 = 5; \ (9)$

Doing $(7) + (8) - (9): 2a + 1 = 6 \Leftrightarrow a = 2 \Rightarrow b = 3 \Rightarrow c = 1 \Rightarrow$

$$(a, b, c) = (tan^2x, tan^2y, tan^2z) = (2,3,1)$$

$$\Leftrightarrow (x, y, z) = (tan^{-1}\left(\pm\sqrt{2}; \pm\frac{\pi}{3} + k\pi; \pm\frac{\pi}{4} + k\pi\right)$$

G.026. Solution (Adrian Popa)

$$cos2x + \frac{sin^3x - cos^3x}{sin^3x + cos^3x} =$$

$$= (cosx - sinx)(cosx + sinx) + \frac{(sinx - cosx)(1 + sinxcosx)}{(sinx + cosx)(1 - sinxcosx)}$$

$$= \frac{sinx - cosx}{sinx + cosx}$$

$$(cosx - sinx)\left(cosx + sinx - \frac{1 + sinxcosx}{(cosx + sinx)(1 - sinxcosx)} + \frac{1}{sinx + cosx}\right) = 0$$

Case 1. $sinx = cosx \Rightarrow tanx = 1 \Rightarrow x = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}$

Case 2.

$$\frac{(1 + 2sinxcosx)(1 - sinxcosx) - 1 - sinxcosx + 1 - sinxcosx}{(sinx + cosx)(sinx - cosx)} = 0$$

$$sinxcosx = t \Rightarrow (1 + 2t)(-t) - 2t = 0 \Leftrightarrow 2t^2 + t - 1 = 0 \Rightarrow$$

$$t_1 = -1, t_2 = \frac{1}{2}$$

$sinxcosx = -1 \Rightarrow \frac{sin2x}{2} = -1 \Rightarrow sin2x = -2$ *(impossible).*

$$sinxcosx = \frac{1}{2} \Rightarrow sin2x = 1 \Rightarrow x = k\pi + \frac{\pi}{4}, k \in \mathbb{Z}.$$

G.027. Solution (Ravi Prakash)

Let: $\vec{x} = a\vec{m} + b\vec{n} + c\vec{p}; \vec{y} = d\vec{m} + e\vec{n} + f\vec{p}; \vec{z} = g\vec{m} + h\vec{n} + i\vec{p}$ then

$$|\vec{x}| = |\vec{y}| = |\vec{z}| = \sqrt[3]{2}$$

$\left|\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}\right| = \left|\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right|$ =volume of parallelepiped with edges $\vec{x}, \vec{y}, \vec{z}$,

hence: $|[\vec{x}\ \ \vec{y}\ \ \vec{z}]| \le |\vec{x}||\vec{y}||\vec{z}| = \sqrt[6]{8} = \sqrt{2}$

Therefore,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \cdot \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \le 2$$

G.028. Solution (Ravi Prakash)

$$\left(1 + \frac{sin^2x}{cos^2y}\right)^{cos^2y} \cdot \left(1 + \frac{cos^2x}{sin^2y}\right)^{sin^2y} =$$

$$=\left[\left(1+\frac{sin^2x}{cos^2y}\right)^{cos^2y}\cdot\left(1+\frac{cos^2x}{sin^2y}\right)^{sin^2y}\right]^{\frac{1}{cos^2y+sin^2y}}\overset{AM-GM}{\leq}$$

$$\leq\frac{1}{sin^2y+cos^2y}\left[cos^2y\left(1+\frac{sin^2x}{cos^2y}\right)+sin^2y(1+\frac{cos^2x}{sin^2y}\right]=2$$

Therefore,

$$\left(1+\frac{sin^2x}{cos^2y}\right)^{cos^2y}\cdot\left(1+\frac{cos^2x}{sin^2y}\right)^{sin^2y}\leq 2$$

G.029. Solution (George Florin Şerban)

$$x^3+y^3\geq xy(x+y),\forall x,y>0$$

$$(x+y)(x^2-xy+y^2)-xy(x+y)\geq 0$$

$$(x+y)(x^2-2xy+y^2)\geq 0\Leftrightarrow (x+y)(x-y)^2\geq 0,\forall x,y\geq 0 \text{ true.}$$

$$\sum_{cyc}\frac{1}{a^3+b^3+abc}\leq\sum_{cyc}\frac{1}{ab(a+b)+abc}=\sum_{cyc}\frac{c}{abc(a+b+c)}=$$

$$=\frac{(a+b+c)}{abc(a+b+c)}=\frac{1}{abc}=\frac{1}{4Rrs}\overset{(1)}{\leq}\frac{\sqrt{3}}{72r^3}$$

$$(1)\Leftrightarrow 4Rrs\sqrt{3}\geq 72r^3\Leftrightarrow s\geq\frac{72r^3}{4Rr\sqrt{3}}=\frac{6\sqrt{3}r^2}{R}\Leftrightarrow$$

$s\geq\frac{6\sqrt{3}r^2}{R}$ but $s\geq 3\sqrt{3}r(Mitrinovic)$

We must prove that: $s\geq 3\sqrt{3}r\geq\frac{6\sqrt{3}r^2}{R}$

$$\Leftrightarrow 3\sqrt{3}Rr\geq 6\sqrt{3}r^2\Rightarrow R\geq 2r\ (Euler).$$

G.030. Solution (Alex Szoros)

Lemma: In ΔABC *the following relationship holds:*

$$am_a\geq 2S+\frac{(b-c)^2}{2}$$

$$(am_a)^2\geq\left[2S+\frac{(b-c)^2}{2}\right]^2\Leftrightarrow a^2m_a^2\geq 4S^2+2S(b-c)^2+\frac{(b-c)^4}{4}$$

$$a^2(2b^2+2c^2-a^2)\geq 16S^2+8S(b-c)^2+(b-c)^4$$

$$2a^2b^2+2a^2c^2-a^4\geq 2\sum a^2b^2-\sum a^4+8S(b-c)^2+(b-c)^4$$

$$b^4-2b^2c^2+c^4\geq 8S(b-c)^2+(b-c)^4$$

$$(b^2-c^2)^2\geq (b-c)^2[8S+(b-c)^2]$$

$$(b-c)^2[(b+c)^2-8S-(b-c)^2]\geq 0$$

$$(b-c)^2(4bc-8S)\geq 0,\qquad (b-c)^2(bc-2S)\geq 0$$

$$(b-c)^2(bc-bc\sin A)\geq 0,\ (b-c)^2bc(1-\sin A)\geq 0 \text{ true.}$$

Therefore,

$$am_a+bm_b+cm_c\geq 6F+\frac{1}{2}((a-b)^2+(b-c)^2+(c-a)^2)$$

G.031. Solution (Adrian Popa)

$$x,y,z>0 \text{ and } 0<x+y+z<\frac{\pi^2}{2} \text{ then } \sqrt{x},\sqrt{y},\sqrt{z}\in\left(0,\frac{\pi}{2}\right)$$

Let be the function $f(x)=\cos\sqrt{x}$

$$f'(x)=-\frac{\sin\sqrt{x}}{2\sqrt{x}};f''(x)=\frac{(\tan\sqrt{x}-\sqrt{x})\cos\sqrt{x}}{4x}>0,$$

$$\forall\sqrt{x}\in\left(0,\frac{\pi}{2}\right),\tan\sqrt{x}-\sqrt{x}\geq 0$$

Applying Jensen Inequality, we get:

$$\frac{x\cos\sqrt{z}+y\cos\sqrt{x}+z\cos\sqrt{y}}{x+y+z}\geq \cos\sqrt{\frac{xy+yz+zx}{x+y+z}}$$

Therefore,

$$\frac{x\cos\sqrt{z}+y\cos\sqrt{x}+z\cos\sqrt{y}}{\cos\sqrt{\frac{xy+yz+zx}{x+y+z}}}\geq x+y+z$$

G.032. Solution (George Florin Şerban)

$$\frac{a^2}{8a^3+(a+b)b^2}\leq\frac{1}{5(a+b)}\Leftrightarrow 5a^3+5a^2b\leq 8a^3+ab^2+b^3$$

$$\Leftrightarrow 3a^3-5a^2b+ab^2+b^3\ge 0\Leftrightarrow 3\left(\frac{a}{b}\right)^3-5\left(\frac{a}{b}\right)^2+\frac{a}{b}+1\ge 0$$

Denote: $t=\frac{a}{b}>0$ *it follows that:*

$$3t^3-5t^2+t+1\ge 0\Leftrightarrow (t-1)^2(3t+1)\ge 0 \text{ true.}$$

Therefore,

$$\sum_{cyc}\frac{a^2}{8a^3+(a+b)b^2}\le\sum_{cyc}\frac{1}{5(a+b)}=\frac{1}{5}\sum_{cyc}\frac{1}{a+b}=\frac{1}{5}\cdot\frac{5s^2+r^2+4Rr}{2s(s^2+r^2+2Rr)}$$

$$=\frac{5s^2+r^2+4Rr}{10s(s^2+r^2+2Rr)}$$

G.033. Solution (Abdul Hannan)

Note that $3(ab+bc+ca)\le(a+b+c)^2=4s^2;\ (1)$

$$72\sum_{cyc}\left(\frac{1}{b+c}+\frac{2}{c+a}\right)\left(\frac{1}{c+a}+\frac{2}{a+b}\right)\left(\frac{1}{a+b}+\frac{2}{b+c}\right)=$$

$$=216\left(\frac{1}{b+c}+\frac{2}{c+a}\right)\left(\frac{1}{c+a}+\frac{2}{a+b}\right)\left(\frac{1}{a+b}+\frac{2}{b+c}\right)\overset{AGM}{\le}$$

$$\le\frac{216}{27}\left(\frac{1}{b+c}+\frac{2}{c+a}+\frac{1}{c+a}+\frac{2}{a+b}+\frac{1}{a+b}+\frac{2}{b+c}\right)^3=$$

$$=\frac{216}{27}\left(\frac{3}{b+c}+\frac{3}{c+a}+\frac{3}{a+b}\right)^3=216\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)^3\overset{CBS}{\le}$$

$$\le 216\left(\frac{1}{4}\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{4}\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)\right)^3=$$

$$=216\left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right)^3=216\left(\frac{ab+bc+ca}{2abc}\right)^3=216\left(\frac{ab+bc+ca}{8Rrs}\right)^3\overset{(1)}{\le}$$

$$\le 216\left(\frac{4s^2}{24Rrs}\right)^3=\left(\frac{s}{Rr}\right)^3$$

G.034. Solution (Adrian Popa)

$$\prod_{cyc}(2b^2+2c^2+a^2)\overset{Holder}{\ge}4(a^2+b^2+c^2)^3$$

$$s^3 r_a r_b r_c = s^3 \cdot \frac{S}{s-a} \cdot \frac{S}{s-b} \cdot \frac{S}{s-c} = s^4 S$$

$$a^2+b^2+c^2 > 4\sqrt{3}S \Rightarrow S < \frac{a^2+b^2+c^2}{4\sqrt{3}}$$

$$\Rightarrow\ 4096 s^3 r_a r_b r_c < 2^{12} \cdot \frac{a^2+b^2+c^2}{2^4} \cdot \frac{a^2+b^2+c^2}{2^2\sqrt{3}}$$

$$= \frac{2^6}{\sqrt{3}}(a+b+c)^4(a^2+b^2+c^2)$$

$$a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3} \Rightarrow (a^2+b^2+c^2)^2 \geq \frac{(a+b+c)^4}{9}$$

$$\Rightarrow\ 4096 s^3 r_a r_b r_c < \frac{2^6}{\sqrt{3}} \cdot 9(a^2+b^2+c^2)^3$$

We must show that:

$$81\sqrt{3} \cdot 4(a^2+b^2+c^2)^3 > \frac{2^6}{\sqrt{3}} \cdot 9(a^2+b^2+c^2)^3 \Leftrightarrow 3^3 > 2^4 \text{ −true.}$$

G.035. Solution (Șerban George Florin)

We denote $x=(b+c)^2, y=(a+c)^2, z=(a+b)^2$

$$\Rightarrow \frac{4yz+9xz+xy}{xyz} \cdot \frac{9yz+xz+4xy}{xyz} > \frac{49\cdot(x+y+z)}{xyz}$$

$$\Rightarrow 36y^2z^2 + 4xyz^2 + 16xy^2z + 81xyz^2 + 9x^2z^2 + 36x^2yz + 9xy^2z + x^2yz + 4x^2y^2 >$$

$$> 49x^2yz + 49xy^2z + 49xyz^2, 36y^2z^2 + 9x^2z^2 + 4x^2y^2 - 12x^2yz - 24xy^2z + 36xyz^2 > 0$$

$$(6yz+3xz-2xy)^2 > 0, \textit{ true if}$$

$6yz+3xz-2xy \neq 0$. *We prove that* $6yz+3xz-2xy > 0 |: xyz$

$$\frac{6}{x} + \frac{3}{y} > \frac{2}{z} \Rightarrow 6(a+c)^2(a+b)^2 + 3(b+c)^2(a+b)^2 > 2(b+c)^2(a+c)^2$$

$$6\left(a^2+\sum ab\right)^2 + 3\left(b^2+\sum ab\right)^2 > 2\left(c^2+\sum ab\right)^2, \sum ab = s$$

$6(a^2+s)^2+3(b^2+s)^2$

$> 2(c^2+s)^2, 6a^4+12a^2s+6s^2+3b^4+6b^2s+3s^2 >$

$> 2c^4+4c^2s+2s^2, (6a^4+3b^4+7s^2)+(12a^2+6b^2)s > 2c^4+4c^2s$

We will prove that: $(12a^2+6b^2)s > 4c^2s$ *and* $6a^4+3b^3+7s^2 > 2c^4$. *We prove that*

$(12a^2+6b^2)s > 4c^2s|: s \Rightarrow 12a^2+6b^2 > 4c^2$. $\Delta ABC \Rightarrow a+b > c \Rightarrow 2a+2b > 2c$

$\Rightarrow (2a+2b)^2 > (2c)^2, 4a^2+8ab+4b^2 > 4c^2$

$12a^2+6b^2 \geq 4a^2+8ab+4b^2 > 4c^2 \Rightarrow 8a^2-8ab+2b^2 \geq 0|: 2$

$4a^2-4ab+b^2 \geq 0, (2a-b)^2 \geq 0$, *true.*

We prove that: $6a^4+3b^4+7s^2 > 2c^4$

$s = ab+bc+ac = ab+c(a+b) > ab+c\cdot c = ab+c^2, a+b \geq c$

$\Rightarrow s > ab+c^2, 6a^4+3b^4+7s^2 > 6a^4+3b^4+7(ab+c^2)^2 =$

$= 6a^4+3b^4+7a^2b^2+14abc^2+7c^4 > 2c^4$, *true, because* $7c^4 > 2c^4$.

G.036.

$$f: [0,1] \to \mathbb{R}, f(x) = x\left(x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3\right) - log(x^2-x+1)$$

$$f(0) = f(1) = 0, f'(x) = \frac{(8x^3-19x^2+25x-5)(x-1)^2}{6(x^2-x+1)}$$

$$g: [0,1] \to \mathbb{R}, g(x) = 8x^3-19x^2+25x-5, g'(x) = 24x^2-38x+25 > 0$$

g pass from negative to positive values $(g(0) = -5, g(1) = 9)$ g *−increasing* $\Rightarrow$ *same thing its happened with* f'. *But* $f(0) = f(1) = 0$ *hence* $f(x) \leq 0$.

$$x\left(x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3\right) \leq log(x^2-x+1) \ (1)$$

$$h: [0,1] \to \mathbb{R}, h(x) = logx - x + 1 + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3$$

$$h'(x) = -\frac{(x-1)^3}{x}, \lim_{\substack{x\to 0\\ x>0}} h(x) = -\infty, h(1) = 0 \Rightarrow h(x) \leq 0$$

$$logx \leq x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3 \overset{(1)}{\leq} \frac{log(x^2-x+1)}{x}$$

$$xlogx \leq log(x^2-x+1) \Rightarrow logx^x \leq log(x^2-x+1)$$

$x^x \leq x^2-x+1, \forall x \in [0,1]$ *(PERFETTI'S INEQUALITY-2014)*

$$0 < x, y \leq \frac{\pi}{4} \Rightarrow 0 < tanx, tany \leq 1$$

By Perfetti's inequality:

$$(tanx)^{tanx} \leq tan^2x - tanx + 1 = \frac{1}{cos^2x} - \frac{sinx}{cosx} = \frac{1 - sinxcosx}{cos^2x}$$

$$(tany)^{tany} \leq tan^2y - tany + 1 = \frac{1}{cos^2y} - \frac{siny}{cosy} = \frac{1 - sinycosy}{cos^2y}$$

$$(tanx)^{tanx} \cdot (tany)^{tany} \leq \frac{1 - sinxcosx}{cos^2x} \cdot \frac{1 - sinycosy}{cos^2y}$$

$$cos^2x \cdot cos^2y \cdot (tanx)^{tanx} \cdot (tany)^{tany} \leq (1 - sinxcosx)(1 - sinycosy)$$

Equality holds for $tanx = tany = 1 \leftrightarrow x = y = \frac{\pi}{4}$

G.037. Solution (Florentin Vişescu)

$$15cosx \cdot cosy \cdot cosz + 4\sum_{cyc} cos5x \cdot cosy \cdot cosz = 0 \Leftrightarrow$$

$$\sum_{cyc}(5cosx \cdot cosy \cdot cosz + 4cos5x \cdot cosy \cdot cosz) = 0$$

$$\sum_{cyc} cosy \cdot cosz(5cosx + 4cos5x) = 0$$

$$\sum_{cyc} cosy \cdot cosz(5cosx + 64cos^5x - 80cos^3x + 20cosx) = 0$$

$$\sum_{cyc} cosx \cdot cosy \cdot cosz(64cos^4x - 80cos^2x + 25) = 0$$

$$cosx \cdot cosy \cdot cosz \sum_{cyc}(8cos^2x - 5)^2 = 0$$

For $-\frac{\pi}{2} < x, y, z < \frac{\pi}{2}$ we have: $cosx \cdot cosy \cdot cosz \neq 0$ then

$$\sum_{cyc}(8cos^2x - 5)^2 = 0$$

$$8cos^2x - 5 = 8cos^2y - 5 = 8cos^2z - 5 = 0;\ -\frac{\pi}{2} < x, y, z < \frac{\pi}{2}$$

$$cox = cosy = cosz = \sqrt{\frac{5}{8}}$$

SOLUTIONS

ANALYSIS

AN.001. Solution (Avishek Mitra)

$$\Leftrightarrow \frac{x+y}{2\sqrt{xy}}+\frac{2\sqrt{xy}}{(x+y)}=\frac{(x+y)^2+4xy}{2\sqrt{xy}(x+y)} \overset{AM-GM}{\leq} \frac{(x+y)^2+4xy}{4xy}$$

$$\Rightarrow Need\ to\ show$$

$$\Leftrightarrow \frac{(x+y)^2+4xy}{4xy}\leq \frac{x^2+y^2}{xy} \Rightarrow x^2+y^2+6xy\leq 4x^2+4y^2$$

$$\Rightarrow 3(x-y)^2\geq 0(*\ true) \Leftrightarrow \frac{x+y}{2\sqrt{xy}}+\frac{2\sqrt{xy}}{x+y}\leq \frac{x}{y}+\frac{y}{x}$$

$$\Rightarrow \frac{1}{2}\int_a^b\int_a^b \frac{(x+y)}{\sqrt{xy}}dxdy+2\int_a^b\int_a^b \frac{\sqrt{xy}}{(x+y)}dxdy\leq \int_a^b\int_a^b\left(\frac{x}{y}+\frac{y}{x}\right)dxdy$$

$$\Leftrightarrow \Omega\leq \int_a^b\int_a^b \frac{x}{y}dxdy+\int_a^b\int_a^b \frac{y}{x}dxdy=\left[\frac{x^2}{2}\right]_a^b[\log y]_a^b+\left[\frac{y^2}{2}\right]_a^b[\log x]_a^b$$

$$=2\times\frac{1}{2}\times(b^2-a^2)(\log b-\log a)=\log\left(\frac{b}{a}\right)^{(b^2-a^2)} \Leftrightarrow \Omega\leq \log\left(\frac{b}{a}\right)^{(b^2-a^2)}$$

AN.002. Solution (Ravi Prakash)

$$\Omega(a)=\int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx;\quad (1)$$

$$\Omega(a)=\int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx \overset{x=-t}{=}$$

$$=\int_{-a}^{a}\left(\frac{e^{3t^2}}{1+e^{-t}}-6te^{3t^2}log(1+e^{-t})\right)dt$$

$$= \int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x} - 6xe^{3x^2}(log(1+e^x)-x)\right)dx; \quad (2)$$

Adding (1),(2) we get:

$$2\Omega(a) = \int_{-a}^{a}(e^{3x^2}+6x^2e^{3x^2})dx = 2\int_{0}^{a}(e^{3x^2}+6xe^{3x^2}\cdot x)dx =$$

$$= 2\int_{0}^{a} e^{3x^2}dx + 2xe^{3x^2}\Big|_0^a = 2ae^{3a^2}$$

$$\Omega(a)+\Omega(b)+\Omega(c) \geq 3\sqrt[3]{abc}\cdot\sqrt[3]{e^{3a^2+3b^2+3c^2}} = 3e^{a^2+b^2+c^2}$$

AN.003. Solution (Ali Jaffal)

$$Let:\ I_n = \frac{1}{n^2}\sum_{k=1}^{n}\frac{k^3}{3k^2-3nk+n^2} = \frac{1}{n}\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)^3}{3\left(\frac{k}{n}\right)^2-3\left(\frac{k}{n}\right)+1}$$

$$So,\ \lim_{n\to\infty} I_n = \int_0^1 \frac{x^3}{3x^2-3x+1}dx = I > 0$$

$$Let:\ J_n = \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2} = \frac{1}{n}\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)^2}{2\left(\frac{k}{n}\right)^2-2\left(\frac{k}{n}\right)+1}$$

$$\lim_{n\to\infty} J_n = \int_0^1 \frac{x^2}{2x^2-2x+1}dx = J > 0$$

Let:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\left(\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}\right)\left(\sum_{k=1}^{n}\frac{k^3}{3k^2-3nk+n^2}\right)}$$

$$= \sqrt[n]{n^2 I_n\cdot nJ_n} = \sqrt[n]{n^3}\cdot\sqrt[n]{I_n\cdot J_n}$$

$$\lim_{n\to\infty}\sqrt[n]{n^3} = \lim_{n\to\infty} e^{\frac{1}{n}log(n^3)} = e^{\lim_{n\to\infty}\frac{3logn}{n}} = 1$$

$$\lim_{n\to\infty}\sqrt[n]{I_n\cdot J_n} = \lim_{n\to\infty} e^{\frac{1}{n}log(I_nJ_n)} = e^{log(I\cdot J)\cdot\lim_{n\to\infty}\frac{1}{n}} = 1$$

Then: $\Omega = 1.$

AN.004. Solution (Remus Florin Stanca)

$x \to e^{x^2}$ continuous function, because $f(x) = e^x, g(x) = x^2$ are both continuous as elementary functions.

$$\int_{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}^{\frac{\pi^2}{6}} e^{x^2}dx = \left(\frac{\pi^2}{6} - \frac{1}{1^2} - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)e^{c_n^2}$$

and $c_n \in \left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}, \frac{\pi^2}{6}\right]$ then:

$$\Omega = \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n} \cdot ne^{c_n^2} \cdot \left(\frac{\pi^2}{6} - \frac{1}{1^2} - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)\right) \overset{Stolz-Cesaro}{=}$$

$$= \frac{1}{e} \cdot \lim_{n\to\infty} \frac{-\frac{1}{(n+1)^2}}{-\frac{1}{n(n+1)}} \cdot e^{\left(\frac{\pi^2}{6}\right)^2} = e^{\frac{\pi^4-36}{36}}$$

AN.005. Solution (Tran Hong)

Because: $a + b + c = 9$, inequality becomes as:

$$\int_0^3 e^{x^2}dx + \frac{1}{a}\int_0^a e^{x^2}dx + \frac{1}{b}\int_0^b e^{x^2}dx + \frac{1}{c}\int_0^c e^{x^2}dx \overset{(*)}{\geq}$$

$$\geq 4\left(\frac{1}{b+c}\int_0^{\sqrt{bc}} e^{x^2}\,dx + \frac{1}{a+b}\int_0^{\sqrt{ab}} e^{x^2}\,dx + \frac{1}{a+c}\int_0^{\sqrt{ac}} e^{x^2}\,dx\right)$$

Let: $\varphi(x) = \frac{1}{x}\int_0^x e^{t^2}dt$; $\forall x > 0$

$$\varphi'(x) = -\frac{1}{x^2}\int_0^x e^{t^2}dt + \frac{e^{x^2}}{x} = \frac{1}{x}\left(e^{x^2} - \frac{1}{x}\int_0^x e^{t^2}dt\right)$$

$$\varphi''(x) = -\frac{1}{x^2}\left(e^x - \frac{1}{x}\int_0^x e^{t^2}dt\right) + \frac{1}{x}\left(2xe^{x^2} + \frac{1}{x^2}\int_0^x e^{t^2}dt - \frac{e^{x^2}}{x}\right)$$

$$= 2e^{x^2} + \frac{2}{x^3}\int_0^x e^{t^2}dt > 0, \forall x > 0$$

By Popoviciu's inequality:

$$\varphi(a) + \varphi(b) + \varphi(c) + 3\varphi\left(\frac{a+b+c}{3}\right) \geq 2\varphi\left(\frac{a+b}{2}\right) + 2\varphi\left(\frac{b+c}{2}\right) + 2\varphi\left(\frac{c+a}{2}\right)$$

$$\Leftrightarrow (a)+\varphi(b)+\varphi(c)+3\varphi(3)\geq 2\varphi\left(\frac{a+b}{2}\right)+2\varphi\left(\frac{b+c}{2}\right)+2\varphi\left(\frac{c+a}{2}\right)$$

$$\frac{1}{a}\int_0^a e^{x^2}dx+\frac{1}{b}\int_0^b e^{x^2}dx+\frac{1}{c}\int_0^c e^{x^2}dx+\int_0^3 e^{x^2}dx\geq$$

$$\geq 2\left(\frac{2}{a+b}\int_0^{\frac{a+b}{2}} e^{x^2}dx+\frac{2}{b+c}\int_0^{\frac{b+c}{2}} e^{x^2}dx+\frac{2}{c+a}\int_0^{\frac{c+a}{2}} e^{x^2}dx\right)$$

$$=4\left(\frac{1}{a+b}\int_0^{\frac{a+b}{2}} e^{x^2}dx+\frac{1}{b+c}\int_0^{\frac{b+c}{2}} e^{x^2}dx+\frac{1}{c+a}\int_0^{\frac{c+a}{2}} e^{x^2}dx\right)\overset{Am-Gm}{\geq}$$

$$\geq 4\left(\frac{1}{b+c}\int_0^{\sqrt{bc}} e^{x^2}\,dx+\frac{1}{a+b}\int_0^{\sqrt{ab}} e^{x^2}\,dx+\frac{1}{a+c}\int_0^{\sqrt{ac}} e^{x^2}\,dx\right)$$

AN.006. Solution (Abner Chinga Bazo)

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{sin2x}dx=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{\frac{\pi}{2}-x}{sin2\left(\frac{\pi}{2}-x\right)}dx=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{\frac{\pi}{2}-x}{sin2x}dx$$

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{sin2x}dx=\frac{\pi}{2}\left(\frac{1}{2}log(tanx)\right)\Bigg|_{\frac{\pi}{5}}^{\frac{3\pi}{10}}$$

$$=\frac{\pi}{8}\left(log\left(tan\left(\frac{3\pi}{10}\right)\right)-log\left(tan\left(\frac{\pi}{5}\right)\right)\right)=\frac{\pi}{8}log\left(\frac{log\left(tan\left(\frac{3\pi}{10}\right)\right)}{log\left(tan\left(\frac{\pi}{5}\right)\right)}\right)$$

AN.007. Solution (Remus Florin Stanca)

$$\Omega=\lim_{n\to\infty}\left(n\cdot\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}}\frac{\sqrt[n]{e^x}}{x}dx\right)=\lim_{n\to\infty}n\cdot\left|\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right|\cdot\frac{\sqrt[n]{e^{c_n}}}{c_n};\ \ (1)$$

Where c_n *is between* $\frac{n^2}{\sqrt[n]{n!}}$ *and* $\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}$ *then* $\frac{n}{c_n}$ *is between*

$$\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^2}\cdot n \text{ and } \frac{\sqrt[n]{n!}}{n} \quad (2)$$

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^2}\cdot n=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}$$

$$=\frac{1}{e};\ \ (3)\overset{(2),(3)}{\Longrightarrow}\lim_{n\to\infty}\frac{n}{c_n}=\frac{1}{e};\ \ (4)$$

$$\lim_{n\to\infty}\sqrt[n]{e^{c_n}}=\lim_{n\to\infty}e^{\frac{c_n}{n}}=e^e;(5)$$

$$\overset{(1),(4),(5)}{\Longrightarrow}\Omega=e^{e-1}\cdot\left|\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)\right|$$

$$=e^{e-1}\cdot\left|\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{n!}}\cdot\left(\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}-1\right)\right|$$

$$=e^{e-1}\cdot e\cdot\left|\lim_{n\to\infty}n\cdot\log\left(\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)\right|$$

$$=e^e\cdot\left|\lim_{n\to\infty}\log\left(\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{n!}{(n+1)!}\cdot\sqrt[n+1]{(n+1)!}\right)\right|=e^e\cdot\left|\log\left(e^2\cdot\frac{1}{e}\right)\right|$$

$$=e^e\Rightarrow\Omega=e^e.$$

AN.008. Solution (Nassim Nicholas Taleb)

We must prove that:

$$\int_a^1 x^x dx\ge\log(2-a)=\int_a^1\frac{1}{2-x}dx$$

It is sufficient to prove dominance of integrands, that is $x^x\ge\frac{1}{2-x}$ *for all* $0\le x\le 1,\frac{1-2x^x}{-2+x}\ge 0$ *then* $\frac{1+x^{1+x}}{x^x}\le 2\Leftrightarrow x+x^{-x}\le 2$

$$\int_a^1 x^x dx+\int_b^1 x^x dx+\int_c^1 x^x dx\ge\log\big((2-a)(2-b)(2-c)\big)$$

AN.009. Solution (Naren Bhandari)

Here $\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}=\sum_{0\le i\le n}\binom{n}{i}^2+2\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}$ *givin us*

$$2\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}=\frac{3}{2}\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}-\sum_{0\le i\le n}\binom{n}{i}^2$$

Further

$$\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}=\left(\sum_{0\le i\le n}\binom{n}{i}\right)\left(\sum_{0\le j\le n}\binom{n}{j}\right)=(1+1)^n(1+1)^n=4^n$$

Therefore we can write:

$$\Omega=\lim_{n\to\infty}\left(\frac{3\cdot 4^n}{2}-\sum_{0\le i\le n}\binom{n}{i}^2\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{3\cdot 4^n}{2}-\binom{2n}{n}\right)^{\frac{1}{n}}$$

We use asymptotic expansion for central binomial coefficients ie

$\binom{2n}{n}\sim\frac{4^n}{\sqrt{\pi n}}$ *giving us*

$$\Omega\sim\lim_{n\to\infty}\left(\frac{3\cdot 4^n}{2}-\frac{4^n}{\sqrt{\pi n}}\right)^{\frac{1}{n}}=4\cdot\lim_{n\to\infty}\left(1+\left(\frac{1}{2}-\frac{1}{\sqrt{\pi n}}\right)\right)^{\frac{1}{n}}$$

$$=4\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\left(\frac{1}{2}-\frac{1}{\sqrt{\pi n}}\right)+O\left(\frac{1}{n^2}\right)\right)=4$$

By Vandermonde's identity

$$\binom{m+n}{k}=\sum_{k=0}^{n}\binom{m}{k}\binom{n}{n-k}\Rightarrow\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2,m=n=k$$

AN.010. Solution (Precious Itsuokor)

$$4(\sqrt{a}-1)^2\le 4\left(\frac{a}{2}-1+\frac{1}{2}\right)^2=(a-1)^2$$

$$4\left(\sqrt{a}-1\right)^2+\left(\int_1^a\sqrt{1-\frac{1}{x}}dx\right)^2$$

$$\overset{C-B-S}{\le}4(\sqrt{a}-1)^2+\left(\int_1^a\left(1-\frac{1}{x}\right)dx\right)\left(\int_1^a dx\right)$$

$$\leq 4\left(\sqrt{a}-1\right)^2+(x-logx)|_1^a(a-1)$$

$$\overset{Am-Gm}{\leq} 4\left(\frac{a}{2}-\frac{1}{2}\right)^2+(a-loga-1)(a-1)$$

$$=(a-1)^2+\left((a-1)-loga\right)(a-1)$$

$$=(a-1)^2+(a-1)^2-(a-1)loga$$

Since $e^a \geq 1+a \Rightarrow a \geq log(1+a) \Rightarrow loga \leq a-1 \Rightarrow$

$$(a-1)^2+(a-1)^2-(a-1)loga \leq 2(a-1)^2-(a-1)^2=(a-1)^2$$

$$\Rightarrow 4(\sqrt{a}-1)^2+\left(\int_1^a \sqrt{1-\frac{1}{x}dx}\right)^2 \leq (a-1)^2$$

AN.011. Solution (Sanong Huayrerai)

For $x,y>0$ and $0<a\leq b$ we have:

$$\sqrt{\left(1+\frac{1}{x^4}\right)\left(1+\frac{1}{y^4}\right)} \geq \sqrt{\left(1+\frac{1}{(xy)^2}\right)^2}=1+\frac{1}{(xy)^2} \Rightarrow$$

$$\int_a^b\int_a^b \sqrt{\left(1+\frac{1}{x^4}\right)\left(1+\frac{1}{y^4}\right)}dxdy \geq \int_a^b\int_a^b \left(1+\frac{1}{(xy)^2}\right)dxdy=\left(xy+\frac{1}{xy}\right)\Big|_a^b$$

$$=(b-a)(b-a)+\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right)=(b-a)^2+\frac{(b-a)^2}{(ab)^2} \geq \frac{2(b-a)^2}{ab}$$

AN.012. Solution (Naren Bhandari)

Inductively we show that $1 \leq k^4-k^2+1 \leq k^4, \forall k \geq 1$

We have: $\frac{1}{k^4} \leq \frac{1}{k^4-k^2+1} \leq 1$ and we have:

$$\lim_{n\to\infty}\sum_{1\leq k\leq n}\frac{n^2}{k^4} \leq \Omega \leq \lim_{n\to\infty}\sum_{1\leq k\leq n} n^2=\lim_{n\to\infty} n^3$$

Clearly $\lim_{n\to\infty} n^3=\infty$ and

$$\lim_{n\to\infty}\sum_{1\leq k\leq n}\frac{n^2}{k^4}=\lim_{n\to\infty} n^2H_n^{(4)}=\lim_{n\to\infty}\zeta(4)n^2-0=\infty$$

Thus by Squeeze theorem $\Omega = \infty$

AN.013. Solution (Ravi Prakash)

Let $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{4}(I_2 + B)$, where $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$B^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}, \forall n \in \mathbb{N}, n \ge 1$$

$$A^n = \frac{1}{4^n} \cdot (I_2 + B)^n = \frac{1}{4^n} \cdot (I_2 + \binom{n}{1} B + \binom{n}{2} B^2 + \cdots + \binom{n}{n} B^n =$$

$$= \frac{1}{2^{2n}} \begin{pmatrix} \sum_{r=1}^{n} \binom{n}{r} 2^{r-1} + 1 & \sum_{r=1}^{n} \binom{n}{r} 2^{r-1} \\ \sum_{r=1}^{n} \binom{n}{r} 2^{r-1} & \sum_{r=1}^{n} \binom{n}{r} 2^{r-1} + 1 \end{pmatrix}$$

$$= \frac{1}{2^{2n}} \begin{pmatrix} \frac{1}{2}(3^n + 1) & \frac{1}{2}(3^n - 1) \\ \frac{1}{2}(3^n - 1) & \frac{1}{2}(3^n + 1) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \cdot \left(\frac{3}{4}\right)^n + \frac{1}{2} \cdot \left(\frac{1}{4}\right)^n & \frac{1}{2} \cdot \left(\frac{3}{4}\right)^n + \frac{1}{2} \cdot \left(\frac{1}{4}\right)^n \\ \frac{1}{2} \cdot \left(\frac{3}{4}\right)^n - \frac{1}{2} \cdot \left(\frac{1}{4}\right)^n & \frac{1}{2} \cdot \left(\frac{3}{4}\right)^n + \frac{1}{2} \cdot \left(\frac{1}{4}\right)^n \end{pmatrix} \to$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot A^n = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Where

$$2a = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{3}{4}\right)^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{1}{4}\right)^n =$$

$$= log\left(1 + \frac{3}{4}\right) + log\left(1 + \frac{1}{4}\right) = log\left(\frac{35}{16}\right)$$

$$2b = log\left(1 + \frac{3}{4}\right) - log\left(1 + \frac{1}{4}\right) = log\left(\frac{7}{5}\right)$$

AN.014. Solution (Marian Ursărescu and Florică Anastase)

We show that: $F_{2n+2}=F_{n+1}^2-F_n^2, \forall n\in\mathbb{N}^*; (1)$

Let: $r_1=\frac{1+\sqrt{5}}{2};\ r_2=\frac{1-\sqrt{5}}{2}\rightarrow$

$$F_{n+1}^2-F_n^2=\frac{1}{5}(r_1^{n+2}-r_2^{n+2})^2-\frac{1}{5}(r_1^n-r_2^n)^2=$$

$$=\frac{1}{5}\left[r_1^{2n+2}\left(r_1^2-\frac{1}{r_1^2}\right)+r_2^{2n+2}\left(r_2^2-\frac{1}{r_2^2}\right)+2(r_1r_2)^n(1-(r_1r_2)^2)\right]=$$

$$=\frac{1}{5}[r_1^{2n+2}(r_1^2-r_2^2)+r_2^{2n+2}(r_1^2-r_2^2)]=$$

$$=\frac{1}{5}(r_1-r_2)(r_1+r_2)(r_1^{2n+2}-r_2^{2n+2})=\frac{\sqrt{5}}{5}(r_1^{2n+2}-r_2^{2n+2})=F_{2n+2}$$

We show that: $sin(F_{2n+2})+sin(F_n^2)+cos(F_{n+2}^2)=$

$$=sin(F_{n+2}^2-F_n^2)+sin(F_n^2)+cos(F_{n+2}^2)\le\frac{3}{2};(2)$$

Let: $x=F_{n+2}^2; y=F_n^2\overset{(2)}{\rightarrow}sin(x-y)+siny+cosx\le\frac{3}{2}$

$$3-2sin(x-y)-2siny-2cosx\ge 0$$

$$3-2(sin(x-y)+siny)-2cos\left(2\cdot\frac{x}{2}\right)\ge 0$$

$$3-4sin\frac{x}{2}cos\left(\frac{x-2y}{2}\right)-2\left(1-2sin^2\frac{x}{2}\right)\ge 0$$

$$4sin^2\frac{x}{2}-4sin\frac{x}{2}cos\left(\frac{x-2y}{2}\right)+1\ge 0$$

$$4sin^2\frac{x}{2}-4sin\frac{x}{2}cos\left(\frac{x-2y}{2}\right)+cos^2\left(\frac{x-2y}{2}\right)+sin^2\left(\frac{x-2y}{2}\right)\ge 0$$

$$\left[2sin\frac{x}{2}-cos\left(\frac{x-2y}{2}\right)\right]^2+sin^2\left(\frac{x-2y}{2}\right)\ge 0 \textit{ true.}$$

$$\frac{a^4}{sin^3(F_{2n+2})}+\frac{b^4}{sin^3(F_n^2)}+\frac{c^4}{cos^3(F_{n+2}^2)}\ge$$

$$\overset{Radon}{\ge}\frac{(a+b+c)^4}{\left(sin(F_{2n+2})+sin(F_n^2)+cos(F_{n+2}^2)\right)^3}\overset{(2)}{\ge}\frac{9^4}{\left(\frac{3}{2}\right)^3}=72$$

AN.015. Solution (Samir HajAli)

$$\Omega_n = \prod_{k=1}^{n}\left(\frac{k^{\frac{1}{k+1}}}{2}\right) \leq \prod_{k=1}^{n}\frac{n^{\frac{1}{k+1}}}{2} = \frac{n^{H_{n+1}-1}}{2^n}$$

$$\Omega \leq \lim_{n\to\infty}\frac{n^{H_{n+1}-1}}{2^n} = \lim_{n\to\infty}\frac{n^{log(n+1)+\gamma+\delta_n-1}}{2^n} =$$

$$\lim_{n\to\infty}\frac{n^{\gamma+\delta_n-1}}{2^n}\cdot\lim_{n\to\infty}\frac{n^{log(n+1)}}{2^n} = 0$$

Where, $\lim_{n\to\infty}\frac{n^{log(n+1)}}{2^n} = \lim_{n\to\infty}exp(log\left(2^{-n}n^{log(n+1)}\right) =$

$$= \lim_{n\to\infty}exp(-nlog2 + log(n+1)logn) =$$

$$= \lim_{n\to\infty}exp\left(-nlog2 - \frac{log(n+1)logn}{n}\right) = 0$$

$$0 \leq \Omega \leq 0$$

AN.016. Solution (Adrian Popa)

$$\Omega = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1 + \underbrace{\frac{e-1}{n}log\left(1+\frac{(e-1)k}{n}\right)}_{=a_{n,k}}\right)\right) = e^{\lim_{n\to\infty}\sum_{k=1}^{n}log\left(1+a_{n,k}\right)} =$$

$$= e^{\lim_{n\to\infty}\sum_{k=1}^{n}(a_{n,k})}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{e-1}{n}log(1+\frac{(e-1)k}{n}\right) =$$

$$= \lim_{n\to\infty}(e-1)\cdot\frac{1}{n}\sum_{k=1}^{n}log\left(1+(e-1)\cdot\frac{k}{n}\right) =$$

$$= (e-1)\int_0^1 log(1+(e-1)x)dx = (e-1)I$$

$$\begin{cases} f(x) = log(1+(e-1)x) \\ g'(x) = 1 \end{cases} \rightarrow \begin{cases} f'(x) = \frac{e-1}{1+(e-1)x} \\ g(x) = x \end{cases}$$

$$I=x\log(1+(e-1)x)\Big|_0^1-\int_0^1\frac{(e-1)x}{1+(e-1)x}dx=$$

$$=1-\int_0^1\frac{(e-1)x+1-1}{1+(e-1)x}dx=\frac{1}{e-1}\left(\log\frac{e}{e-1}-\log\frac{1}{e-1}\right)=\frac{1}{e-1}\to$$

$$\Omega=e^{(e-1)\cdot\frac{1}{e-1}}=e$$

AN.017. Solution (Paulo Sergio Lino)

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+3)(2n+5)}=$$

$$\Omega=4\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+2)(n+2)(2n+3)(2n+5)}=$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\frac{(-1)^n}{\binom{2n+5}{4}}=\frac{2}{3}\sum_{n=1}^{\infty}(-1)^n\frac{3!\,(2n+1)!}{(2n+5)!}=\frac{2}{3}\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(4)\Gamma(2n+2)}{\Gamma(2n+6)}=$$

$$=\frac{2}{3}\sum_{n=1}^{\infty}(-1)^nB(2n+2,4)=\frac{2}{3}\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n+1}(1-x)^3dx=$$

$$=\frac{2}{3}\int_0^1 x(1-x)^3\sum_{n=1}^{\infty}(-x^2)^ndx=-\frac{2}{3}\int_0^1\frac{x^3(1-x)^3}{1+x^2}dx=$$

$$=\frac{2}{3}\int_0^1\frac{x^6-3x^5+3x^4-x^3}{1+x^2}dx=$$

$$=\frac{2}{3}\int_0^1(x^4-3x^3+2x^2+2x-2)dx+\frac{4}{3}\int_0^1\frac{1}{1+x^2}dx-\frac{4}{3}\int_0^1\frac{x}{1+x^2}dx=$$

$$=-\frac{53}{90}+\frac{\pi}{3}-\frac{2\log 2}{3}$$

AN.018. Solution (Tran Hong)

For $a,b,c,d\ge 1\Rightarrow x=\log a\ge 1, y=\log b\ge 1, z=\log c\ge 1,$

$t=\log d\ge 1$. *We have:*

$$5log(ae)\cdot log(be)\cdot log(ce)\cdot log(de)\geq log(abcde)^{16}\Leftrightarrow$$
$$5[(loga+1)(logb+1)(logc+1)(logd+1)]\geq$$
$$\geq 16(loga+logb+logc+logd+1)$$
$$5(x+1)(y+1)(z+1)(t+1)\geq 16(x+y+z+t+1)\Leftrightarrow$$
$$5(xyzt+xyz+yzt+xyt+xzt+xy+yz+zt+xz+ty+xt)\overset{(*)}{\geq}$$
$$\geq 11(x+y+z+t)$$

Because: $x,y,z,t\geq 1\Rightarrow (x-1)(y-1)(z-1)(t-1)\geq 0\Leftrightarrow$

$$xyzt\geq xyz+xzt+yzt+xyt-xy-yz-zt-xz-ty-xt\ +x+y+z$$
$$+t-1$$
$$5(xyzt+xyz+yzt+xyt+xzt+xy+yz+zt+xz+ty+xt)\geq$$
$$\geq 5[2(xyz+yzt+xyt+xzt)+x+y+z+t-1]$$

So,we need to prove:

$$10(xyz+yzt+xyt+xzt)\geq 6(x+y+z+t)+16$$

But: $xyz+yzt+xyt+xzt\overset{x,y,z,t\geq 1}{\geq} x\cdot 1\cdot 1+y\cdot 1\cdot 1+z\cdot 1\cdot 1+t\cdot 1\cdot 1=$
$=x+y+z+t\Rightarrow$

$$10(xyz+yzt+xyt+xzt)\geq 10(x+y+z+t)\overset{(*)}{\geq}$$
$$\geq 6(x+y+z+t)+16$$
$$(*)\Leftrightarrow 4(x+y+z+t)\geq 16\Leftrightarrow x+y+z+t\geq 4$$

Which is true,because: $x,y,z,t\geq 1\Rightarrow x+y+z+t\geq 4\Rightarrow (*)$ *is*

AN.019.Solution (Adrian Popa)

Let: $f(x)=(1+x)(1-x)^{1-x};x\in(0,1]$

$$\Rightarrow logf(x)=log(1+x)+(1-x)log(1-x)=g(x)$$
$$g'(x)=\frac{1}{x+1}-log(1-x)-1$$
$$g(x)=\frac{x^2+3x}{(x+1)^2(1-x)};g''(x)=0\Leftrightarrow x=0,(x\neq -3)$$

x	0 1
$g''(x)$	+ + + + + + + + + + + + + + +
$g'(x)$	0 +↗ + ↗+ ↗+ ↗+
$g(x)$	0 ↗ ↗ ↗ ↗ ↗ ↗ ↗ ↗

So, $g(x) > 0, \forall x > 0 \Rightarrow log f(x) > 0 \Rightarrow f(x) > 1$

$$\Rightarrow (1+x)(1-x)^{1-x} > 1; \forall x \in (0,1]$$

If $x = cos^2 a \Rightarrow (1+cos^2 a)(1-cos^2 a)^{1-cos^2 a} > 1$

$\Rightarrow (1+\cos^2 a)(\sin^2 a)^{\sin^2 a} > 1 \Rightarrow (1+\cos^2 a)(\sin a)^{2\sin^2 a} > 1$ and analogs.Then:

$$(1+cos^2 a)(1+cos^2 b)(1+cos^2 c)(sina)^{2sin^2 a}(sinb)^{2sin^2 b}(sinc)^{2sin^2 c} \geq 1$$

AN.020. Solution (Avishek Mitra)

$$\text{erf}(kx) = \frac{2}{\sqrt{\pi}}\int_0^{kx} e^{-t^2}\,dt \Rightarrow erf'(kx) = \frac{2k}{\sqrt{\pi}} e^{-k^2x^2} > 0 \ \ \textit{[for all } x \in \mathbb{R}\textit{]}$$

$$\Leftrightarrow erf'(kx) > 0 \ in\ x \in [a,b] - \{0\}$$

$\Leftrightarrow$ *Also, we know for* $x_1 < x_2 \Rightarrow \text{erf}(kx_1) < \text{erf}(kx_2) \Rightarrow \therefore$ *for any* a_1, b_1 *in the interval*

$[a,b] - \{0\}$ *if* $a_1 \leq b_1, \text{erf}(ka_1) < \text{erf}(kb_1)\,; [k = \overline{1,n}]$

$\Leftrightarrow f(x) = \text{erf}(kx)$ *is monotonically increasing in* $[a,b] - \{0\}$

$$\Leftrightarrow also\ I = \int_a^b \text{erf}(kx)\,dx = \frac{1}{k}\int_a^b \text{erf}(z)\,dz = \frac{1}{k}\left[z\,\text{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}\right]_a^b$$

$$= \frac{1}{k}\left[b\,\text{erf}(b) - a\,\text{erf}(a) + \frac{e^{-b^2} - e^{-a^2}}{\sqrt{\pi}}\right]$$

$\Leftrightarrow f(x) = \text{erf}(kx)$ *is the integrable between* $[a,b] - \{0\}$

$$\Leftrightarrow (b-a)^{n-1}\int_a^b f_1(x)\cdot f_2(x)\cdot \ldots \cdot f_n(x)dx \overset{Chebyshev}{\geq} \int_a^b f_1(x)\,dx \cdot \int_a^b f_2(x)\,dx \ldots \int_a^b f_n(x)\,dx$$

[Let us denote $f_1(x) = \text{erf}(x)\,, f_2(x) = \text{erf}(2x) \ldots f_n(x) = \text{erf}_n(x)$*]*

$$\Leftrightarrow (b-a)^{n-1}\int_a^b \text{erf}(x)\,\text{erf}(2x)\ldots\text{erf}(nx)\,dx$$

$$\geq \int_a^b \text{erf}(x)\,dx \int_a^b \text{erf}(2x)\,dx \ldots \int_a^b \text{erf}(nx)\,dx$$

$$\Leftrightarrow (b-a)^{n-1}\int_a^b \prod_{k=1}^{n} \text{erf}(kx)\,dx \geq \prod_{k=1}^{n}\int_a^b \text{erf}(kx)\,dx$$

AN.021. Solution (Gilmer Lopez)

$$\Omega = \int (10tan^3x + 7tan^2x + 12tanx + 9)e^x dx =$$

$$= \int (10tanx(tan^2x + 1) + 7sec^2x - 7 + 2tanx + 9)e^x dx =$$

$$= \int (10tanxsec^2x + 7sec^2x + 2tanx + 2)e^x dx =$$

$$= 10 \underbrace{\int e^x tanxsec^2xdx}_{\substack{u=e^x, du=e^x dx \\ dv=tanxsec^2xdx, v=\frac{1}{2}tan^2x}} + 7\int sec^2xe^x dx + 2 \underbrace{\int tanxe^x dx}_{\substack{u=tanx, du=sec^2xdx \\ dv=e^x dx, v=e^x}} + 2$$

$$= \int e^x dx = 10\left(\frac{1}{2}e^x tan^2x - \frac{1}{2}\int tan^2xe^x dx\right) + 7\int sec^2xe^x dx +$$

$$+2\left(tanxe^x - \int sec^2e^x dx\right) + 2e^x =$$

$$= 5e^x tan^2x - 5\int tan^2xe^x dx + 7\int sec^2xe^x dx + 2tanxe^x$$

$$- 2\int sec^2xe^x dx + 2e^x =$$

$$= 5e^x tan^2x - 5\int tan^2xe^x dx + 5\int sec^2xe^x dx + 2tanxe^x + 2e^x =$$

$$= 5e^x tan^2x - 5\int (sec^2x - 1)e^x dx + 5\int sec^2xe^x dx + 2tanxe^x + 2e^x =$$

$$= 5e^x tan^2x - 5\int sec^2xe^x dx + 5\int sec^2xe^x dx + 5\int e^x dx + 2tanxe^x$$

$$+ 2e^x =$$

$$= 5e^x tan^2x + 5e^x + 2tanxe^x + 2e^x + C =$$

$$= e^x(5tan^2x + 2tanx + 7) + C$$

AN.022. Solution (Adrian Popa)

$\frac{\Gamma'(x)}{\Gamma(x)} = \psi(x)$ *−digamma function.*

Lemma: $logx - \frac{1}{x} < \psi(x) < logx - \frac{1}{2x}$

Proof: We starting from the first representation of Binet integral for $log(\Gamma(x))$ and we have:

$$log(\Gamma(x)) = \left(x-\frac{1}{2}\right)logx - x + log\sqrt{2\pi} + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-tx}dt \Rightarrow$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = logx + \left(1-\frac{1}{2x}\right) - 1 - \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-tx}dt \Rightarrow$$

$$\psi(x) = logx - \frac{1}{2x} - \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-tx}dt$$

We need to prove: $\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1} \ge 0, \forall t > 0$

$$\frac{t(e^t-1)-2(e^t-1)+2t}{2t(e^t-1)} \overset{(?)}{\ge} 0 \Leftrightarrow \frac{te^t-2e^t+t+2}{\underbrace{2t(e^t-1)}_{>0}} \ge 0$$

Let: $f(t) = te^t - 2e^t + t + 2, t > 0,\ f'(t) = te^t - e^t + 1$

$$f''(t) = te^t > 0;\ \forall t > 0$$

So, f –convex and increasing function, then

$$\int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-tx}dt > 0, \forall x > 0 \Rightarrow \psi(x) < logx - \frac{1}{2x}$$

Now, we have:

$$e^{9+\frac{\Gamma'(a)}{\Gamma(a)}+\frac{\Gamma'(b)}{\Gamma(b)}+\frac{\Gamma'(c)}{\Gamma(c)}} = e^9 \cdot e^{\psi(a)+\psi(b)+\psi(c)}$$

$$< e^9 \cdot e^{loga-\frac{1}{2a}} \cdot e^{logb-\frac{1}{2b}} \cdot e^{logc-\frac{1}{2c}} = e^9 \cdot \frac{abc}{e^{\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}}}$$

$$= e^9 \cdot \frac{abc}{e^{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}} = e^9 \cdot \frac{abc}{e^{\frac{18}{2}}} = abc$$

AN.023. Solution (Khanh Hung Vu)

We have: $f'(x) = f'\left(\frac{\pi}{2}-x\right) \Rightarrow \left[f'\left(\frac{\pi}{2}-x\right)\right]' = -f'\left(\frac{\pi}{2}-x\right) = -f'(x) =$

$$= [-f(x)]' \Rightarrow f(x) + f\left(\frac{\pi}{2}-x\right) = C$$

Moreover, we have: $f(0) = 0, f\left(\frac{\pi}{2}\right) = 96$ *so* $f(x) + f\left(\frac{\pi}{2}-x\right) = 96$

Put $t=\frac{\pi}{2}-x\Rightarrow dt=-dx$ *we have:*

$$\Omega=\int_{0}^{\frac{\pi}{2}}x\left(\frac{\pi}{2}-x\right)f(x)dx=\int_{\frac{\pi}{2}}^{0}-t\left(\frac{\pi}{2}-t\right)f\left(\frac{\pi}{2}-t\right)dt=$$

$$=\int_{0}^{\frac{\pi}{2}}t\left(\frac{\pi}{2}-t\right)f\left(\frac{\pi}{2}-t\right)dt=\int_{0}^{\frac{\pi}{2}}x\left(\frac{\pi}{2}-x\right)f\left(\frac{\pi}{2}-x\right)dx$$

$$2\Omega=\int_{0}^{\frac{\pi}{2}}x\left(\frac{\pi}{2}-x\right)f(x)dx+\int_{0}^{\frac{\pi}{2}}x\left(\frac{\pi}{2}-x\right)f\left(\frac{\pi}{2}-x\right)dx=$$

$$=\int_{0}^{\frac{\pi}{2}}x\left(\frac{\pi}{2}-x\right)\left[f(x)+f\left(\frac{\pi}{2}-x\right)\right]dx=\int_{0}^{\frac{\pi}{2}}96x\left(\frac{\pi}{2}-x\right)dx$$

$$\Omega=\int_{0}^{\frac{\pi}{2}}48x\left(\frac{\pi}{2}-x\right)dx=\pi^{3}$$

AN.024. Solution (Ali Jaffal)

$$Let\ U_n=\sqrt[n]{\prod_{k=1}^{k=n}\sin^2\left(\frac{k}{n}\right)}$$

$$\log U_n=\frac{1}{n}\sum_{k=1}^{k=n}\sin^2\left(\frac{k}{n}\right)$$

$$So,\ \lim_{n\to+\infty}\log U_n=\int_0^1\sin^2 x\,dx=\frac{1}{2}x-\frac{\sin(2x)}{4}\Big]_0^1=\frac{1}{2}-\frac{\sin(2)}{4}$$

$$then\ \lim_{n\to+\infty}V_n=e^{\frac{1}{2}-\frac{\sin(2)}{4}}$$

We know that

$$2\sum_{1\le i<j\le n}\sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right)+\sum_{i=1}^{i=n}\sin\left(\frac{i}{n}\right)\sin\left(\frac{i}{n}\right)=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right)$$

$$then\ \frac{1}{n}\sum_{1\le i<j<n}\sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right)=\frac{1}{2n}\sum_{i=1}^{i=n}\sin^2\left(\frac{1}{n}\right)+\left(\frac{1}{n}\sum_{i=1}^{i=n}\sin\left(\frac{i}{n}\right)\right)\times\frac{1}{2}\sum_{i=1}^{i=n}\sin\left(\frac{i}{n}\right)$$

$$but\ \lim_{n\to+\infty}\sin\left(\frac{i}{n}\right)=\lim_{n\to+\infty}n\times\frac{1}{n}8\sum_{i=1}^{i=n}\sin\left(\frac{i}{n}\right)$$

$$=+\infty\times\int_0^1\sin x\,dx=+\infty$$

$$So,\ \lim_{n\to+\infty}\frac{1}{n}\sum_{1\le i<j\le n}\sin\left(\frac{1}{n}\right)\sin\left(\frac{i}{n}\right)=\frac{1}{2}\int_0^1\sin^2(x)\,dx+\infty=+\infty$$

$$Therefore\ \Omega=\frac{e^{\frac{1}{2}-\frac{\sin(2)}{4}}}{+\infty}=0$$

AN.025.Solution (Michael Sterghiou)

$$If\ 0<x\le b<\frac{\pi}{10}$$

$$tan\left(\frac{5}{2}\left(\sqrt{a}+\sqrt{b}\right)\sqrt[4]{ab}\right)tan\left(\frac{3}{2}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\right)\le tan\left(\frac{3}{2}\left(\sqrt{a}+\sqrt{b}\right)\sqrt[4]{ab}\right)$$

$$\cdot\, tan\left(\frac{5}{2}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\right)\ (1)$$

$$Write\ (1)\ as\ \frac{tan(5x)}{tan(3x)}\le\frac{\tan\,(5y)}{\tan\,(3y)}\ where\ x=\frac{\left(\sqrt{a}+\sqrt{b}\right)\sqrt[4]{ab}}{2}$$

$$y=\left(\frac{\left(\sqrt{a}+\sqrt{b}\right)}{2}\right)^2\ and\ x\le y\to\sqrt{a}+\sqrt{b}-2\sqrt[4]{ab}\ge0\ or\ \left(\sqrt{a}-\sqrt{b}\right)^2\ge0$$

$$The\ function\ f(t)=\frac{tan(5t)}{tan(3t)};0<t<\frac{\pi}{10}^{*}\ with$$

$$f'(t)=\frac{8sin(2t)+3sin(6t)}{(2cos(2t)+1)^2\cdot\theta}$$

$$Were\ \theta=(2cos(2t)-2cos(4t)-1)^2\ so\ f'(t)>0\ and\ f(t)\uparrow$$

$$x\le y\to f(x)\le f(y)\to(2)true.Done.Equality\ for\ a=b$$

$$*\,0<t<\frac{\pi}{10}\ So\ sin(6t)>0,cos(2t)>0,cos(4t)>0$$

AN.026. Solution (Tran Hong)

$$\textit{Because: } x, y \in [a, b], (a, b \geq 0) \Rightarrow b \geq x, y \geq a \Rightarrow$$

$$bx + ay - ab \geq ba + a^2 - ab = a^2 \geq 0 \Rightarrow$$

$$|ay - ab + bx| = bx + ay - ab$$

$$\int_a^b \int_a^b |ay - ab + bx| dydx = \int_a^b \int_a^b |bx + ay - ab| dydx =$$

$$= \int_a^b \int_a^b (bx + ay - ab) dxdy =$$

$$= b \int_a^b \int_a^b xdxdy + a \int_a^b \int_a^b ydxdy - ab \int_a^b \int_a^b dxdy =$$

$$= b \int_a^b xdx \int_a^b dy + a \int_a^b ydy \int_a^b dx - ab \int_a^b \int_a^b dxdy =$$

$$= b \cdot \frac{b^2 - a^2}{2} \cdot (b - a) + a \cdot \frac{b^2 - a^2}{2} \cdot (b - a) - ab(b - a)^2 =$$

$$= (b - a)^2 \left[\frac{b(b + a)}{2} + \frac{a(b + a)}{2} - ab\right] =$$

$$= (b - a)^2 \left[\frac{(b + a)^2}{2} - ab\right] = \frac{(b - a)^2(b^2 + a^2)}{2} \overset{(*)}{\leq} (a^2 + b^2)^2$$

$$(*) \Leftrightarrow (b - a)^2 \leq 2(b^2 + a^2) \Leftrightarrow b^2 - 2ab + a^2 \leq 2a^2 + 2b^2 \Leftrightarrow$$

$$a^2 + b^2 + 2ab \geq 0 \Leftrightarrow (a + b)^2 \geq 0 \textit{ true.}$$

AN.027. Solution (Samir HajAli)

$\prod_{n=1}^{\infty}(1 + b_n) = 0 \Leftrightarrow \sum_{n=1}^{\infty} log(1 + b_n) = -\infty \Leftrightarrow b_n < 1$ *and* $\sum_{n=1}^{\infty} b_n$ *is divergent.* $\Omega = \prod_{n=1}^{\infty} log\left(2 + \frac{1}{n}\right) log\left(2 - \frac{1}{n+1}\right) =$

$$= \prod_{n=1}^{\infty}\left[1 + log\left(2 + \frac{1}{n}\right) log\left(2 - \frac{1}{n + 1}\right) - 1\right]$$

Where $b_n = log\left(2 + \frac{1}{n}\right) log\left(2 - \frac{1}{n+1}\right) - 1$

So, $log\left(2 + \frac{1}{n}\right) log\left(2 - \frac{1}{n+1}\right) \leq log3log\left(2 - \frac{1}{n+1}\right) < log2log3 \approx 0.7,$

$\forall n \in \mathbb{N}$. *Therefore* $b_n < 0, \forall n \in \mathbb{N}, n \geq 1$ *and*
$\sum_{n=1}^{\infty}\left[log\left(2+\frac{1}{n}\right)log\left(2-\frac{1}{n+1}\right)-1\right]$ *is diverge, because*

$$\lim_{n\to\infty}\left(log\left(2+\frac{1}{n}\right)log\left(2-\frac{1}{n+1}\right)-1\right)=log^2 2-1\neq 0 \Rightarrow$$

$b_n < 0$ *and* $\sum_{n=1}^{\infty} b_n$ *is diverge. Depending on theory we can conclude:*

$$\Omega=\prod_{n=1}^{\infty} log\left(2+\frac{1}{n}\right)log\left(2-\frac{1}{n+1}\right)=0$$

AN.028. Solution (Sergio Esteban)

$$I=\int_{\frac{1}{a}}^{a}\frac{x+logx}{xf\left(x-\frac{1}{x}\right)}dx \overset{x=\frac{1}{u};dx=-\frac{du}{u^2}}{=} \int_{a}^{\frac{1}{a}}\frac{\frac{1}{u}-logu}{f\left(u-\frac{1}{u}\right)\frac{1}{u}}\cdot\left(-\frac{du}{u^2}\right)$$

$$=\int_{\frac{1}{a}}^{a}\frac{\frac{1}{u}-logu}{f\left(u-\frac{1}{u}\right)\cdot u}du$$

$$I=\int_{\frac{1}{a}}^{a}\frac{\frac{1}{u^2}+1-\left(\frac{logu}{u}+1\right)}{f\left(u-\frac{1}{u}\right)}du=\int_{\frac{1}{a}}^{a}\frac{\frac{1}{u^2}+1}{f\left(u-\frac{1}{u}\right)}du-\underbrace{\int_{\frac{1}{a}}^{a}\frac{u+logu}{f\left(u+\frac{1}{u}\right)\cdot u}du}_{=I}$$

$$2I=\int_{\frac{1}{a}}^{a}\frac{\frac{1}{u^2}+1}{f\left(u-\frac{1}{u}\right)}du \overset{t=u-\frac{1}{u};dt=\left(1+\frac{1}{u^2}\right)du}{=} \int_{\frac{1+\sqrt{1+4a^2}}{2a}}^{\frac{a+\sqrt{1+4a^2}}{2}}\frac{dt}{f(t)}\Rightarrow$$

$$\int_{\frac{1}{a}}^{a}\frac{x+logx}{xf\left(x-\frac{1}{x}\right)}dx=\frac{1}{2}\int_{\frac{1+\sqrt{1+4a^2}}{2a}}^{\frac{a+\sqrt{1+4a^2}}{2}}\frac{dx}{f(x)}$$

AN.029. Solution (Remus Florin Stanca)

$$\frac{2sin(\sqrt{ab})}{(a+b)sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}=\frac{\frac{sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}}\cdot\frac{2}{\pi}$$

Let: $f:\left(0;\frac{\pi}{2}\right] \to \mathbb{R}, f(x)=\frac{sinx}{x}; f'(x)=\frac{xcosx-sinx}{x^2}$; let $g(x)=xcosx-sinx \Rightarrow$

$g'(x)=-xsinx+cosx-cosx \le 0, g(0)=0 \Rightarrow g(x)\le 0 \Rightarrow f'(x)\le 0 \Rightarrow$ f *−decreasing.*

$$\frac{\pi}{a+b} \ge 1 \Rightarrow \frac{\pi\sqrt{ab}}{a+b} \ge \sqrt{ab} \Rightarrow f\left(\frac{\pi\sqrt{ab}}{a+b}\right) \le f(\sqrt{ab}) \Rightarrow$$

$$\frac{sin(\sqrt{ab})}{\sqrt{ab}} \ge \frac{sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}} \Rightarrow \frac{\frac{sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}} \cdot \frac{2}{\pi} \ge \frac{2}{\pi}; (1)$$

We must prove that:

$$\frac{\frac{sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}} \cdot \frac{2}{\pi} \le \frac{sin\frac{a+b}{2}}{\frac{a+b}{2}} \Leftrightarrow \frac{\frac{sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}} \le \frac{\frac{sin\frac{a+b}{2}}{\frac{a+b}{2}}}{\frac{sin\frac{\pi}{2}}{\frac{\pi}{2}}}$$

Let: $h(x)=\frac{\frac{sinx}{x}}{\frac{sin\left(\frac{\pi x}{a+b}\right)}{\frac{\pi x}{a+b}}}=\frac{sinx}{x}\cdot\frac{\pi x}{a+b}\cdot\frac{1}{sin\frac{\pi x}{a+b}}=\frac{\pi}{a+b}\cdot\frac{sinx}{sin\frac{\pi x}{a+b}}; (2)$

Let: $f_1(x)=\frac{sinx}{sin\left(\frac{\pi x}{a+b}\right)} \Rightarrow f'_1(x)=\frac{cosx\cdot sin\frac{\pi x}{a+b}-sinx\cdot cos\frac{\pi x}{a+b}\cdot\frac{\pi x}{a+b}}{sin^2\frac{\pi x}{a+b}}$

Let: $f_2(x)=cosx\cdot sin\frac{\pi x}{a+b}-sinx\cdot cos\frac{\pi x}{a+b}\cdot\frac{\pi x}{a+b} \Rightarrow$

$$f'_2(x)=sinx\cdot sin\frac{\pi x}{a+b}\left(\frac{\pi^2}{a+b}-1\right)$$

$$\ge 0\left(we\ take\ the\ function\ for\ x \le \frac{a+b}{2}\right)$$

$\Rightarrow f_2$ *−increasing,* $f_2(0)=0 \Rightarrow f_1$ *−increasing* $\underset{(2)}{\Rightarrow} h(x)$ *−increasing.*

$$\sqrt{ab}\le\frac{a+b}{2}\Rightarrow h(\sqrt{ab})\le h\left(\frac{a+b}{2}\right)\Rightarrow\frac{\frac{sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}}\le\frac{\frac{sin\frac{a+b}{2}}{\frac{a+b}{2}}}{\frac{sin\frac{\pi}{2}}{\frac{\pi}{2}}}\Leftrightarrow$$

$$\frac{2sin(\sqrt{ab})}{(a+b)sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}\le\frac{2sin\left(\frac{a+b}{2}\right)}{a+b};(4)\overset{(1),(4)}{\Longrightarrow}$$

$$\frac{2}{\pi}\le\frac{2sin(\sqrt{ab})}{(a+b)sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}\le\frac{2sin\left(\frac{a+b}{2}\right)}{a+b}$$

AN.030. Solution (Remus Florin Stanca)

$$125^a\cdot(4a+b)^{a+4b}\le125^b\cdot(a+4b)^{4a+b}\Leftrightarrow$$

$$5^{3a}\cdot(4a+b)^{a+4b}\le5^{3b}\cdot(a+4b)^{4a+b}\overset{\cdot5^{a+b}}{\Longleftrightarrow}$$

$$5^{4a+b}\cdot(4a+b)^{a+4b}\le5a+4^b\cdot(a+4b)^{4a+b}\Leftrightarrow$$

$$\left(\frac{4a+b}{5}\right)^{a+4b}\le\left(\frac{a+4b}{5}\right)^{4a+b}\Leftrightarrow log\left(\frac{4a+b}{5}\right)^{a+4b}\le log\left(\frac{a+4b}{5}\right)^{4a+b}\Leftrightarrow$$

$$\frac{log\left(\frac{4a+b}{5}\right)}{4a+b}\le\frac{log\left(\frac{a+4b}{5}\right)}{a+4b};(1)$$

Let $f:(1,\infty)\to\mathbb{R}$ *such that* $f(x)=\frac{log\left(\frac{x}{5}\right)}{x}\Rightarrow f'(x)=\frac{1+log5}{x^2}>0\Rightarrow$

f *–is increasing*

$$a<b\Rightarrow3a<3b\Rightarrow4a+b<a+4b\Rightarrow f(4a+b)<f(a+4b)\Rightarrow$$

$$\frac{log\left(\frac{4a+b}{5}\right)}{4a+b}\le\frac{log\left(\frac{a+4b}{5}\right)}{a+4b};(2)$$

From (1),(2) we get: $125^a\cdot(4a+b)^{a+4b}\le125^b\cdot(a+4b)^{4a+b}$

AN.031. *Let be* $f:[2,3]\to\mathbb{R},f(x)=\sqrt{x^x}+\frac{x}{2}-\frac{x^2}{2}-1$

$$f'(x)=\left(x^{\frac{x}{2}}-\frac{x^2}{2}+\frac{x}{2}-1\right)'=\frac{1}{2}\cdot x^{\frac{x}{2}}\cdot\log x+\frac{x}{2}\cdot x^{\frac{x}{2}-1}-x+\frac{1}{2}=$$

$$=\frac{1}{2}x^{\frac{x}{2}}(\log x+1)-x+\frac{1}{2}=\frac{x^{\frac{x}{2}}(\log x+1)-2x+1}{2}$$

$$Let\ be\ g:[2,3]\to\mathbb{R};\ g(x)=x^{\frac{x}{2}}(\log x+1)-2x+1$$

$$g'(x)=\left(x^{\frac{x}{2}}\right)'(1+\log x)+\frac{1}{x}x^{\frac{x}{2}}-2$$

$$g'(x)=\left(\frac{1}{2}\sqrt{x^x}\log x+\frac{1}{2}\sqrt{x^x}\right)(1+\log x)+\frac{1}{x}\sqrt{x^x}-2$$

$$g'(x)=\sqrt{x^x}(\log x+1)^2+\frac{1}{x}\sqrt{x^x}-2$$

$$g'(x)=\sqrt{x^x}\left(\frac{1}{2}(\log x+1)^2+\frac{1}{x}\right)-2$$

$$x\ge 2\Rightarrow x^{\frac{x}{2}}\ge 2^{\frac{2}{2}}=2$$

$$x\le 3\Rightarrow\frac{1}{x}\ge\frac{1}{3};\ 1+\log x\ge 1+\log 2$$

$$g'(x)=x^{\frac{x}{2}}\left(\frac{1}{2}(\log x+1)^2+\frac{1}{x}\right)-2\ge 2\left(\frac{1}{2}(1+\log 2)^2+\frac{1}{3}\right)-2=$$

$$=(1+\log 2)^2+\frac{2}{3}-2=1+2\log 2+\log^2 2-\frac{4}{3}=$$

$$=\log^2 2+2\log 2-\frac{1}{3}>0$$

$$g'(x)>0\Rightarrow f'(x)>0\Rightarrow f\ increasing$$

$$\Rightarrow f(x)\ge f(2)=\sqrt{2^2}+\frac{2}{2}-\frac{2^2}{2}-1=0$$

$$f(x)\ge 0\Rightarrow\sqrt{a^a}+\frac{a}{2}-\frac{a^2}{2}-1\ge 0\quad (1)$$

$$\sqrt{b^b}+\frac{b}{2}-\frac{b^2}{2}-1\ge 0\ \ (2),\ \sqrt{c^c}+\frac{c}{2}-\frac{c^2}{2}-1\ge 0\quad (3)$$

By adding (1); (2); (3):

$$2\left(\sqrt{a^a}+\sqrt{b^b}+\sqrt{c^c}\right)+a+b+c-(a^2+b^2+c^2)-6\ge 0$$

$$2\left(\sqrt{a^a}+\sqrt{b^b}+\sqrt{c^c}\right)+a+b+c\geq a^2+b^2+c^2+6$$

AN.032. Solution (Ravi Prakash)

$$Let\ f_1(x)=\sin^2 x+\tan^2 x-2x^2, 0\leq x<\frac{\pi}{2}$$

$$f_1'(x)=2\sin x\cos x+2\tan x\sec^2 x-4x$$

$$\geq 2(2)\sqrt{\sin x\cos x\tan x\sec^2 x}-4x$$

$$=4(\tan x-x)>0\ for\ 0<x<\frac{\pi}{2}$$

$$\Rightarrow f_1\ is\ strictly\ increasing\ on\ \left[0,\frac{\pi}{2}\right)\Rightarrow f_1(x)>f_1(0)=0\ for\ 0<x<\frac{\pi}{2}$$

$$\Rightarrow\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^2>2\ for\ 0<x<\frac{\pi}{2}. As\ 0<x<\tan x\ for\ 0<x<\frac{\pi}{2}\Rightarrow$$

$$\frac{\tan x}{x}>1$$

$$\Rightarrow\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^4>\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^2>2\ for\ 0<x<\frac{\pi}{2}. Next,\ let$$

$$f_2(y)=\sin^3 y+\tan^3 y-2y^3;\ 0\leq y<\frac{\pi}{2}$$

$$f_2'(y)=3\sin^2 y\cos y+3\tan^2 y\sec^2 y-6y^2;\ 0<y<\frac{\pi}{2}$$

$$f_2''(y)=-3\sin^3 y+6\sin y\cos^2 y+6\tan y\sec^4 y+6\tan^3 y\sec^2 y-12y$$

$$\geq 3[2\tan^3 y\sec^2 y-\sin^3 y]+12\sqrt{\sin y\cos^2 y\tan y\sec^4 y}-12y$$

$$=3\sin^3 y\,(2\sec^5 y-1)+12(\tan y\sec y-y)>0+12(\tan y-y)>0$$

$$[\because \sec y>1\ for\ 0<y<\frac{\pi}{2}]$$

$$Thus,\ f_2''(y)>0\ for\ 0<y<\frac{\pi}{2}\Rightarrow f_2'(y)>f_2'(0)=0,\ for\ 0<y<\frac{\pi}{2}$$

$$\Rightarrow f_2\ is\ strictly\ increasing\ on\ \left[0,\frac{\pi}{2}\right)\Rightarrow f_2(y)>f_2(0)=0,\ for\ 0<y<\frac{\pi}{2}$$

$$\Rightarrow \sin^3 y+\tan^3 y>2y^3\ for\ 0<y<\frac{\pi}{2}\Rightarrow\left(\frac{\sin y}{y}\right)^3+\left(\frac{\tan y}{2}\right)^3>2\ for\ 0<y<\frac{\pi}{2}$$

$$As\ \frac{\tan y}{y}>1\ for\ 0<y<\frac{\pi}{2}\Rightarrow\left(\frac{\sin y}{y}\right)^3+\left(\frac{\tan y}{y}\right)^5$$

$$> \left(\frac{\sin y}{y}\right)^3 + \left(\frac{\tan y}{y}\right)^3 > 2 \textit{ for } 0 < y < \frac{\pi}{2}$$

From (1), (2) we get: $\left[\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^4\right]\left[\left(\frac{\sin y}{y}\right)^3 + \left(\frac{\tan y}{y}\right)^5\right] > 4 \textit{ for } 0 < x, y < \frac{\pi}{2}$

AN.033. Solution (Naren Bhandari)

Denote

$$S(n) = \sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\sin k \sin j}{\sin j + \sin k} = \sin 1\sum_{j=1}^{n}\frac{\sin j}{\sin 1 + \sin j} +$$

$$+\sin 2\sum_{j=1}^{n}\frac{\sin j}{\sin 2 + \sin j} + \cdots + \sin k\sum_{j=1}^{n}\frac{\sin j}{\sin k + \sin j}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\frac{\sin^2 k}{\sin k} + 2\sum_{1\le k<l\le n}\frac{\sin k \sin j}{\sin j + \sin k} = \frac{1}{2}\sum_{k=1}^{n}\sin k + a$$

Thus we have

$$\Omega = \lim_{n\to\infty}\exp\left(\frac{S(n)}{n^3} - \frac{2}{n^3}\sum_{1\le k<j\le n}\frac{\sin i \sin j}{\sin i + \sin j}\right)$$

$$= \lim_{n\to\infty}\exp\left(\frac{1}{2n^3}\sum_{k=1}^{n}\sin k\right) \quad (1)$$

Here

$$\sum_{k=1}^{n}\sin k = \mathcal{T}\sum_{k=1}^{n}e^{ik} = \mathcal{T}\left(\frac{e^i(e^{in}-1)}{e^i - 1}\right)$$

$$= \mathcal{T}\left(\frac{e^i e^{\frac{in}{2}}\left(e^{\frac{ni}{2}} - e^{-\frac{ni}{2}}\right)}{e^{\frac{i}{2}}\left(e^{\frac{i}{2}} - e^{-\frac{i}{2}}\right)}\right) = \mathcal{T}\left(e^{\frac{(n+1)i}{2}}\cdot\frac{\sin\frac{n}{2}}{\sin\frac{1}{2}}\right) = \frac{\sin\frac{n+1}{2}\sin\frac{n}{2}}{\sin\frac{1}{2}}$$

Plugging in 1 we have:

$$\lim_{n\to\infty}\exp\left(\frac{1}{2n^3}\cdot\frac{\sin\frac{n+1}{2}\sin\frac{n}{2}}{\sin\frac{1}{2}}\right) = e^0 = 1$$

AN.034. Solution (Tran Hong)

Using Cauchy – Schwarz inequality:

$$\left(\int_0^1 \sqrt{f(x)}\,dx\right)^2 = \left(\int_0^1 1\sqrt{f(x)}\,dx\right)^2 \leq \int_0^1 1^1 dx \cdot \int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx$$

$$\left(\int_0^1 \sqrt{f(x)}\,dx\right)^2 = \left(\int_0^1 \sqrt[6]{f(x)} \cdot \sqrt[6]{f(x)} \cdot \sqrt[6]{f(x)}\,dx\right)^2$$

$$\leq \int_0^1 \left[\sqrt[8]{f(x)}\right]^2 dx \cdot \int_0^1 \left[\sqrt[6]{f(x)}\right]^2 dx \cdot \int_0^1 \left[\sqrt[6]{f(x)}\right]^2 dx = \left(\int_0^1 \sqrt[3]{f(x)}\,dx\right)^3$$

$$\Rightarrow 2\left(\int_0^1 \sqrt{f(x)}\,dx\right)^2 \leq \int_0^1 f(x)\,dx + \left(\int_0^1 \sqrt[3]{f(x)}\,dx\right)^3$$

AN.035. Solution (Ravi Prakash)

$$For\ 0 < x < \frac{\pi}{4}, let\ f(x) = \frac{tan(2x)}{tan\ x} = \frac{2}{1 - tan^2 x}$$

$$f'(x) = -2(1 - tan^2 x)^{-2}(-2tan\ x\ sec^2 x) = \frac{4tan\ x\ sec^2 x}{(1 - tan^2 x)^2} > 0$$

$$\Longrightarrow f(x)\ is\ strictly\ increasing\ on\ \left(0, \frac{\pi}{4}\right)$$

$$For\ 0 < a \leq b < \frac{\pi}{4}, \sqrt{ab} \leq \frac{a+b}{2}$$

$$\therefore \frac{tan\left(2\sqrt{ab}\right)}{tan\left(\sqrt{ab}\right)} \leq \frac{\tan(a+b)}{tan\left(\frac{a+b}{2}\right)}$$

$$\Longrightarrow tan\left(2\sqrt{ab}\right) tan\left(\frac{a+b}{2}\right) \leq tan\left(\sqrt{ab}\right) tan(a+b)$$

AN.036. Solution (Ali Jaffal)

We have: $x - \frac{x^2}{2} \leq log(1+x) \leq x, \forall x \geq 0;\ \ (*)$

$$\text{Let: } S_n=\sum_{k=1}^{n}log\left(1+\frac{ntan^{-1}\left(\frac{k}{n}\right)}{k^2+n^2}\right)\text{ and } P_n=\prod_{k=1}^{n}\left(1+\frac{ntan^{-1}\left(\frac{k}{n}\right)}{k^2+n^2}\right)$$

By $(*)$ we have:

$$\frac{ntan^{-1}\left(\frac{k}{n}\right)}{k^2+n^2}-\frac{1}{2}\cdot\frac{n^2\left(tan^{-1}\left(\frac{k}{n}\right)\right)^2}{(k^2+n^2)^2}\le log\left(1+\frac{ntan^{-1}\left(\frac{k}{n}\right)}{k^2+n^2}\right)\le\frac{ntan^{-1}\left(\frac{k}{n}\right)}{k^2+n^2}$$

$$\frac{1}{n}\sum_{k=1}^{n}\frac{tan^{-1}\left(\frac{k}{n}\right)}{1+\left(\frac{k}{n}\right)^2}-\frac{1}{2}\left(\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{tan^{-1}\left(\frac{k}{n}\right)}{1+\left(\frac{k}{n}\right)^2}\right)^2\right)\le S_n\le\frac{1}{n}\sum_{k=1}^{n}\frac{tan^{-1}\left(\frac{k}{n}\right)}{1+\left(\frac{k}{n}\right)^2}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{tan^{-1}\left(\frac{k}{n}\right)}{1+\left(\frac{k}{n}\right)^2}=\int_0^1 f(x)dx=\int_0^1\frac{tan^{-1}x}{1+x^2}dx=\frac{1}{2}(tan^{-1}x)^2\Big|_0^1$$

$$=\frac{1}{2}((tan^{-1}1)^2-(tan^{-1}0)^2)=\frac{\pi^2}{32}$$

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{tan^{-1}\left(\frac{k}{n}\right)}{1+\left(\frac{k}{n}\right)^2}\right)^2=0\cdot\left(\int_0^1\frac{tan^{-1}x}{1+x^2}dx\right)^2=0$$

$$\lim_{n\to\infty}S_n=\frac{\pi^2}{32}\Rightarrow\lim_{n\to\infty}P_n=e^{\frac{\pi^2}{32}}$$

AN.037. Solution (Tran Hong)

$$We\ have: f(u)=tan^{-1}u, u\epsilon\mathbb{R}\longrightarrow f'(u)=\frac{1}{1+u^2}\to f(u)\uparrow on\ \mathbb{R}$$

$$\bullet\ \frac{x+2y}{3}\le\frac{|x+2y|}{3}\le\frac{\sqrt{5}\cdot\sqrt{x^2+y^2}}{3}=\frac{\sqrt{5}\cdot\sqrt{3}}{3}=\frac{\sqrt{15}}{3}$$

$$\to tan^{-1}\left(\frac{x+2y}{3}\right)\le tan^{-1}\left(\frac{\sqrt{15}}{3}\right)$$

Similary:

$$tan^{-1}\left(\frac{x+3y}{4}\right)\le tan^{-1}\left(\frac{\sqrt{30}}{4}\right),\qquad tan^{-1}\left(\frac{x+4y}{5}\right)\le tan^{-1}\left(\frac{\sqrt{51}}{5}\right)$$

$$tan^{-1}\left(\frac{\sqrt{15}}{3}\right)+tan^{-1}\left(\frac{\sqrt{30}}{4}\right)+tan^{-1}\left(\frac{\sqrt{51}}{5}\right)\overset{Jensen}{\leq}$$

$$3tan^{-1}\left(\frac{\frac{\sqrt{15}}{3}+\frac{\sqrt{20}}{4}+\frac{\sqrt{51}}{5}}{3}\right)\underset{<}{\frac{\sqrt{15}}{3}+\frac{\sqrt{30}}{4}+\frac{\sqrt{51}}{5}}\approx 4.09<5$$

$$3tan^{-1}\left(\frac{5}{3}\right)\approx 3.09<\pi\approx 3.14$$

AN.038. Solution (Florentin Visescu)

$$Let\ be\ f:\left(0,\frac{\pi}{2}\right]\to\mathbb{R}, f(x)=sin\,x-x-\frac{8-4\pi}{\pi^3}x^3$$

$$f'(x)=\cos x-1-3\frac{8-4\pi}{\pi^3}x^2,\qquad f''(x)=-sin\,x-6\frac{8-4\pi}{\pi^3}x$$

$$f'''(x)=-cos\,x-6\frac{8-4\pi}{\pi^3},\qquad f^{IV}=sin\,x>0,\forall x\epsilon\left(0,\frac{\pi}{2}\right]$$

x	0 $\quad$ r_1 $\quad$ r_2 $\quad$ r_3 $\quad$ $\frac{\pi}{2}$
f^{IV}	+ + + + + + + +
f'''	- - 0 + + + +
f''	0 - - $f''(?)$ - 0 + + +
f'	0 - - - - - 0 + +
f	0 ↘ ↘ − ↗ 0

$$f'''\left(\frac{\pi}{2}\right)=6\frac{4\pi-8}{\pi^3};\ \lim_{\substack{x\to 0\\ x>0}}f'''(x)=-1+6\frac{4\pi-8}{\pi^3}=\frac{24\pi-48-\pi^3}{\pi^3}<0$$

$$\lim_{\substack{x\to 0\\ x>0}}f''(x)=0;\quad f''\left(\frac{\pi}{2}\right)=-1-6^3\frac{8-4\pi}{\pi^3}\cdot\frac{\pi}{2}=\frac{-\pi^2-24+12\pi}{\pi^2}>0$$

$$\lim_{\substack{x\to 0\\ x>0}}f'(x)=0;\quad f'\left(\frac{\pi}{2}\right)=-3\frac{8-4\pi}{\pi^3}\cdot\frac{\pi^2}{4}=-3\frac{2-\pi}{\pi}=\frac{-6+3\pi}{\pi}>0$$

$$\lim_{\substack{x\to 0\\ x>0}} f(x)=0;\quad f\left(\frac{\pi}{2}\right)=1-\frac{\pi}{2}-\frac{8-4\pi}{\pi^3}\cdot\frac{\pi^3}{8}=1-\frac{\pi}{2}-\frac{2-\pi}{2}$$

$$=\frac{2-\pi+\pi-2}{2}=0$$

$$So\ f(x)=0.\forall x\in\left(0,\frac{\pi}{2}\right]\ equality\ just\ for\ x=\frac{\pi}{2}\Rightarrow\frac{\sin x}{x}+\frac{\sin y}{y}+\frac{\sin z}{3}$$

$$\le 3+\frac{8-4\pi}{\pi^3}(x^2+y^2+z^2)$$

$$\forall\ x,y,z\in\left(0,\frac{\pi}{2}\right]\ equality\ just\ for\ x=y=z=\frac{\pi}{2}$$

AN.039. Solution (Remus Florin Stanca)

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k(k+1)e^{\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}}}{n(n+1)(n+2)}$$

$$Let\ x_k=\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}\Rightarrow x_{k+1}-x_k=\frac{6k(k+1)}{n(n+1)(n+2)}\Rightarrow\left||\Delta_n|\right|\overset{\max}{\underset{k\le n}{=}}\frac{6}{n+2}\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\left||\Delta_n|\right|=0\Rightarrow\lim_{n\to\infty}\sum_{k=1}^{n}(x_{k+1}-x_k)\,f(\zeta_k)=\int_a^b f(x)\,dx,\zeta_k\in[x_k;x_{k+1}]$$

$$let\ \zeta_k=x_k and$$

$$a=\lim_{n\to\infty}x_1\ and\ b=\lim_{n\to\infty}x_n\Rightarrow\lim_{n\to\infty}\frac{1}{6}\sum_{k=1}^{n}\frac{6(k+1)k}{n(n+1)(n+2)}e^{\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}}=$$

$$\frac{1}{6}\int_0^2 e^x\,dx=\frac{e^2-1}{6}\Rightarrow\Omega=\frac{e^2-1}{6}$$

AN.040. Solution (Serban George Florin)

$$We\ prove\ that\ from\ u\le v\Rightarrow(1+u)^v\ge(1+v)^u;(\forall)u,v>0$$

$$(1+u)^v\ge(1+v)^u\Rightarrow\ln(1+u)^v\ge\ln(1+v)^u$$

$$v\ln(1+u)\ge u\ln(1+v),\frac{\ln(1+u)}{u}\ge\frac{\ln(1+v)}{v}$$

$$f:(0,\infty)\to\mathbb{R},f(x)=\frac{\ln(1+x)}{x},f'(x)=\frac{\frac{x}{x+1}-\ln(x+1)}{x^2}$$

$$\varphi:[0,\infty)\to\mathbb{R},\varphi(x)=\frac{x}{x+1}-\ln(x+1),\varphi'(x)=\frac{x+1-x}{(x+1)^2}-\frac{1}{x+1}=$$

$$= \frac{1}{(x+1)^2} - \frac{1}{x+1} = \frac{1-x-1}{(x+1)^2} = -\frac{x}{(x+1)^2} \leq 0$$

$$\Rightarrow \varphi'(x) \leq 0 \ on\ [0,\infty) \Rightarrow \varphi \searrow on\ [0,\infty), x \geq 0 \Rightarrow \varphi(x) \leq \varphi(0) = 0, \varphi(x) \leq 0 \Rightarrow$$

$$\Rightarrow f'(x) \leq 0 \ on\ [0,\infty) \Rightarrow f \searrow on\ (0,\infty); u \leq v, u, v \in (0,\infty) \Rightarrow f(u) \geq f(v) \Rightarrow$$

$\Rightarrow \frac{\ln(1+u)}{u} \geq \frac{\ln(1+v)}{v}$ *true. Applying this inequality:*

(1) $\left(1+\frac{2ab}{a+b}\right)^{\sqrt{ab}} \geq \left(1+\sqrt{ab}\right)^{\frac{2ab}{a+b}}$, *where* $u = \frac{2ab}{a+b} \leq v = \sqrt{ab}$ *(*$M_h \leq M_g$*)*

(2) $\left(1+\sqrt{ab}\right)^{\frac{a+b}{2}} \geq \left(1+\frac{a+b}{2}\right)^{\sqrt{ab}}$

$u' = \sqrt{ab} \leq \frac{a+b}{2} = v'$ *(*$M_g \leq M_a$*)*

Multiplying the two inequalities (1) and (2) we obtain:

$$\left(1+\frac{2ab}{a+b}\right)^{\sqrt{ab}} \cdot \left(1+\sqrt{ab}\right)^{\frac{a+b}{2}} \geq \left(1+\sqrt{ab}\right)^{\frac{2ab}{a+b}} \cdot \left(1+\frac{a+b}{2}\right)^{\sqrt{ab}}$$

AN.041. Solution (Sergio Esteban)

We can notice that: $f(x) = \tan^{-1}(x) \cdot x$ *and* $g(x) = e^{-x^2}$

as $f''(x) = \frac{2}{(x^2+1)^2} > 0 \to f$ *is strictly convex*

By integral form of Jensen's inequality

$$\int_a^b f \circ g \, dx \geq f\left(\int_a^b g(x)\,dx\right) = \int_a^b \frac{1}{e^{x^2}} dx \cdot \tan^{-1}\left(\int_a^b \frac{1}{e^{x^2}} dx\right)$$

AN.042. Solution (Florentin Visescu)

$$f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, f(x) = \sin x - \frac{3x}{\pi} + \frac{4x^3}{\pi^3}$$

$$f'(x) = \cos x - \frac{3}{\pi} + \frac{12x^2}{\pi^3}, f''(x) = -\sin x + \frac{24x}{\pi^3}$$

$$f'''(x) = -\cos x + \frac{24}{\pi^3}; f^{IV}(x) = \sin x > 0, x \in \left(0; \frac{\pi}{2}\right) \lim_{\substack{x\to 0\\ x>0}} f'''(x) = \frac{24}{\pi^3} - 1$$

$$< 0; \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f'''(x) = \frac{2}{\pi^3} > 0$$

So, f''' *has a root* r_1 *in* $\left(0; \frac{\pi}{2}\right)$, $\lim_{\substack{x\to 0\\ x>0}} f''(x) = 0;$

$$\lim_{\substack{x\to 0\\ x<0}} f''(x) = \frac{12}{\pi^2} - 1 > 0$$

So $f''(r_1) < 0$ *and* f'' *has a root* r_2 *in* $\left(r_1, \frac{\pi}{2}\right)$

$$\lim_{\substack{x\to 0\\ x>0}} f'(x) = 1 - \frac{3}{\pi} > 0; \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f'(x) = 0$$

So, $f'(r_2) < 0$ *and* f' *has a root* r_3 *in* $(0; r_2)$. *It doesn't matter where is towards* r_1

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = 0; \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f(x) = 0$$

So, $f(x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$. *Then:* $\sin x > \frac{3x}{\pi} - \frac{4x^3}{\pi^3}; \frac{\sin x}{x} > \frac{3}{\pi} - \frac{4x^2}{\pi^3}$

$$\int_a^b \frac{\sin x}{x} dx > \frac{3}{\pi}(b-a) - \frac{4}{3\pi^3}(b^3 - a^3)$$

$$\frac{4}{3}(b^3 - a^3) + \pi^3 \int_a^b \frac{\sin x}{x} dx > 3\pi^2(b-a)$$

AN.043. Solution (Kamel Benaicha)

$$\Omega = \lim_{n\to+\infty} \sqrt[n]{\int_e^n \frac{x^{n-1}e^x(x\ln(x) + n\ln(x) - n)}{\ln^{n+1}(x)} dx}$$

$$Put\ I(n)=\int_e^n \frac{x^{n-1}e^x(x\ln(x)+n\ln(x)-n)}{\ln^{n+1}(x)}dx$$

$$=\int_e^n \frac{x^ne^x+nx^{n-1}e^x}{\ln^n(x)}dx-n\int_e^n \frac{x^ne^x}{x\ln^{n+1}(x)}dx$$

$$n\int_e^n \frac{x^ne^x}{x\ln^n(x)}dx \overset{IBP}{=} -\left(\frac{n^ne^n}{\ln^n(n)}-e^{n+e}\right)+\int_e^n \frac{x^ne^x+nx^{n-1}e^x}{\ln^n(x)}dx$$

$$\therefore I(n)=\left(\frac{ne}{\ln(n)}\right)^n-e^{n+e}$$

$$\therefore \Omega=\lim_{n\to+\infty}\left(I(n)\right)^{\frac{1}{n}}=\lim_{n\to+\infty}\frac{ne}{\ln(n)}\left(1-e^e\left(\frac{\ln(n)}{n}\right)^n\right)^{\frac{1}{n}}$$

$$=\lim_{n\to+\infty}\left(\frac{ne}{\ln(n)}\right)e^{\frac{1}{n}\ln\left(1-e^e\left(\frac{\ln(n)}{n}\right)^n\right)}$$

$$\lim_{n\to+\infty}\left(\frac{\ln(n)}{n}\right)^n=\lim_{n\to+\infty}e^{n\ln\left(\frac{\ln(n)}{n}\right)}=0\left(\lim_{n\to+\infty}\left(\frac{\ln(n)}{n}\right)=0_+\right)$$

$$\therefore \Omega=\lim_{n\to+\infty}\left(\frac{ne}{\ln(n)}\right)e^{-\frac{e^e}{n}\left(\frac{\ln(n)}{n}\right)^n}$$

$$=+\infty\left(\lim_{n\to+\infty}\frac{n}{\ln(n)}=+\infty,\ \lim_{n\to+\infty}\frac{e^e}{n}\left(\frac{\ln(n)}{n}\right)^n=0\right)$$

$$\therefore \lim_{n\to+\infty}\sqrt[n]{\int_e^n \frac{x^{n-1}e^x(x\ln(x)+n\ln(x)-n)}{\ln^{n+1}(x)}dx}=+\infty$$

AN.044. Solution (Marian Ursărescu)

From Huygens inequality, we have:

$$\left(H_n^{(2)}+H_n^{(6)}\right)\left(H_n^{(4)}+H_n^{(8)}\right)\ge\left(\sqrt{H_n^{(2)}\cdot H_n^{(4)}}+\sqrt{H_n^{(6)}\cdot H_n^{(8)}}\right)^2\Rightarrow$$

We must show: $\sqrt{H_n^{(2)}\cdot H_n^{(4)}}+\sqrt{H_n^{(6)}\cdot H_n^{(8)}}\ge H_n^{(3)}+H_n^{(7)}$ *(1)*

From Cauchy's inequality, we have:

$$H_n^{(2)}\cdot H_n^{(4)}=\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\left(\frac{1}{1^4}+\frac{1}{2^4}+\cdots+\frac{1}{n^4}\right)\ge$$

$$\geq\left(\frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{n^3}\right)^2\Rightarrow\sqrt{H_n^{(2)}\cdot H_n^{(4)}}\geq H_n^{(3)}\quad (2)$$

$$H_n^{(6)}\cdot H_n^{(8)}=\left(\frac{1}{1^6}+\frac{1}{2^6}+\cdots+\frac{1}{n^6}\right)\left(\frac{1}{1^8}+\frac{1}{2^8}+\cdots+\frac{1}{n^8}\right)\geq$$

$$\geq\left(\frac{1}{1^7}+\frac{1}{2^2}+\cdots+\frac{1}{n^7}\right)^2\Rightarrow\sqrt{H_n^{(6)}\cdot H_n^{(8)}}\geq H_n^{(7)}\quad (3)$$

From (2)+(3)⇒ (1) it is true.

AN.045. Solution (Serban George Florin)

$$\frac{x^{\sin^2 t}}{1+x^{\cos^2 t}}+\frac{x^{\cos^2 t}}{1+x^{\sin^2 t}}\geq 2\sqrt{\frac{x^{\sin^2 t}+\cos^2 t}{\left(1+x^{\cos^2 t}\right)\left(1+x^{\sin^2 t}\right)}}=$$

$$=2\sqrt{\frac{x}{\left(1+x^{\cos^2 t}\right)\left(1+x^{\sin^2 t}\right)}}\geq 2\sqrt{\frac{x}{(1+1)(1+1)}}=\frac{2\sqrt{x}}{2}=\sqrt{x}$$

$$\textit{because: } x^{\cos^2 t}\leq 1=x^0,(\forall)x\in[0,1],0\leq\cos^2 t\leq 1 \textit{ and}$$

$$x^{\sin^2 t}\leq 1=x^0,(\forall)x\in[0,1],0\leq\sin^2 t\leq 1$$

$$\Rightarrow\Omega_1(t)+\Omega_2(t)=\int_0^1\left(\frac{x^{\sin^2 t}}{1+x^{\cos^2 t}}+\frac{x^{\cos^2 t}}{1+x^{\sin^2 t}}\right)dt\geq\int_0^1\sqrt{t}\,dt=$$

$$=\int_0^1 t^{\frac{1}{2}}\,dt=\left.\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^1=\left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1=\frac{2}{3}\geq\frac{4-\sin 2t}{8+\sin 2t}$$

$$\sin 2t=y\geq 0,t\in\left[0,\frac{\pi}{2}\right]\Rightarrow 2t\in[0,\pi]\Rightarrow 8+y>0 \textit{ and}$$

$$4-y>0 \textit{ because } y\in[0,1]\Rightarrow 2(8+y)\geq 3\cdot(4-y),16+2y\geq 12-3y$$

$$\Rightarrow 4+5y\geq 0, \textit{ true, because } y\geq 0$$

AN.046. Solution (Adrian Popa)

$$f(x)+f(y)\geq 3f(x+y)\Rightarrow f(x+y)\leq\frac{f(x)+f(y)}{3}\Rightarrow$$

$$\Rightarrow f(x+y+z) \le \frac{f(x+y)+f(z)}{3} \le \frac{\frac{f(x)+f(y)}{3}+f(z)}{3}$$

$$= \frac{f(x)+f(y)+3f(z)}{9}$$

$$\textit{Similarly: } f(x+y+z) \le \frac{f(x)+f(z)+3f(y)}{9}$$

$$f(x+y+z) \le \frac{f(y)+f(z)+3f(x)}{9}$$

$$\textit{So, } 3f(x+y+z) \le \frac{5\big(f(x)+f(y)+f(z)\big)}{9}$$

$$3\int_0^1\int_0^1\int_0^1 f(x+y+z)\,dx\,dy\,dz \le \frac{5}{9}\int_0^1\int_0^1\int_0^1 \big(f(x)+f(y)+f(z)\big)dx\,dy\,dz =$$

$$= \frac{5}{9}\cdot 3\int_0^1 f(x)\,dx = \frac{5}{3}\int_0^1 f(x)\,dx \le 5\int_0^1 f(x)\,dx$$

AN.047. Solution (Ali Jaffal) *We have by GM-AM inequality:*

$$\log^n\left(\frac{(1+H_1)^2+(1+H_2)^2+\cdots+(1+H_n)^2}{n}\right) \ge$$

$$\log^n\left(\sqrt[n]{(1+H_1)^2(1+H_2)^2\cdot\ldots\cdot(1+H_n)^2}\right) \ge$$

$$\left(\frac{2}{n}\right)^n[\log(1+H_1)(1+H_2)\ldots(1+H_n)]^n \ge$$

$$\left(\frac{2}{n}\right)^n[\log(1+H_1)+\log(1+H_2)+\cdots+\log(1+H_n)]^n \quad (*)$$

$$\textit{And } \log(1+H_1)+\cdots+\log(1+H_n) \le \left(\frac{1}{n}\right)^n[\log(1+H_1)+\cdots+$$

$$\log(1+H_n)]^n \quad (**)$$

$$\textit{Let } U_n = \frac{\log_n^n\big((1+H_1)^2+(1+H_2)^2+\cdots+(1+H_n)^2\big)}{\log_n(1+H_1)\cdot\log_n(1+H_2)\cdot\ldots\cdot\log_n(1+H_n)}$$

$$= \frac{\frac{1}{(\ln n)^n}\log^n\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}{\frac{1}{(\ln n)^n}[\log(1+H_1)\log(1+H_2)\cdot\ldots\cdot\log(1+H_n)]} \overset{by\ (*)\ and\ (**)}{\ge} \frac{\left(\frac{2}{n}\right)^n}{\left(\frac{1}{n}\right)^n} \ge 2^n. \textit{ So, } U_n \ge 2^n$$

$$\textit{but } \lim_{n\to+\infty} 2^n = +\infty \textit{ then } \lim_{n\to\infty} U_n \ge +\infty \textit{ so, } \lim_{n\to\infty} U_n = +\infty$$

AN.048. *Let be* $x, y \in [a, b]; a \leq x \leq y \leq b.$

By Schweitzer's inequality: $(x+y)\left(\frac{1}{x}+\frac{1}{y}\right) \leq \frac{(a+b)^2 \cdot 2^2}{4ab}$

$$1+\frac{x}{y}+\frac{y}{x}+1 \leq \frac{4(a^2+b^2+2ab)}{4ab}$$

$$\frac{x}{y}+\frac{y}{x}+2 \leq \frac{a^2+b^2}{ab}+2, \qquad \frac{x}{y}+\frac{y}{x} \leq \frac{a}{b}+\frac{b}{a}$$

$$\int_a^b \int_a^b \left(\frac{x}{y}+\frac{y}{x}\right) dx\, dy \leq \int_a^b \int_a^b \left(\frac{a}{b}+\frac{b}{a}\right) dx\, dy$$

$$\int_a^b \int_a^b \frac{x}{y} dx\, dy + \int_a^b \int_a^b \frac{y}{x} dx\, dy \leq \left(\frac{a}{b}+\frac{b}{a}\right)(b-a)^2$$

$$2\left(\int_a^b x\, dx\right)\left(\int_a^b \frac{1}{y} dy\right) \leq \left(\frac{a}{b}+\frac{b}{a}\right)(b-a)^2$$

$$(b^2-a^2)(\log b - \log a) \leq \left(\frac{a^2+b^2}{ab}\right)(b-a)^2$$

$$(b+a)\log\left(\frac{b}{a}\right) \leq \frac{(a^2+b^2)(b-a)}{ab}$$

$$\log\left(\frac{b}{a}\right) \leq \frac{(a^2+b^2)(b-a)}{ab} \Rightarrow \frac{b}{a} \leq e^{\frac{(a^2+b^2)(b-a)}{ab}}$$

AN.049. Solution (Adrian Popa)

$$\Omega = \sum_{n=2}^{\infty}\left(\frac{H_n}{\left(\sum_{i=1}^{n-1} H_i\right)\left(\sum_{j=1}^{n} H_j\right)}\right) =$$

$$= \sum_{n=2}^{\infty}\left(\frac{1}{H_1+H_2+\cdots+H_{n-1}} - \frac{1}{H_1+H_2+\cdots+H_{n-1}+H_n}\right) =$$

$$= \frac{1}{H_1} - \frac{1}{H_1+H_2} + \frac{1}{H_1+H_2} - \frac{1}{H_1+H_2+H_3} + \cdots + \frac{1}{H_1+H_2+\cdots+H_{n-1}} -$$

$$- \frac{1}{H_1+H_2+\cdots+H_n} + \cdots = \lim_{n\to\infty} \frac{1}{H_1} - \frac{1}{H_1+H_2+\cdots+H_n}$$

We know that $H_n = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n} \to \infty \Rightarrow H_1+H_2+\cdots+H_n \to \infty \Rightarrow$

$$\Rightarrow \frac{1}{H_1+H_2+\dots+H_n}\to 0\Rightarrow \Omega=\frac{1}{H_1}=\frac{1}{1}=1$$

AN.050. Solution (Florentin Vișescu)

$$f:\mathbb{R}\to\mathbb{R}, 2f^4(x)+2f^2(x)+2\le 3f^3(x)+3f(x), \forall x\in\mathbb{R}$$

$$2(f^4(x)+f^2(x)+1)\le 3[f^3(x)+f(x)]$$

$$2f^4(x)-3f^3(x)+2f^2(x)-3f(x)+2\le 0; \forall x\in\mathbb{R}$$

$$2f^2(x)-3f(x)+2-3\frac{1}{f(x)}+\frac{2}{f^2(x)}\le 0; \forall x\in\mathbb{R}$$

$$2\left(f^2(x)+\frac{1}{f^2(x)}\right)-3\left(f(x)+\frac{1}{f(x)}\right)+2\le 0$$

$$f(x)+\frac{1}{f(x)}=g(x),\qquad 2(g^2(x)-2)-3g(x)+2\le 0$$

$$2g^2(x)-3g(x)-2\le 0$$

$$\Delta=9+16=25; g(x)_{1;2}=\frac{3\pm 5}{4}=\left\langle \begin{matrix}2\\ -\frac{1}{2}\end{matrix}\right.$$

$$g(x)\in\left[-\frac{1}{2};2\right]$$

$$f(x)+\frac{1}{f(x)}\in\left[-\frac{1}{2};2\right];\ -\frac{1}{2}\le f(x)+\frac{1}{f(x)}\le 2\Rightarrow f(x)=1; \forall x\in\mathbb{R}$$

$$\Omega=\frac{1}{2}\left(\int_0^1 1\,dx\right)^2=\frac{1}{2}\left(x\Big|_0^1\right)^2=\frac{1}{2}$$

$$\sum_{1\le i<j\le n}\left(f\left(e+\frac{\pi i}{n}\right)f\left(e+\frac{nj}{n}\right)\right)=\frac{\left(\sum_{i=1}^n f\left(e+\frac{\pi i}{n}\right)\right)^2-\sum_{i=1}^n f^2\left(e+\frac{\pi i}{n}\right)}{2}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{1\le i<j\le n}\left(f\left(e+\frac{\pi i}{n}\right)f\left(e+\frac{\pi j}{n}\right)\right)\right)=$$

$$=\frac{1}{2}\lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^n f\left(e+\frac{\pi i}{n}\right)\right]^2-\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n}\sum_{i=1}^n f^2\left(e+\frac{\pi i}{n}\right)$$

$$=\frac{1}{2}\left(\int_0^1 f(e+\pi x)\,dx\right)^2-\frac{1}{2}\int_0^1 f^2(e+\pi x)\,dx\cdot\lim_{n\to\infty}\frac{1}{n}=\frac{1}{2}(f(e+\pi x)dx)^2$$

AN.051. Solution (Adrian Popa)

$$\Omega=\sum_{n=2}^{\infty}\left(\frac{H_n}{\left(\sum_{i=1}^{n-1}H_i\right)\left(\sum_{j=1}^{n}H_j\right)}\right)=$$

$$=\sum_{n=2}^{\infty}\left(\frac{1}{H_1+H_2+\cdots+H_{n-1}}-\frac{1}{H_1+H_2+\cdots+H_{n-1}+H_n}\right)=$$

$$=\frac{1}{H_1}-\frac{1}{H_1+H_2}+\frac{1}{H_1+H_2}-\frac{1}{H_1+H_2+H_3}+\cdots+\frac{1}{H_1+H_2+\cdots+H_{n-1}}-$$

$$-\frac{1}{H_1+H_2+\cdots+H_n}+\cdots=\lim_{n\to\infty}\frac{1}{H_1}-\frac{1}{H_1+H_2+\cdots+H_n}$$

$$\textit{We know that } H_n=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\to\infty\Rightarrow H_1+H_2+\cdots+H_n\to\infty\Rightarrow$$

$$\Rightarrow\frac{1}{H_1+H_2+\cdots+H_n}\to 0\Rightarrow\Omega=\frac{1}{H_1}=\frac{1}{1}=1$$

AN.052. Solution (Naren Bhandari)

$$\lim_{n\to\infty}\left(\sqrt[4]{(2n+3)^3+\sqrt[n+1]{(n+1)!}}-\sqrt[4]{(2n+1)^3\cdot\sqrt[n]{n!}}\right)$$

$$\sim\lim_{n\to\infty}\left((2n+3)^{\frac{3}{4}}\left(\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}\right)^{\frac{1}{4(n+1)}}\right.$$

$$\left.-(2n+1)^{\frac{3}{4}}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\right)^{\frac{1}{4}}\right)$$

$$=\sqrt[4]{\frac{8}{e}}\lim_{n\to\infty}\left(\left(n+\frac{3}{2}\right)^{\frac{3}{4}}\sqrt[4]{n+1}(n+1)^{\frac{1}{8(n+1)}}-\left(n+\frac{1}{2}\right)^{\frac{3}{4}}\sqrt[4]{n}n^{\frac{1}{8n}}\right)$$

$$=\sqrt[4]{\frac{8}{e}}\lim_{n\to\infty}\left((n+1)^{\frac{3}{4}+\frac{1}{4}}\left(1+\frac{1}{2(n+1)}\right)^{\frac{3}{4}}e^{\frac{\log(n+1)}{8(n+1)}}-n^{\frac{3}{4}+\frac{1}{4}}\left(1+\frac{1}{2n}\right)^{\frac{3}{4}}e^{\frac{\log n}{8n}}\right)$$

$$=\sqrt[4]{\frac{8}{e}}\lim_{n\to\infty}\left(n+1\left(1+\frac{3}{8(n+1)}+O\left(\frac{1}{(n+1)^2}\right)\right)-n\left(1+\frac{3}{8n}+O\left(\frac{1}{n^2}\right)\right)\right)$$

$$=\sqrt[4]{\frac{8}{e}}\lim_{n\to\infty}\left(n+1+\frac{3}{8}-n-\frac{3}{8}\right)=\sqrt[4]{\frac{8}{e}}$$

AN.053. Solution (Ravi Prakash)

$$Let\ I=\int_e^x t^{\log t}(1+2\log t)dt=\int_e^x t^{\log t}\left(\frac{2\log t}{t}\right)t\,dt+\int_e^x t^{\log t}\,dt$$

$$Let\ t^{\log t}=y\Rightarrow \log y=(\log t)(\log t)$$

$$\frac{1}{y}\frac{dy}{dt}=\frac{2(\log t)}{t}\ \therefore\ \int t^{\log t}\frac{(2\log t)}{t}dt=\int dy=y=t^{\log t}$$

Thus,

$$I=t\cdot t^{\log t}\Big]_e^x-\int_e^x t^{\log t}\,dt+\int_e^x t^{\log t}\,dt$$

$$=x\cdot x^{\log x}-e\cdot e'=x^{\log x+1}-e^2$$

$$Thus,\ the\ given\ equation\ becomes\ e^2+x^{1+\log x}-e^2=x^4$$

$$\Rightarrow x\cdot x^{\log x}=x^4\Rightarrow x^{\log x}=x^3\Rightarrow x=1\ or\ \log x=3\Rightarrow x=1\ or\ x=e^3$$

AN.054. Solution (Marian Ursarescu)

$$Let\ f:[0,1]\to\mathbb{R}, f(x)=\arctan x\,; f-\ Riemann\ integrability$$

$$Let\ \Delta_n=\left(0,\frac{1-2}{n(n+1)},\dots\frac{k(k+1)}{n(n+1)},\dots\frac{n(n+1)}{n(n+1)}=1\right)$$

$\left||\Delta_n|\right|=\frac{1}{n(n+1)}\max\limits_{1\le k\le n}\left(k(k+1)-(k-1)(k)\right)=\frac{1}{n(n+1)}\cdot 2k=\frac{2n}{n(n+1)}\left||\Delta_n|\right|\to$ $0.$ *Let* $\xi_k^n\in[x_{k-1}^n,x_k^n]$, *so that:*

$$\xi_k^n=\frac{k(n+1)}{n(n+1)}\Rightarrow\sigma_{\Delta_n}(f,\xi_k^n)=\sum_{k=1}^n f\left(\frac{k(k+1)}{n(n+1)}\right)\cdot\frac{2k}{n(n+1)}=$$

$$=\frac{2}{n(n+1)}\sum_{k=1}^n k\cdot\arctan\frac{k^2+k}{n^2+n}\Rightarrow$$

$$\Omega=\frac{1}{2}\lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{k=1}^n k\arctan\frac{k^2+k}{n^2+n}=\frac{1}{2}\int_0^1\arctan x\,dx$$

$$=\frac{1}{2}\int_0^1 x'\arctan x\,dx=\frac{1}{2}x\arctan x\Big|_0^1-\frac{1}{2}\int_0^1\frac{x}{1+x^2}dx$$

$$=\frac{\pi}{8}-\frac{1}{4}\ln(1+x^2)\Big|_0^1=\frac{\pi}{8}-\frac{1}{4}\ln 2$$

AN.055. Solution (Florentin Visescu)

$$f(t)=\arctan t\,;f'(t)=\arctan t+\frac{t}{1+t^2}$$

$$f''(t)=\frac{1}{1+t^2}+\frac{1+t^2-2t^2}{(1+t^2)^2}=\frac{1}{1+t^2}+\frac{1-t^2}{(1+t^2)^2}=\frac{1+t^2+1-t^2}{(1+t^2)^2}$$

$$=\frac{2}{(1+t^2)^2}>0\Rightarrow f\ convexe\Rightarrow f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2}$$

$$\frac{x+y}{2}\arctan\frac{x+y}{2}\le\frac{1}{2}(x\arctan x+y\arctan y)$$

$$(x+y)\arctan\frac{x+y}{2}\le x\arctan x+y\arctan y\,;\forall x,y\in\mathbb{R}$$

$$x\to f(x);y\to f(y)$$

$$\int_a^b\int_a^b[f(x)+f(y)]\arctan\frac{f(x)+f(y)}{2}\,dx\,dy$$

$$\le\int_a^b\int_a^b f(x)\arctan(x)+f(y)\arctan(y)\,dx\,dy$$

$$\int_a^b\int_a^b\big(f(x)+f(y)\big)\arctan\frac{f(x)+f(y)}{2}\,dx\,dy\le 2(b-a)\int_a^b f(x)\arctan(x)\,dx$$

AN.056. Solution (Khaled Abd Imouti)

$$\int_a^b(4\csc(2f(x))+\cos\left(\frac{\pi}{4}-f(x)\right)dx\overset{?}{\ge}5(b-a)$$

Let be the function:

$$g(x)=4\csc(2x)+\cos\left(\frac{\pi}{4}-x\right)=\frac{4}{\sin 2x}+\cos\left(\frac{\pi}{4}-x\right),x\in\left(0,\frac{\pi}{2}\right)$$

$$\lim_{\substack{x\to 0\\ >}}[g(x)]=+\infty,\lim_{\substack{x\to\frac{\pi}{2}\\ <}}[g(x)]=+\infty$$

$$g'(x)=\frac{-8\cos 2x}{\sin^2 2x}+\sin\left(\frac{\pi}{4}-x\right),\qquad g'(x)=0\Rightarrow x=\frac{\pi}{4},g\left(\frac{\pi}{4}\right)=5$$

x	0 $\qquad\qquad\qquad\qquad\frac{\pi}{4}\qquad\qquad\qquad\qquad\frac{\pi}{2}$
$g'(x)$	$----------0+++++++++$
$g(x)$	$+\infty \searrow 5 \nearrow +\infty$

So: $\int_a^b\left(4\csc(2f(x))\right)+\cos\left(\frac{\pi}{4}-f(x)\right)dx \geq \int_a^b 5\,dx = 5(b-a)$

AN.057. Solution (Remus Florin Stanca)

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{(i+n)^4}{3+(i+n)^5+\cot^{-1}(i+n)}-\sum_{i=1}^{n}\frac{1}{1+n}\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{(i+n)^5-3-(i+n)^5-\cot^1-1\,(i+n)}{(3+(i+n)^5+\cot^{-1}(i+n))(i+n)}\right)=$$

$$=-\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{3+\cot^{-1}(i+n)}{(3+(i+n)^5+\cot^{-1}(i+n))(i+n)}\right)$$

We know that $\frac{3+\cot^{-1}(i+n)}{(3+(i+n)^5+\cot^{-1}(i+n))(i+n)} \leq \frac{3+\pi}{(3+(i+)^5)(i+n)} \leq \frac{3+\pi}{(3+(n+1)^5)(n+1)} \Rightarrow$

$$\Rightarrow \sum_{i=1}^{n}\frac{3+\cot^{-1}(i+n)}{(3+(i+n)^5+\cot^{-1}(i+n))(i+n)} \leq$$

$$\leq \frac{n(3+\pi)}{(n+1)(3+(n+1)^5)}$$

$\lim\limits_{n\to\infty}\frac{n(3+\pi)}{(n+1)(3+(n+1)^5)}=0 \Rightarrow \lim\limits_{n\to\infty}\left(\sum_{i=1}^{n}\frac{(i+n)^4}{3+(i+n)^5+\cot^{-1}(i+n)}-\sum_{i=1}^{n}\frac{1}{1+n}\right)=0$ *(*)*

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i+n}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\frac{i}{n}+1}\ \textit{(1)}$$

$f(x)=\frac{1}{x+1}=$ *continuous, let* $x_k=\frac{k}{n}\Rightarrow x_{k+1}-x_k=\frac{1}{n}\Rightarrow \lim\limits_{n\to\infty}\left||\Delta_n|\right|=0\Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty}\sum_{i=1}^{n}(x_{k+1}-x_k)\,f(\zeta_k)=\int_a^b f(x)\,dx$ *where* $\zeta_k\in[x_k;x_{k+1}]$ *and*

$a=\lim\limits_{n\to\infty}x_1$ *and* $b=\lim\limits_{n\to\infty}x_n$ *and let* $\zeta_k=\frac{k}{n}\Rightarrow \lim\limits_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\frac{i}{n}+1}=\int_0^1\frac{1}{x+1}dx=$

$$\ln(2)\overset{(1)}{\Rightarrow}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{i+n}=\ln(2)\overset{(*)}{\Rightarrow}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(i+n)^4}{3+(i+n)^5+\cot^{-1}(i+n)}=\ln(2)\Rightarrow$$

$a=\lim\limits_{n\to\infty}x_1$ *and* $b=\lim\limits_{n\to\infty}x_n$ *and let* $\zeta_k=\frac{k}{n}\Rightarrow\lim\limits_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\frac{i}{n}+1}=\int_0^1\frac{1}{x+1}dx=$

$$\ln(2)\overset{(1)}{\Rightarrow}$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i+n}=\ln(2)\overset{(*)}{\Rightarrow}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(i+n)^4}{3+(i+n)^5+\cot^{-1}(i+n)}=\ln(2)\Rightarrow$$

$$\Rightarrow\Omega\overset{\infty-0}{=}\lim_{n\to\infty}n\left(\ln(2)-\sum_{i=1}^{n}\frac{(i+n)^4}{3+(i+n)^5+\cot^{-1}(i+n)}\right)\overset{\frac{0}{0}}{=}$$

$$=\lim_{n\to\infty}\frac{\ln(2)-\sum_{i=1}^{n}\frac{(i+n)^4}{3+(i+n)^5+\cot^{-1}(i+n)}}{\frac{1}{n}}\overset{Stolz-Cesaro}{\underset{\frac{0}{0}}{=}}$$

$$\overset{Stolz-Cesaro}{\underset{\frac{0}{0}}{=}}\lim_{n\to\infty}\left(\frac{(2n+2)^4}{3+(2n+2)^5+\cot^{-1}(2n+2)}+\frac{(2n+1)^4}{3+(2n+1)^5+\cot^{-1}(2n+1)}-\frac{(n+1)^4}{3+(n+1)^5+\cot^{-1}(n+1)}\right)n(n+1)=$$

$$=\lim_{n\to\infty}\left(\frac{n(2n+2)^4}{3+(2n+2)^5+\cot^{-1}(2n+2)}-\frac{1}{2}+\frac{n(2n+1)^4}{3+(2n+1)^5+\cot^{-1}(2n+1)}-\frac{1}{2}+1-\frac{n(n+1)^4}{3+(n+1)^5+\cot^{-1}(n+1)}\right)n$$

$$=\lim_{n\to\infty}\left(\frac{n(2n+2)^4}{3+(2n+2)^5+\cot^{-1}(2n+2)}-\frac{1}{2}\right)n+$$

$$+\lim_{n\to\infty}\left(\frac{n(2n+1)^4}{3+(2n+1)^5+\cot^{-1}(2n+1)}\right)n+$$

$$+\lim_{n\to\infty}\left(1-\frac{n(n+1)^4}{3+(n+1)^5+\cot^{-1}(n+1)}\right)n=-\frac{1}{2}-\frac{1}{4}+1=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

AN.058. Solution (Adrian Popa)

$$\big(1+f(x)\big)\big(1+f(y)\big)\overset{MG\le MA}{\le}\left(\frac{1+f(x)+1+f(y)}{2}\right)^2$$

$$=\left(1+\frac{f(x)+f(y)}{2}\right)^2\Rightarrow$$

$$\Rightarrow\frac{\big(1+f(x)\big)\big(1+f(y)\big)}{\left(1+\frac{f(x)+f(y)}{2}\right)^2}\le 1\Rightarrow\ln\frac{\big(1+f(x)\big)\big(1+f(y)\big)}{\left(1+\frac{f(x)+f(y)}{2}\right)^2}<0$$

$$a<b\Rightarrow\int_a^b\int_a^b\ln\frac{\big(1+f(x)\big)\big(1+f(y)\big)}{\left(1+\frac{f(x)+f(y)}{2}\right)^2}dx\,dy<0\quad (1)$$

$$\left(\int_a^b(f(x)\cdot 1)dx\right)^2\overset{C.B.S.}{\le}\int_a^b f^2(x)\,dx\cdot\int_a^b 1\,dx=x\Big|_a^b\cdot\int_a^b f^2(x)\,dx=$$

$$=(b-a)\int_a^b f^2(x)\,dx\Rightarrow(b-a)\int_a^b f^2(x)\,dx-\left(\int_a^b f(x)\,dx\right)^2\ge 0\quad (2)$$

From (1) and (2) $\Rightarrow\int_a^b\int_a^b\ln\frac{\big(1+f(x)\big)\big(1+f(y)\big)}{\left(1+\frac{f(x)+f(x)}{2}\right)^2}dxdy\le(b-a)\int_a^b f^2(x)\,dx-\left(\int_a^b f(x)\,dx\right)^2$

AN.059. Solution (Adrian Popa) *Let be* $\underset{f'(x)=e^x+xe^x}{\underset{\Downarrow}{f(x)=xe^x}}$ *and* $\underset{f'(x)=e^x}{\underset{\Downarrow}{g(x)=e^x}}$

We apply Cauchy's theorem to f *and* g *on the interval* $[a,b]$.

$$(\exists)c\in(a,b):\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}\Rightarrow\frac{be^b-ae^a}{e^b-e^a}=\frac{e^c+ce^c}{e^c}\Rightarrow$$

$$\Rightarrow\frac{be^b-ae^a}{e^b-e^a}-1=\frac{e^c+ce^c}{e^c}-1=\frac{e^c+ce^c-e^c}{e^c}=\frac{ce^c}{e^c}=c$$

$$a<c<b\Rightarrow a<\left(\frac{be^b-ae^a}{e^b-e^a}-1\right)<b\quad (1)$$

Let be $\underset{f_1'(x)=\frac{e^x-xe^x}{e^{2x}}}{\underset{\Downarrow}{f_1(x)=\frac{x}{e^x}}}$ *and* $\underset{g_1'(x)=\frac{-e^x}{e^{2x}}}{\underset{\Downarrow}{g_1(x)=\frac{1}{e^x}}}$

We apply Cauchy's theorem to f_1 *and* g_1 *on the interval* $[a,b]\Rightarrow$

$$\frac{f_1(b)-f_1(a)}{g_1(b)-g_1(a)}=\frac{f_1(c_1)}{g_1(c_1)},\qquad a<c_1<b$$

$$\frac{\frac{b}{e^b}-\frac{a}{e^a}}{\frac{1}{e^b}-\frac{1}{e^a}}=\frac{\frac{e^{c_1}-ce^{c_1}}{e^{2c_1}}}{-\frac{e^{c_1}}{e^{2c_1}}}\Rightarrow\frac{\frac{be^a-ae^b}{e^a\cdot e^b}}{\frac{e^a-e^b}{e^a\cdot e^b}}=\frac{ce^{c_1}-e^{c_1}}{e^{c_1}}\Rightarrow\frac{ae^b-be^a}{e^b-e^a}$$

$$=\frac{ce^{c_1}-e^{c_1}}{e^{c_1}}$$

$$\Rightarrow\frac{ae^b-be^a}{e^b-e^a}+1=\frac{c_1e^{c_1}-e^{c_1}}{e^{c_1}}+1=c_1\Rightarrow\frac{ae^b-be^a}{e^b-e^a}<b\quad(2)$$

Multiplying (1) and (2) ⇒ the relationship from enunciation.

AN.060. Solution (Lazaros Zachariadis)

$f(x)=e^{x^2}, x>0, f''(x)=e^{x^2}\cdot(4x^2+2)>0, \forall x>0\Rightarrow f$ *convex, so:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx\overset{Hermite}{\underset{Hadamard}{\geq}} f\left(\frac{a+b}{2}\right)=e^{\frac{(a+b)^2}{4}}$$

$$Thus\ LHS>\sum e^{\frac{(a+b)^2}{4}}\overset{AM-GM}{>}3\cdot\sqrt[3]{e^{\frac{(a+b)^2}{4}}\cdot e^{\frac{(b+c)^2}{4}}\cdot e^{\frac{(c+d)^2}{4}}}$$

$$=3\cdot\sqrt[3]{e^{\frac{(a+b)^2}{4}+\frac{(b+c)^2}{4}+\frac{(c+d)^2}{4}}}\overset{Andreescu}{>}3\sqrt[3]{e^{\frac{(a+b+b+c+c+d)^2}{4+4+4}}}$$

$$=3\cdot\sqrt[36]{e^{(a+2b+2c+d)}}=RHS$$

AN.061. Solution (Igor Soposki)

$$\arctan\left(\sqrt{\frac{x+3}{x+2}}\right)=t$$

$$\frac{1}{1+\left(\sqrt{\frac{x+3}{x+2}}\right)^2}\cdot\frac{1}{2\sqrt{\frac{x+3}{x+2}}}\cdot\frac{x+2-(x+3)}{(x+2)^2}dx=dt$$

$$-\frac{1}{1+\frac{x+3}{x+2}}\cdot\frac{1}{2\sqrt{\frac{x+3}{x+2}}}\cdot\frac{1}{(x+2)^2}dx=dt,$$

$$-\frac{1}{\frac{(2x+5)}{x+2}}\cdot\frac{1}{2\sqrt{\frac{x+3}{x+2}}}\cdot\frac{1}{(x+2)^2}dx=dt$$

$$-\frac{1}{(2x+5)}\cdot\frac{1}{2\sqrt{(x+2)^2\left(\frac{x+3}{x+2}\right)}}dx=dt$$

$$-\frac{1}{(2x+5)}\cdot\frac{1}{2\sqrt{(x+2)(x+3)}}dx=dt$$

$$\frac{dx}{(2x+5)\sqrt{(x+2)(x+3)}}=-2dt\Rightarrow I=\int\frac{-2\,dt}{(4t-\pi)^2}=-2\int\frac{dt}{(4t-\pi)^2}=$$

$$\Rightarrow\left\{\begin{matrix}u=4t-\pi\\du=4dt\end{matrix}\right\}=-2\int\frac{1}{4}\cdot\frac{du}{u^2}=-\frac{1}{2}\int\frac{du}{u^2}=-\frac{1}{2}\int u^{-2}\,du=$$

$$=-\frac{1}{2}\cdot\frac{u^{-2+1}}{-2+1}=\frac{u^{-1}}{2}=\frac{1}{2u}=\frac{1}{2(4t-\pi)}=\frac{1}{2\left(4\arctan\sqrt{\frac{x+3}{x+2}}-\pi\right)}+c$$

AN.062.

$$\frac{5f(x)+3}{f(x)+7}\leq\frac{f(x)+1}{2}\Leftrightarrow$$

$$10f(x)+6\leq$$

$$\leq f^2(x)+8f(x)+7\Leftrightarrow f^2(x)-2f(x)+1\geq 0\Leftrightarrow(f(x)-1)^2\geq 0$$

$$\frac{5f(x)+3}{f(x)+7}\geq\frac{f(x)+1}{2}\quad(1)$$

$$\frac{6f(x)+4}{f(x)+9}\leq\frac{f(x)+1}{2}\Leftrightarrow 12f(x)+8\leq f^2(x)+10f(x)+9$$

$$\Leftrightarrow f^2(x)-2f(x)+1\geq 0\Leftrightarrow(f(x)-1)^2\geq 0$$

$$\frac{6f(x)+4}{f(x)+9}\leq\frac{f(x)+1}{2}\quad(2)$$

$$\frac{7f(x)+5}{f(x)+11}\leq\frac{f(x)+1}{2}\Leftrightarrow 14f(x)+10\leq f^2(x)+12f(x)+11$$

$$\Leftrightarrow f^2(x)-2f(x)+1\geq 0\Leftrightarrow(f(x)-1)^2\geq 0$$

$$\frac{7f(x)+5}{f(x)+11}\leq\frac{f(x)+1}{2}\quad(3)$$

By adding (1); (2); (3):

$$\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\leq\frac{3}{2}(f(x)+1)$$

$$\int_a^b \left(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\right)dx \le$$

$$\le \frac{3}{2}\left(\int_a^b f(x)\,dx+\int_a^b dx\right)=\frac{3}{2}\big(5(b-a)+(b-a)\big)=9(b-a)$$

Equality holds for $a=b$.

AN.063. *By Huygens' inequality:*

$$\left(H_n^{(2p)}+H_n^{(2q)}\right)\left(H_n^{(2r)}+H_n^{(2s)}\right)\ge\left(\sqrt{H_n^{(2p)}\cdot H_n^{(2r)}}+\sqrt{H_n^{(2q)}\cdot H_n^{(2s)}}\right)^2=$$

$$=\left(\sqrt{\left(\sum_{k=1}^n\frac{1}{k^{2p}}\right)\left(\sum_{k=1}^n\frac{1}{k^{2r}}\right)}+\sqrt{\left(\sum_{k=1}^n\frac{1}{k^{2q}}\right)\left(\sum_{k=1}^n\frac{1}{k^{2s}}\right)}\right)^2\ge$$

$$\overset{Cauchy-Schwarz}{\ge}\left(\sqrt{\left(\sum_{k=1}^n\frac{1}{k^p}\cdot\frac{1}{k^r}\right)^2}+\sqrt{\left(\sum_{k=1}^n\frac{1}{k^q}\cdot\frac{1}{k^s}\right)^2}\right)^2=$$

$$=\left(\sum_{k=1}^n\frac{1}{k^{p+r}}+\sum_{k=1}^n\frac{1}{k^{q+s}}\right)^2=\left(H_n^{(p+r)}+H_n^{(q+s)}\right)^2$$

AN.064. Solution (Ali Jaffal)

We have $\varphi(t)=e^t$ is continuous and convex function on $\mathbb{R}$. Since $\varphi''(t)=e^t>0$ for all $x\in\mathbb{R}$ then

$$\varphi\left(\frac{1}{b-a}\int_a^b\log\left(\frac{\sin x}{x}\right)dx\right)\le\frac{1}{b-a}\int_a^b\varphi\left(\log\left(\frac{\sin x}{x}\right)\right)dx$$

But $\varphi\left(\log\left(\frac{\sin x}{x}\right)\right)=\frac{\sin x}{x}$

So, $\varphi\left(\frac{1}{b-a}\int_a^b\log\left(\frac{\sin x}{x}\right)dx\right)\le\frac{1}{b-a}\int_a^b\frac{\sin x}{x}dx$

We have

$$(b-a)^2+(b-a)\left(\int_a^b \frac{x}{\sin x}dx\right)e^{\frac{1}{b-a}\int_a^b \log\left(\frac{\sin x}{x}\right)}dx \le$$

$$\le (b-a)^2+(b-a)\left(\int_a^b \frac{x}{\sin x}dx\right)\left(\frac{1}{b-a}\right)\int_a^b \frac{\sin x}{x}dx \le$$

$$\le (b-a)^2+\left(\int_a^b \frac{x}{\sin x}\right)\left(\int_a^b \frac{\sin x}{x}dx\right)$$

$$But\ (b-a)^2=\left(\int_a^b 1\,dx\right)^2=\left(\int_a^b \left(\frac{x}{\sin x}\right)^{\frac{1}{2}}\times\left(\frac{\sin x}{x}\right)^{\frac{1}{2}}dx\right)^2$$

$$\le\left(\int_a^b \frac{x}{\sin x}dx\right)\left(\int_a^b \frac{\sin x}{x}dx\right)-CBS$$

$$Therefore:\ (b-a)^2+\left(\int_a^b \frac{x}{\sin x}\right)\left(\int_a^b \frac{\sin x}{x}dx\right)\le 2\left(\int_a^b \frac{\sin x}{x}dx\right)\left(\int_a^b \frac{x}{\sin x}dx\right)$$

$$Therefore\ (b-a)^2+(b-a)\left(\int_a^b \frac{x}{\sin x}dx\right)e^{\frac{1}{b-a}\int_a^b \log\left(\frac{\sin x}{a}\right)dx}\le$$

$$2\int_a^b \frac{\sin x}{x}dx\int_a^b \frac{x}{\sin x}dx$$

AN.065. Solution (Ali Jaffal)

$$We\ have\ by\ Bernoulli's\ inequality:\ 1\le(1+n)^\alpha\le 1+\alpha n$$

$$for\ all\ n>0\ and\ 0\le\alpha\le 1\ then\ 1\le\left(1+\frac{1}{n\phi^k}\right)^{\frac{1}{k}}\le 1+\frac{1}{nk\phi^k}$$

$$Let\ \Omega_n=\sum_{k=1}^{k=n}\left(1+\frac{1}{n\phi^k}\right)^{\frac{1}{k}}-n.\ So,\ 0\le\Omega_n\le\frac{1}{n}\sum_{k=1}^{k=n}\frac{1}{k\phi^k}$$

$$but\ \frac{1}{k}\le 1\ then\ 0\le\Omega_n\le\frac{1}{n}\sum_{k=1}^{k=n}\frac{1}{\phi^k}\ we\ know\ that\ \sum_{k=1}^{k=n}\left(\frac{1}{\phi}\right)^k=\frac{1}{\phi}\cdot\frac{1-\left(\frac{1}{\phi}\right)^n}{1-\frac{1}{\phi}}$$

$$and\ \lim_{n\to\infty}\left(\frac{1}{\phi}\right)^n=0\ since\ 0<\frac{1}{\phi}<1\ then\ 0\le\lim_{n\to\infty}\Omega_n\le 0$$

$$therefore\ \lim_{n\to\infty}\Omega_n=0$$

AN.066. Solution (Ali Jaffal)

$$\sum_{k=0}^{n-1}(n^2-nk)^2\cdot\binom{2n}{k}=n^2\binom{2n}{0}+(n^2-n)^2\cdot\binom{2n}{1}+(n^2-2n)^2\cdot\binom{2n}{2}+$$

$$+(n^2-3n)^2\cdot\binom{3n}{3}+\dots+\left(n^2-n(n-1)\right)^2\cdot\binom{2n}{n-1}$$

$$=n^2\binom{2n}{0}+n^2(n-1)^2\binom{2n}{1}+n^2(n-2)^2\binom{2n}{2}+n^2(n-3)^2\binom{2n}{3}+$$

$$+\dots+n^2\cdot\left(n-(n-1)\right)^2\cdot\binom{2n}{n-1}$$

$$=n^2\cdot\left[\binom{2n}{0}+(n-1)^2\binom{2n}{1}+(n-2)^2\binom{2n}{2}+(n-3)^2\binom{2n}{3}+\dots\right.$$

$$\left.+(1)^2\binom{2n}{n-1}\right]$$

$$\le n^2\left[\binom{2n}{n-1}[(n-1)^2+(n-2)^2+(n-3)^2+\dots+1]\right]$$

$$\frac{1}{n^2}\sqrt[n]{\sum_{k=0}^{n-1}(n^2-nk)^2\binom{2n}{k}}\le n^{\frac{2}{n}-2}\sqrt[n]{\binom{2n}{n-1}\cdot\frac{n(n+1)(2n+1)}{6}}$$

$$\textit{Degree of}\binom{2n}{n-1}\cdot\frac{n(n+1)(2n+1)}{6}\textit{ is equal to }\frac{3}{n}$$

$$\textit{So: degree of } n^{\frac{2}{n}-2}\cdot\sqrt[n]{\binom{2n}{n-1}\cdot\frac{n(n+1)(2n+1)}{6}}\textit{ is } n^{\frac{2}{n}-2}\cdot n^{\frac{3}{n}}=n^{\frac{5}{n}-2}$$

$$\lim_{n\to+\infty}\left(\frac{1}{n^2}\cdot\sqrt[n]{\sum_{k=0}^{n-1}(n^2-nk)^2\binom{2n}{k}}\right)\le\lim_{n\to+\infty}\left(\frac{n^{\frac{5}{n}}}{n^2}\right)\quad (*)$$

$$\textit{But }\lim_{n\to+\infty}\left(\frac{n^{\frac{5}{n}}}{n^2}\right)=\lim_{n\to+\infty}\left(\frac{e^{\frac{5}{n}\ln(n)}}{n^2}\right)=\lim_{n\to+\infty}\left(\frac{5-\frac{\ln(n)}{n}}{n^2}\right)=0$$

$$\textit{So: }\lim_{n\to+\infty}\sum_{k=0}^{n-1}(n^2-nk)^2\binom{2n}{k}=0$$

AN.067. Solution (Tran Hong)

$$\textit{Let } f(x)=u;\ f(y)=v;\ f(z)=t\Rightarrow u,v,t>0$$

$$\textit{We choose } u,v,t>0 \textit{ such that } u+v+t=1$$

Using Jensen's Inequality:

$$u\ln(t+v)+v\ln(u+t)+t\cdot\ln(u+v)\le\ln(2[uv+vt+tu])$$

$$\Rightarrow (u+v)^t\cdot(u+t)^u\cdot(t+v)^u\le e^{\ln(2[uv+vt+tu])}=2(uv+vt+tu)\le$$

$$\le 2\cdot\frac{(u+v+t)^2}{3}=\frac{2}{3}(u+v+t)$$

$$\Rightarrow \int_0^1\int_0^1\int_0^1 \big(f(x)+f(y)\big)^{f(z)}\big(f(y)+f(z)\big)^{f(x)}\big(f(x)+f(z)\big)^{f(y)}\,dx\,dy\,dz\le$$

$$\le\frac{2}{3}\int_0^1\int_0^1\int_0^1[f(x)+f(y)+f(z)]dx\,dy\,dz$$

$$=\frac{2}{3}\left[\left\{\int_0^1 f(x)\,dx\int_0^1 dy\int_0^1 dz\right\}+\left\{\int_0^1 f(y)\,dy\int_0^1 dz\int_0^1 dx\right\}\right.$$

$$\left.+\left\{\int_0^1 f(z)\,dz\int_0^1 dx\int_0^1 dy\right\}\right]$$

$$=\frac{2}{3}\left[\int_0^1 f(x)\,dx+\int_0^1 f(y)\,dy+\int_0^1 f(z)\,dz\right]$$

$$=\frac{2}{3}\cdot 3\cdot\int_0^1 f(x)\,dx=2\int_0^1 f(x)\,dx\Rightarrow K=2\Rightarrow\Omega=2$$

AN.068. ***Solution(Samir HajAli)***

We have: $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}=f(x)$

$\Rightarrow\sum_{n=1}^{\infty}n\cdot x^{n-1}=\frac{1}{(1-x)^2}$ *because the series converge uniformly when* $|x|<1$

$\Rightarrow\sum_{n=0}^{\infty}(n+1)\cdot x^n=\frac{1}{(1-x)^2}$ *(Similarly)*

We find: $\sum_{n=1}^{\infty}n(n+1)\,x^{n-1}=\frac{2(1-x)}{(1-x)^4}=\frac{1\cdot 2}{(1-x)^3}=\frac{2!}{(1-x)^3}$

Then: $\sum_{n=0}^{\infty}(n+1)(n+2)\,x^n=\frac{1\cdot 2}{(1-x)^3};|x|<1$

Therefore: $\sum_{n=1}^{\infty}n(n+1)(n+2)x^{n-1}=\frac{1\cdot 2\cdot 3}{(1-x)^4}=\frac{3!}{(1-x)^4}$

Similarly step by step we find:

$$\sum_{n=1}^{\infty} n(n+1)(n+2)\dots(n+a-1)x^{n-1} = \frac{a!}{(1-x)^{a+1}}; |x|<1$$

Hence: $f(x) = \sum_{n=1}^{\infty} \prod_{k=0}^{a-1}(n-k)x^{n-1} = \frac{a!}{(1-x)^{a+1}}$

$$\lim_{b\to\infty} \Omega(a) = \lim_{b\to\infty} f\left(\frac{1}{b}\right) = \lim_{b\to\infty} \frac{a!}{\left(1+\frac{1}{b}\right)^{a+1}} = a!$$

and: $\Omega = \sum_{a=2}^{\infty} \frac{1}{\Omega(a)} = \sum_{a=2}^{\infty} \frac{1}{a!} = \sum_{a=0}^{\infty} \frac{1}{a!} - 2 = e - 2 \Rightarrow \Omega = e - 2$

AN.069. Solution (Naren Bhandari)

$$I(n) = \int_0^1 e^{\frac{x^2}{n}}\,dx = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^2}{n}\right)^k\right)dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)k!\,n^k}$$

and hence we have the limit:

$$\lim_{n\to\infty} \left(I(n)\right)^n = \exp\left(\lim_{n\to\infty} n\log\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)k!\,n^k}\right)\right)$$

$$= \exp\left(\lim_{n\to\infty} n\log\left(1+\sum_{k=1}^{\infty} \frac{1}{(2k+1)k!\,n^k}\right)\right)$$

$$= \exp\left(\lim_{n\to\infty} n\left((-1)^{k+1}\sum_{k=1}^{\infty} \frac{1}{(2k+1)n^k}\right)^k\right)$$

$$= \exp\left(\lim_{n\to\infty} n\left(\frac{1}{3n} + \frac{1}{2!\,5n^2} + \cdots\right)\right) - O(n^2) = \sqrt[3]{e}$$

AN.070. Solution (Chris Kyriazis)

It holds that $\sqrt{\frac{f^2(x)+f^2(y)}{2}} \le \frac{f(x)+f(y)}{2}, \forall x,y \in \mathbb{R}$ *and*

$$\sqrt{f(x)f(y)} \le \frac{f(x)+f(y)}{2}, \forall x,y \in \mathbb{R}$$

So, adding those inequalities, we have:

$$\sqrt{\frac{f^2(x)+f^2(y)}{2}} + \sqrt{f(x)f(y)} \le f(x)+f(y), \forall x,y \in \mathbb{R}$$

Integrating from a to b, we have:

$$\int_a^b\int_a^b\left[\sqrt{\frac{f^2(x)+f^2(y)}{2}}+\sqrt{f(x)f(y)}\right]dx\,dy\le\int_a^b\int_a^b[f(x)+f(y)]\,dx\,dy=$$

$$=2(b-a)\int_a^b f(x)\,dx\ \ (*)$$

$$(*)\int_a^b\left\{\int_a^b[f(x)+f(y)]\,dx\right\}dy=\int_a^b\left[\int_a^b f(x)\,dx+f(y)(b-a)\right]dy=$$

$$=\int_a^b f(x)dx\,(b-a)+\int_a^b f(y)\,dy(b-a)=2(b-a)\int_a^b f(x)\,dx$$

AN.071. Solution (Ali Jaffal)

$$Let\ \varphi(x)=\arctan x-\log x,\ where\ x\in\,]0,+\infty[$$

$$\varphi''(x)=-\frac{2x}{(1+x^2)^2}+\frac{1}{x^2}=\frac{x^4-2x^3+2x^2+1}{x^2(1+x^2)^2}$$

$$Let\ \psi(x)=x^4-2x^3+2x^2+1, x\in\mathbb{R}$$

$\psi'(x)=2x(2x^2-3x+2)$ *has the same sign as* x *since* $2x^2-3x+2>0$ *for all*

$x\in\mathbb{R}$. *We have* $2x^2-3x+2=2\left[x^2-\frac{3x}{2}+1\right]=2\left[\left(x-\frac{3}{4}\right)^2+\frac{7}{16}\right]>0$. *So,*

x	$-\infty$ 0 $+\infty$
$\psi'(x)$	$-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,0\,+\,+\,+\,+\,+\,+\,+\,+\,+\,+$
$\psi(x)$	↘ 1 ↗

then $\psi(x)>0$ *for all* $x\in\mathbb{R}$ *then* $\varphi''(x)>0$ *and* φ *is convex.*

By Jensen's inequality: $\varphi\left(\frac{ax+by}{a+b}\right)\le\frac{a\varphi(x)+b\varphi(y)}{a+b}; x>0, y>0$

$$\int_a^b\int_a^b\varphi\left(\frac{ax+by}{a+b}\right)dx\,dy\le\frac{a(b-a)}{a+b}\int_a^b(\arctan x-\log x)\,dx+\frac{a(b-a)}{a+b}\int_a^b\varphi(y)\,dy$$

$$\le\left(\frac{a(b-a)}{a+b}+\frac{a(b-a)}{a+b}\right)\left(\int_a^b(\arctan x-\log x)\,dx\right)$$

$$\le(b-a)\int_a^b\arctan x\,dx-(b-a)\int_a^b\log x\,dx$$

$$\textit{Therefore: } \int_a^b \int_a^b \arctan\left(\frac{ax+by}{a+b}\right) + (b-a)\int_a^b \log x\,dx \le$$

$$\le (b-a)\int_a^b \arctan x\,dx + \int_a^b \int_a^b \log\left(\frac{ax+by}{a+b}\right) dx\,dy$$

AN.072. Solution (Ali Jaffal)

$$\textit{Let } f_n(x) = (\sin x)^{\frac{1}{n}}; n \ge 2,\ f_n'(x) = \frac{1}{n}\cos x\,(\sin x)^{\frac{1}{n}-1}$$

$$f_n''(x) = -\frac{1}{n}(\sin x)^{\frac{1}{n}} + \frac{1}{n}\left(\frac{1}{n}-1\right)\cos^2 x\,(\sin x)^{\frac{1}{n}-2}$$

we have $f_n''(x) \le 0$ *for all* $x \in [0,\pi]$ *then* f_n *is concave on* $[0,\pi]$

Let $x, y, z \in [0,\pi]$, *so, by Jensen's inequality we have:*

$$f_n\left(\frac{px+qy+rz}{p+q+r}\right) \ge \frac{pf(x)+qf(y)+rf(z)}{p+q+r}$$

$$\int_a^b \int_a^b \int_a^b f_n\left(\frac{px+qy+rz}{p+q+r}\right) dx\,dy\,dz \ge \int_a^b \int_a^b \int_a^b \frac{pf(x)+qf(y)+rf(z)}{p+q+r} dx\,dy\,dz$$

$$\int_a^b \int_a^b \int_a^b \frac{pf_n(x)+qf_n(y)+rf_n(z)}{p+q+r} dx\,dy\,dz =$$

$$\frac{p}{p+q+r}\int_a^b dy \int_a^b dy \int_a^b f_n(x)\,dx + \frac{q}{p+q+r}\int_a^b dz \int_a^b dx \int_a^b f_n(y)\,dy +$$

$$+\int_a^b dy \int_a^b dx \int_a^b \frac{rf_n(z)}{p+q+r} dz =$$

$$\frac{p}{p+q+r}(b-a)^2 \int_a^b f_n(n)\,dx + \frac{q}{p+q+r}(b-a)^2 \int_a^b f_n(x)\,dx$$

$$+\frac{(b-a)^2 r}{p+q+r}\int_a^b f_n(x)\,dx =$$

$$\frac{(p+q+r)(b-a)^2}{p+q+r}\int_a^b f_n(x)\,dx = (b-a)^2 \int_a^b f_n(x)\,dx$$

So,

$$\int_a^b\int_a^b\int_a^b f_n\left(\frac{px+qy+rz}{p+q+r}\right)dx\,dy\,dz \ge (b-a)^2\int_a^b f_n(x)\,dx$$

Therefore:

$$\int_a^b\int_a^b\int_a^b \sqrt[n]{\sin\left(\frac{px+qy+rz}{p+q+r}\right)} \ge (b-a)^2\int_a^b \sqrt[n]{\sin x}\,dx$$

AN.073. Solution (Ali Jaffal)

$$\text{Let } S_n = \frac{1}{n}\sum_{i=1}^{i=n} x_i\,;\, h_n = \frac{n}{\sum_{i=1}^{i=n}\frac{1}{x_i}} \text{ and } P_n = \sqrt[n]{\prod_{i=1}^{i=n} x_i}$$

By GM-AM inequality: $h_n < P_n < S_n$ *then:* $(h_n)^n < (P_n)^n < (S_n)^n$

We have $\lim_{n\to\infty}(S_n^{n-1}h_n) = \lim_{n\to\infty} S_n\cdot(h_n)^{n-1} = \omega$. *So,*

$$(n-1)\log S_n + \log h_n = \log\omega + \varphi(n) \quad (*)$$

$$(n-1)\log h_n + \log S_n = \log\omega + \varphi_2(n) \quad (**)$$

where $\lim_{n\to\infty}\varphi(n) = \lim_{n\to\infty}\varphi_2(n) = 0$

by (*) and (**) we have: $n\log S_n + n\log h_n = \log\omega^2 + \varphi_1(n) + \varphi_2(n)$

So: $\lim_{n\to\infty} n\log(S_n\cdot h_n) = \log\omega^2$ *then* $\lim_{n\to\infty}(S_nh_n)^n = \omega^2$

And also, by (*) and (**) we have:

$$\log S_n - (n-1)^2\log S_n = (\log\omega)(2-n) + \varphi_2(n) - (n-1)\varphi_1(n)$$

then $n(2-n)\log S_n = (2-n)\log\omega + \varphi_2(n) - (n-1)\varphi_1(n)$

$$n\log S_n = \log\omega + \frac{\varphi_2(n)}{2-n} + \frac{1-n}{2-n}\varphi_i(n)$$

So, $\lim_{n\to+\infty} n\log S_n = \log\omega$ *then* $\lim_{n\to\infty}(S_n)^n = \omega$ *but* $\lim_{n\to\infty}(S_nh_n)^n = \omega^2$ *then*

$\lim_{n\to\infty}(h_n)^n = \omega$. *we have* $(h_n)^n \le (P_n)^n \le (S_n)^n$

then by Sandwich theorem: $\lim_{n\to\infty}(h_n)^n \le \lim_{n\to\infty}(P_n)^n \le \lim_{n\to\infty}(S_n)^n$ *then*

$$\lim_{n\to\infty}(P_n^n) = \omega \text{ and } \lim_{n\to\infty}\prod_{i=1}^{i=n} x_i = \omega$$

AN.074. *Let be* $f:[0,1]\to\mathbb{R}; f(x)=\left(\sqrt{\frac{a}{b}}\right)^x+\left(\sqrt{\frac{b}{a}}\right)^x$*;*

$$f'(x)=0\Rightarrow\left(\sqrt{\frac{a}{b}}\right)^x\log\frac{b}{a}+\left(\sqrt{\frac{b}{a}}\right)^x\log\frac{b}{a}=0\Rightarrow$$

$$\log\left(\frac{a}{b}\right)\left(\left(\sqrt{\frac{a}{b}}\right)^x-\left(\sqrt{\frac{b}{a}}\right)^x\right)=0\Rightarrow\left(\frac{a}{b}\right)^{\frac{x}{2}}=\left(\frac{a}{b}\right)^{-\frac{x}{2}}\Rightarrow x=0$$

$$f'(x)>0; f\ \textit{increasing;}\ f(0)=2; f(1)=\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}$$

$$\left(\sqrt{\frac{a}{b}}\right)^x+\left(\sqrt{\frac{b}{a}}\right)^x\le\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}};(\forall)x\in[0,1]\quad (1)$$

Replacing in (1) x *by* $\frac{\sin x}{x}<1$ *and* $\frac{x}{\tan x}<1;(\forall)x\in\left(0,\frac{\pi}{2}\right)$

$$\left(\sqrt{\frac{a}{b}}\right)^{\frac{\sin x}{x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{\sin x}{x}}\le\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\quad (2)$$

$$\left(\sqrt{\frac{a}{b}}\right)^{\frac{x}{\tan x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{x}{\tan x}}\le\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\quad (3)$$

By multiplying (2); (3):

$$\left(\left(\sqrt{\frac{a}{b}}\right)^{\frac{\sin x}{x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{\sin x}{x}}\right)\left(\left(\sqrt{\frac{a}{b}}\right)^{\frac{x}{\tan x}}+\left(\sqrt{\frac{b}{a}}\right)^{\frac{x}{\tan x}}\right)\le\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2$$

AN.075. *Let be* $f:[0,1]\to\left[\frac{\pi}{4},\infty\right); f(x)=\tan^{-1}(1+x^2)$

$$f'(x)=\frac{2x}{1+(1+x^2)^2}>0; f\ \textit{increasing}$$

$$(\tan^{-1}(1+x^2))^k\ge(\tan^{-1}(1+0^2))^k=\left(\frac{\pi}{4}\right)^k$$

$$(\tan^{-1}(1+x^2))^{m-k}\cdot(\tan^{-1}(1+x^2))^k\ge\left(\frac{\pi}{4}\right)^k(\tan^{-1}(1+x^2))^m$$

$$\left(\frac{4}{\pi}\right)^k\cdot(\tan^{-1}(1+x^2))^m\ge(\tan^{-1}(1+x^2))^{m-k}\ \ \textit{(1)}$$

Analogous: $\left(\frac{4}{\pi}\right)^l\cdot(\tan^{-1}(1+x^2))^n\ge(\tan^{-1}(1+x^2))^{n-l}$ *(2)*

By multiplying (1); (2):

$$\left(\frac{4}{\pi}\right)^{k+l}(\tan^{-1}(1+x^2))^{m+n}\ge\frac{(\tan^{-1}(1+x^2))^{m+n}}{(\tan^{-1}(1+x^2))^{k+l}}$$

$$\left(\frac{4}{\pi}\right)^{k+l}\int_0^1(\tan^{-1}(1+x^2))^{m+n}\,dx\ge\int_0^1\frac{(\tan^{-1}(1+x^2))^{m+n}}{(\tan^{-1}(1+x^2))^{k+l}}\,dx$$

AN.076.

$$\frac{1+2+\dots+n}{n}\overset{AM-GM}{>}\sqrt[n]{1\cdot2\cdot3\cdot\ldots\cdot n}\Rightarrow\frac{n(n+1)}{2n}>\sqrt[n]{n!}$$

$$\frac{n+1}{2}>\sqrt[n]{n!}\Rightarrow\left(\frac{n+1}{2}\right)^n>n!\Rightarrow\frac{1}{n!}>\left(\frac{2}{n+1}\right)^n\ \ \textit{(1)}$$

$$\binom{n}{i}=n!\cdot\frac{1}{i!}\cdot\frac{1}{(n-1)!}\overset{(1)}{>}n!\cdot\left(\frac{2}{i+1}\right)^i\cdot\left(\frac{2}{n-i+1}\right)^{n-i}=$$

$$=n!\cdot\frac{2^i}{(i+1)^i}\cdot\frac{2^{n-i}}{(n-i+1)^{n-i}}=n!\cdot2^n\cdot\frac{1}{(i+1)^i}\cdot\frac{1}{(n-i+1)^{n-i}}$$

$$\binom{n}{i}>n!\cdot2^n\cdot\frac{1}{(i+1)^i}\cdot\frac{1}{(n-i+1)^{n-i}}$$

$$\sum_{i=0}^n\binom{n}{i}>\sum_{i=0}^n\left(n!\cdot2^n\cdot\frac{1}{(i+1)^i}\cdot\frac{1}{(n-i+1)^{n-i}}\right)$$

$$2^n>n!\cdot2^n\cdot\sum_{i=0}^n\left(\frac{1}{(i+1)^i}\cdot\frac{1}{(n-i+1)^{n-1}}\right)>0$$

$$0<\sum_{i=0}^n\left(\frac{1}{(i+1)^i}\cdot\frac{1}{(n-i+1)^{n-1}}\right)<\frac{1}{n!}$$

$\lim\limits_{n\to\infty}\frac{1}{n!}=0$. *By sandwich's theorem:*

$$\lim_{n\to\infty}\sum_{i=0}^n\left(\frac{1}{(i+1)^i}\cdot\frac{1}{(n-i+1)^{n-1}}\right)=0$$

AN.077.

$$\sqrt[n]{\left(1+\frac{\sqrt[n]{e}}{e}\right)\cdot \underbrace{1\cdot 1\cdot \ldots \cdot 1}_{for\ "n-1"\ times}} \overset{AM-GM}{<} \frac{1+\frac{\sqrt[n]{e}}{e}+\underbrace{1+1+\cdots+1}_{for\ "n-1"\ times}}{n}=$$

$$=\frac{1+\frac{\sqrt[n]{e}}{e}+n-1}{n}=\frac{n+\frac{\sqrt[n]{e}}{e}}{n},\ 0<\sqrt[n]{1+\frac{\sqrt[n]{e}}{e}}<\frac{n+\frac{\sqrt[n]{e}}{e}}{n}\quad (1)$$

$$\sqrt[n]{\left(1-\frac{\sqrt[n]{e}}{e}\right)\cdot \underbrace{1\cdot 1\cdot \ldots \cdot 1}_{for\ "n-1"\ times}} \overset{AM-GM}{<} \frac{1-\frac{\sqrt[n]{e}}{e}+\underbrace{1+1+\cdots+1}_{for\ "n-1"\ times}}{n}$$

$$=\frac{1-\frac{\sqrt[n]{e}}{e}+n-1}{n}=\frac{n-\frac{\sqrt[n]{e}}{e}}{n},\ 0<\sqrt[n]{1-\frac{\sqrt[n]{e}}{e}}<\frac{n-\frac{\sqrt[n]{e}}{e}}{n}\quad (2)$$

By adding (1); (2):

$$0<\sqrt[n]{1+\frac{\sqrt[n]{e}}{e}}+\sqrt[n]{1-\frac{\sqrt[n]{e}}{e}}<\frac{n+\frac{\sqrt[n]{e}}{e}+n-\frac{\sqrt[n]{e}}{e}}{n}=2$$

$$0<\frac{1}{n}\left(\sqrt[n]{1+\frac{\sqrt[n]{e}}{e}}+\sqrt[n]{1-\frac{\sqrt[n]{e}}{e}}\right)<\frac{2}{n}$$

$\lim\limits_{n\to\infty}\frac{2}{n}=0$. *By sandwich's theorem:*

$$\lim_{n\to\infty}\left(\frac{1}{n}\left(\sqrt[n]{1+\frac{\sqrt[n]{e}}{e}}+\sqrt[n]{1-\frac{\sqrt[n]{e}}{e}}\right)\right)=0$$

AN.078.

$$\int_1^x \left(\frac{\log t-1}{t^2-\log^2 t}\right)dt=\int_1^x \frac{\log^2 t\left(\frac{\log t-1}{\log^2 t}\right)}{\log^2 t\left(\left(\frac{t}{\log t}\right)^2-1\right)}dt=$$

$$=\int_1^x \frac{\frac{\log t-1}{\log^2 t}}{\left(\frac{t}{\log t}\right)^2-1}dt=\int_1^x \frac{\left(\frac{t}{\log t}\right)'}{\left(\frac{t}{\log t}\right)^2-1}dt=$$

$$=\frac{1}{2}\log\left(\frac{x-\log x}{x+\log x}\right)-\frac{1}{2}\log\left(\frac{1-\log 1}{1+\log 1}\right)$$

Equation becomes:

$$\frac{1}{2}\log\left(\frac{x-\log x}{x+\log x}\right)=\frac{1}{2}\log\left(\frac{e-1}{e+1}\right),\qquad \frac{x-\log x}{x+\log x}=\frac{e-1}{e+1}$$

$$xe+x-e\log x-\log x=ex-x+e\log x-\log x$$

$$2x=2e\log x\Rightarrow x=e\log x,\qquad \frac{\log x}{x}=\frac{1}{e}\Rightarrow x=e$$

$$(f:(0,\infty)\to\mathbb{R}; f(x)=\tfrac{\log x}{x}\ \textit{injective for } x\in(0,e))$$

AN.079. $f\in C^2([0,1]); f$ *convexe* $\Rightarrow f''(x)\geq 0; (\forall)x\in[0,1]$

$$\textit{Let be } g:(0,1]\to\mathbb{R}; g(x)=\tfrac{f(x)}{x}$$

$$g'(x)=\frac{xf'(x)-f(x)}{x^2}=\frac{h(x)}{x^2}; h(x)=xf'(x)-f(x)$$

$$h'(x)=f'(x)+xf''(x)-f'(x)=xf''(x)>0$$

$$h\ \textit{increasing}\ h(x)\geq h(0)=0\Rightarrow g'(x)>0\Rightarrow g\ \textit{increasing}$$

$$\textit{If } x\in\left(0,\tfrac{1}{a}\right]; g(x)\leq g\left(\tfrac{1}{a}\right)\Rightarrow \tfrac{f(x)}{x}\leq\frac{f\left(\frac{1}{a}\right)}{\frac{1}{a}}$$

$$f(x)\leq axf\left(\frac{1}{a}\right)\Rightarrow\int_0^{\frac{1}{a}}f(x)\,dx\leq\int_0^{\frac{1}{a}}ax\,f\left(\frac{1}{a}\right)dx=$$

$$=af\left(\frac{1}{a}\right)\int_0^{\frac{1}{a}}x\,dx=a\cdot\frac{1}{2^{a^2}}f\left(\frac{1}{a}\right)=\frac{1}{2^a}f\left(\frac{1}{a}\right),\qquad f\left(\frac{1}{a}\right)\geq 2a\int_0^{\frac{1}{a}}f(x)\,dx$$

$$\textit{If } x\in\left[\tfrac{1}{a},1\right]; g(x)\geq g\left(\tfrac{1}{a}\right)\Rightarrow\tfrac{f(x)}{x}\geq\frac{f\left(\frac{1}{a}\right)}{\frac{1}{a}},\ f(x)\geq ax\,f\left(\tfrac{1}{a}\right)$$

$$\int_{\frac{1}{a}}^{1} f(x)\,dx \ge \int_{\frac{1}{a}}^{a} ax\, f\left(\frac{1}{a}\right) = af\left(\frac{1}{a}\right)\int_{\frac{1}{a}}^{a} x\,dx =$$

$$= af\left(\frac{1}{2}\right)\cdot a\left(1-\frac{1}{a^2}\right) = \frac{a}{2}\cdot\frac{a^2-1}{a^2} f\left(\frac{1}{a}\right) \ge$$

$$\ge \frac{a^2-1}{2a}\cdot 2a\int_0^{\frac{1}{a}} f(x)\,dx = (a^2-1)\int_0^{\frac{1}{a}} f(x)\,dx$$

$$\int_0^1 f(x)\,dx = \int_0^{\frac{1}{a}} f(x)\,dx + \int_{\frac{1}{a}}^{1} f(x)\,dx \ge$$

$$= \int_0^{\frac{1}{a}} f(x)\,dx + (a^2-1)\int_0^{\frac{1}{a}} f(x)\,dx = a^2\int_0^{\frac{1}{a}} f(x)\,dx$$

$$\int_0^1 f(x)\,dx \ge a^2\int_0^{\frac{1}{a}} f(x)\,dx \quad (1)$$

Analogous: $\int_0^1 f(x)\,dx \ge b^2\int_0^{\frac{1}{b}} f(x)\,dx$ *(2)*

By adding (1); (2): $2\int_0^1 f(x)\,dx \ge a^2\int_0^{\frac{1}{a}} f(x)\,dx + b^2\int_0^{\frac{1}{b}} f(x)\,dx$

AN.080. *Let be* $f:[0,1]\to[0,\infty); f(x)=\log(1+x)$

$$f'(x)=\frac{1}{1+x}>0;\ f \text{ increasing}$$

$$\text{If } x\in\left[0,\frac{1}{2}\right] \Rightarrow f(x)\le f\left(\frac{1}{2}\right) \Rightarrow f(x)-f\left(\frac{1}{2}\right)\le 0$$

$$f\left(x+\frac{1}{2}\right)\ge f\left(\frac{1}{2}\right) \Rightarrow f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\le 0$$

$$\left(f(x)-f\left(\frac{1}{2}\right)\right)\left(f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\right)\le 0$$

$$\left(\log(1+x)-\log\frac{3}{2}\right)\left(\log\left(\frac{3}{2}+x\right)-\log\frac{3}{2}\right)\le 0$$

$$\log(1+x)\log\left(\frac{3}{2}+x\right)-\log\frac{3}{2}\cdot\log(1+x)-$$

$$-\log\frac{3}{2}\cdot\log\left(\frac{3}{2}+x\right)+\log^2\frac{3}{2}\le 0$$

$$\log(1+x)\log\left(\frac{3}{2}+x\right)\le\log\frac{3}{2}\cdot\log(1+x)+\log\frac{3}{2}\cdot\log\left(\frac{3}{2}+x\right)-$$

$$-\log^2\frac{3}{2}$$

$$\int_0^{\frac{1}{2}}\left(\log(1+x)\log\left(\frac{3}{2}+x\right)\right)dx\le\log\frac{3}{2}\int_0^{\frac{1}{2}}\log(1+x)\,dx+$$

$$+\log\frac{3}{2}\int_0^{\frac{1}{2}}\log\left(\frac{3}{2}+x\right)dx-\int_0^{\frac{1}{2}}\log^2\frac{3}{2}dx=$$

$$=\log\frac{3}{2}\int_0^{\frac{1}{2}}\log(1+x)\,dx+\log\frac{3}{2}\int_{\frac{1}{2}}^{1}\log(1+x)\,dx+$$

$$-\frac{1}{2}\log^2\frac{3}{2}=\log\frac{3}{2}\int_0^1\log(1+x)\,dx-\frac{1}{2}\log^2\frac{3}{2}$$

$$2\int_0^{\frac{1}{2}}\left(\log(1+x)\cdot\log\left(\frac{3}{2}+x\right)\right)dx\le 2\log\frac{3}{2}\int_0^1\log(1+x)\,dx-\log^2\left(\frac{3}{2}\right)$$

Remains to prove:

$$2\log\frac{3}{2}\int_0^1\log(1+x)\,dx-\log^2\left(\frac{3}{2}\right)\le\left(\int_0^1\log(1+x)\,dx\right)^2$$

$$\left(\int_0^1\log(1+x)\,dx\right)^2-2\log\frac{3}{2}\int_0^1\log(1+x)\,dx+\log^2\left(\frac{3}{2}\right)\ge 0$$

$$\left(\int_0^1\log(1+x)\,dx-\log\frac{3}{2}\right)^2\ge 0$$

AN.081. *Let be* $f:[0,1]\to[0,\infty); f(x)=x\tan^{-1}x$

$$f'(x)=\tan^{-1}x+\frac{x}{1+x^2};f''(x)=\frac{1}{1+x^2}+\frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$f''(x)=\frac{2}{(1+x^2)^2},\qquad \left(f'(x)-f''(x)\right)^2>0$$

$$\left(f'(x)\right)^2+\left(f''(x)\right)^2>2f'(x)f''(x)$$

$$\int_0^1\left(f'(x)\right)^2dx+\int_0^1\left(f''(x)\right)^2dx>\int_0^1 2f'(x)f''(x)dx=$$

$$=\int_0^1\left(\left(f'(x)\right)^2\right)'dx=\left(f'(1)\right)^2-\left(f'(0)\right)^2=\left(\frac{\pi}{4}+\frac{1}{2}\right)^2=\frac{(\pi+2)^2}{16}$$

$$\int_0^1\left(\tan^{-1}x+\frac{x}{1+x^2}\right)^2dx+4\int_0^1\frac{1}{(1+x^2)^4}dx>\frac{(\pi+2)^2}{16}$$

AN.082. $\log t=u\Rightarrow t=e^u$

$$\int_e^x\left(t^{\log t}(2\log t+1)\right)dt=\int_1^{\log x}\left(e^{u^2}(2u+1)\cdot e^u\right)du$$

$$=\int_1^{\log x}\left(e^{u^2+u}\cdot(2u+1)\right)du=e^{\log^2 x+\log x}-e^2=x^{\log x}\cdot x-e^2=$$

$$=x^{\log x+1}-e^2$$

Equation becomes: $e^2+x^{\log x+1}-e^2=x^4$

$$x^{\log x+1}=x^4,\qquad \log x+1=4,\qquad \log x=3\Rightarrow x=e^3$$

AN.083. *We prove that:* $f^5(x)+1\ge f^3(x)+f^2(x); (\forall)x\in[0,1]$

$$f^5(x)-f^3(x)-f^2(x)+1\ge 0,\qquad f^3(x)(f^2(x)-1)-(f^2(x)-1)\ge 0$$

$$(f^2(x)-1)(f^3(x)-1)\ge 0$$

$$(f(x)-1)(f(x)+1)(f(x)-1)(f^2(x)+f(x)+1)\ge 0$$

$$(f(x)-1)^2(f(x)+1)(f^2(x)+f(x)+1)\ge 0 \text{ (true)}$$

Integrating the relationship:

$$f^5(x)+1\geq f^3(x)+f^2(x)\ \ (1)$$

$$\int_0^1 f^5(x)\,dx+\int_0^1 dx\geq\int_0^1 f^3(x)\,dx+\int_0^1 f^2(x)\,dx$$

$$\int_0^1 f^5(x)\,dx+1\geq\int_0^1 f^3(x)\,dx+7,\qquad \int_0^1 f^5(x)\,dx>6+\int_0^1 f^3(x)\,dx$$

Inequality is strict because (1) is true only for $f(x)=1,(\forall)x\in[0,1]$ *but this function don't verify:* $\int_0^1 f^2(x)\,dx=7$

AN.084. *By absurdum suppose* $5^x\geq 5^{x^2}+5^{x^4}$

$$1\geq 5^{x^2-x}+5^{x^4-x}>5^{x^2-x}$$

$$5^{x^2-x}<1\Rightarrow x^2-x<0\Rightarrow x\in(0,1)$$

$$x<1\Rightarrow 5>5^x\geq 5^{x^2}+5^{x^4}>1+1=2$$

$$5>2.\ \textit{False. Hence: } 5^x<5^{x^2}+5^{x^4}$$

$$\int_a^b 5^{x^2}\,dx+\int_a^b 5^{x^4}\,dx\geq\int_a^b 5^x\,dx=\frac{5^b}{\log 5}-\frac{5^a}{\log 5}=\frac{1}{\log 5}\left(5^b-5^a\right)$$

$$\log 5\cdot\int_a^b 5^{x^2}\,dx+\log 5\cdot\int_a^b 5^{x^4}\geq 5^b-5^a$$

AN.085. *Let be* $f:[0,1]\to\mathbb{R}; f(x)=e^{x^2-1}; f'(x)=2xe^{x^2}$

$$f''(x)=2e^{x^2}+4x^2e^{x^2}>0;\ f-\textit{convexe}$$

$$\textit{Let be } g:(0,1]\to\mathbb{R}; g(x)=\frac{f(x)}{x}$$

$$g'(x)=\frac{xf'(x)-f(x)}{x^2}=\frac{2x^2\cdot e^{x^2}-e^{x^2}}{x^2}=\frac{h(x)}{x^2}$$

$$h:[0,1]\to\mathbb{R}; h(x)=2x^2e^{x^2}$$

$$h'(x)=4xe^{x^2}+4x^3e^{x^2}>0\Rightarrow h\ \textit{increasing}$$

$$h(x) \geq h(0) = 0 \Rightarrow g'(x) = \frac{h(x)}{x^2} > 0 \Rightarrow g \text{ increasing}$$

$$If\, x \in \left(0, \frac{1}{a}\right] \Rightarrow g(x) \leq g\left(\frac{1}{a}\right) \Rightarrow \frac{e^{x^2}-1}{x} \leq \frac{e^{\frac{1}{a^2}}-1}{\frac{1}{a}},\ e^{x^2} - 1 \leq ax\left(e^{\frac{1}{a^2}} - 1\right)$$

$$\int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx \leq a\left(e^{\frac{1}{a^2}} - 1\right)\int_0^{\frac{1}{a}} x\, dx = a \cdot \frac{1}{2a^2}\left(e^{\frac{1}{a^2}} - 1\right)$$

$$\int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx \leq \frac{1}{2a}\left(e^{\frac{1}{a^2}} - 1\right)\ (1)$$

$$If\, x \in \left[\frac{1}{a}, 1\right] \Rightarrow g(x) \geq g\left(\frac{1}{a}\right) \Rightarrow \frac{f(x)}{x} \geq \frac{f\left(\frac{1}{a}\right)}{\frac{1}{a}}$$

$$f(x) \geq axf\left(\frac{1}{a}\right), \qquad \int_{\frac{1}{a}}^{1} f(x)\, dx \geq af\left(\frac{1}{a}\right)\int_{\frac{1}{a}}^{1} x\, dx = \frac{a}{2}f\left(\frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right) =$$

$$= \frac{a}{2}\left(\frac{a^2-1}{a^2}\right) \cdot \left(e^{\frac{1}{a^2}} - 1\right) \overset{(1)}{\geq} \frac{a}{2}\left(\frac{a^2-1}{a^2}\right) \cdot 2a\int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx =$$

$$= (a^2 - 1)\int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx$$

$$\int_0^1 \left(e^{x^2} - 1\right) dx = \int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx + \int_{\frac{1}{a}}^{1} \left(e^{x^2} - 1\right) dx \geq$$

$$\geq \int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx + (a^2 - 1)\int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx = a^2\int_0^{\frac{1}{a}} \left(e^{x^2} - 1\right) dx$$

$$\int_0^1 e^{x^2} dx - 1 \geq a^2\int_0^{\frac{1}{a}} e^{x^2} dx - a^2 \cdot \frac{1}{a} = a^2\int_0^{\frac{1}{a}} e^{x^2} dx - a$$

$$\int_0^1 e^{x^2} dx + a \geq 1 + a^2\int_0^{\frac{1}{a}} e^{x^2} dx \quad (2)$$

Analogous: $\int_0^1 e^{x^2} dx + b \geq 1 + b^2\int_0^{\frac{1}{a}} e^{x^2} dx \quad (3)$

By adding (2); (3):

$$a+b+2\int_0^1 e^{x^2}dx \geq 2+a^2\int_0^{\frac{1}{a}} e^{x^2}dx+b^2\int_0^{\frac{1}{b}} e^{x^2}dx$$

AN.086. *If* $x \leq y$

$$\frac{x+2y}{3}=\frac{1}{3}x+\left(1-\frac{1}{3}\right)y \leq \frac{1}{3}y+y-\frac{1}{3}y=y\leq\sqrt{x^2+y^2}$$

If $x \geq y$

$$\frac{x+2y}{3}=\frac{1}{3}x+\left(1-\frac{1}{3}\right)y \leq \frac{1}{3}x+\left(1-\frac{1}{3}\right)x=x\leq\sqrt{x^2+y^2}$$

Hence: $\frac{x+2y}{3}\leq\sqrt{x^2+y^2}$; $(\forall)x,y\in\mathbb{R}$

$$\tan^{-1}\left(\frac{x+2y}{3}\right)\leq\tan^{-1}\left(\sqrt{x^2+y^2}\right)=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}\quad (1)$$

Analogous: $\tan^{-1}\left(\frac{x+3y}{4}\right)\leq\tan^{-1}\left(\sqrt{x^2+y^2}\right)\leq\frac{\pi}{3}$ *(2)*

$$\tan^{-1}\left(\frac{x+4y}{5}\right)\leq\tan^{-1}\left(\sqrt{x^2+y^2}\right)\leq\frac{\pi}{3}\quad (3)$$

By adding (1); (2); (3):

$$\tan^{-1}\left(\frac{x+2y}{3}\right)+\tan^{-1}\left(\frac{x+3y}{4}\right)+\tan^{-1}\left(\frac{x+4y}{5}\right)<\pi$$

Equality holds in (1); (2); (3) for $x=y=0$ *but* $x^2+y^2=3$ *implies a strictly inequality.*

AN.087.

$$\sum_{i=1}^{p}\left(\sum_{j=1}^{q}\left(\log x-\frac{1}{i^5}\right)\left(\log x-\frac{1}{j^7}\right)\right)=\sum_{i=1}^{p}\left(\left(\log x-\frac{1}{i^5}\right)\cdot\sum_{j=1}^{q}\left(\log x-\frac{1}{j^7}\right)\right)$$

$$=\sum_{i=1}^{p}\left(\left(\log x-\frac{1}{i^5}\right)\left(q\log x-\sum_{j=1}^{p}\frac{1}{j^7}\right)\right)=$$

$$=\sum_{i=1}^{p}\left(q\log^2 x-\log x\left(\sum_{j=1}^{p}\frac{1}{j^7}+\frac{q}{i^7}\right)+\frac{1}{i^5}\sum_{j=1}^{q}\frac{1}{j^7}\right)=$$

$$=pq\log^2 x-\log x\left(p\sum_{j=1}^{q}\frac{1}{j^7}+q\sum_{i=1}^{p}\frac{1}{i^5}\right)+\left(\sum_{i=1}^{p}\frac{1}{i^5}\right)\left(\sum_{j=1}^{q}\frac{1}{j^7}\right)=$$

$$=pq\log^2 x-\left(pH_q^{(7)}+qH_p^{(5)}\right)\log x+H_p^{(5)}H_q^{(7)}=0$$

$$\Delta=\left(pH_q^{(7)}+qH_p^{(5)}\right)^2-4H_p^{(5)}H_q^{(7)}\cdot\left(pH_q^{(7)}-qH_p^{(5)}\right)^2$$

$$\log x=\frac{pH_q^{(7)}+qH_p^{(5)}+pH_q^{(7)}-qH_p^{(5)}}{2pq}=\frac{H_q^{(7)}}{q}$$

$$\log x=\frac{pH_q^{(7)}+qH_p^{(5)}-pH_q^{(7)}+qH_p^{(5)}}{2pq}=\frac{H_p^{(5)}}{p}$$

$$x_1=e^{\frac{H_q^{(7)}}{q}};x_2=e^{\frac{H_p^{(5)}}{p}}$$

$$H_p^{(5)}=\frac{1}{1^5}+\frac{1}{2^5}+\dots+\frac{1}{p^5};H_q^{(7)}=\frac{1}{1^7}+\frac{1}{2^7}+\dots+\frac{1}{q^7}$$

AN.088.

$$f(x)+f(y)+f(z)+f(t)=ef\left(\frac{x+y}{\pi}\right)+ef\left(\frac{z+t}{\pi}\right)=$$

$$=e\cdot e\cdot f\left(\frac{\frac{x+y}{\pi}+\frac{z+t}{\pi}}{\pi}\right)=e^2f\left(\frac{x+y+z+t}{\pi^2}\right)$$

$$Let\ be\ t=\frac{x+y+z}{\pi^2-1}$$

$$f(x)+f(y)+f(z)+f\left(\frac{x+y+z}{\pi^2-1}\right)=e^2f\left(\frac{x+y+z+\frac{x+y+z}{\pi^2-1}}{\pi^2}\right)=$$

$$=e^2f\left(\frac{(\pi^2-1)(x+y+z)+(x+y+z)}{\pi^2(\pi^2-1)}\right)=e^2f\left(\frac{x+y+z}{\pi^2-1}\right)$$

$$f(x)+f(y)+f(z)=e^2f\left(\frac{x+y+z}{\pi^2-1}\right)-f\left(\frac{x+y+z}{\pi^2-1}\right)$$

$$f(x)+f(y)+f(z)=(e^2-1)f\left(\frac{x+y+z}{\pi^2-1}\right)$$

$$\Omega=\int_e^\pi\int_e^\pi\int_e^\pi f\left(\frac{x+y+z}{\pi^2-1}\right)dx\,dy\,dz=\frac{1}{e^2-1}\int_e^\pi\int_e^\pi\int_e^\pi\big(f(x)+f(y)+f(z)\big)dx\,dy\,dz$$

$$=\frac{1}{e^2-1}\cdot 3\cdot(\pi-e)^2\int_e^\pi f(x)\,dx=\frac{3(\pi-e)^2}{e^2-1}\cdot(e^2-1)=$$

$$=3(\pi-e)^2$$

AN.089. *Let be* $A=\begin{pmatrix}4&3\\1&2\end{pmatrix}$; $\det A=5; Tr\ A=6$

Characteristic equation:

$$\lambda^2-\lambda\cdot Tr\ A+\det A=0\Rightarrow\lambda^2-6\lambda+5=0\Rightarrow\lambda_1=1;\lambda_2=5$$

$$A^n=\begin{pmatrix}4&3\\1&2\end{pmatrix}^n=\begin{pmatrix}a_n&b_n\\c_n&d_n\end{pmatrix}=\lambda_1^nB+\lambda_2^nC=1^n\cdot B+5^nC$$

$$A^n=B+5^nC$$

$$A^2=\begin{pmatrix}4&3\\1&2\end{pmatrix}\begin{pmatrix}4&3\\1&2\end{pmatrix}=\begin{pmatrix}19&18\\6&7\end{pmatrix}=B+5^2C=B+25C$$

$$A=B+5C=\begin{pmatrix}4&3\\1&2\end{pmatrix}\Rightarrow B=\begin{pmatrix}4&3\\1&2\end{pmatrix}-5C$$

$$\begin{pmatrix}19&18\\6&7\end{pmatrix}=\begin{pmatrix}4&3\\1&2\end{pmatrix}-5C+25C\Rightarrow 20C=\begin{pmatrix}19&18\\6&7\end{pmatrix}-\begin{pmatrix}4&3\\1&2\end{pmatrix}$$

$$20C=\begin{pmatrix}15&15\\5&5\end{pmatrix}\Rightarrow 4C=\begin{pmatrix}3&3\\1&1\end{pmatrix}\Rightarrow C=\frac{1}{4}\begin{pmatrix}3&3\\1&1\end{pmatrix}$$

$$B=\begin{pmatrix}4&3\\1&2\end{pmatrix}-\frac{5}{4}\begin{pmatrix}3&3\\1&1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}16-15&12-15\\4-5&8-5\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&-3\\-1&3\end{pmatrix}$$

$$\Big(\underbrace{f\circ f\circ\ldots\circ f}_{\text{"}n\text{" times}}\Big)(x)=\frac{a_nx+b_n}{c_nx+d_n}$$

$$A^n=\begin{pmatrix}a_n&b_n\\c_n&d_n\end{pmatrix}=B+5^nC=\frac{1}{4}\begin{pmatrix}1&-3\\-1&3\end{pmatrix}+\frac{5^n}{4}\begin{pmatrix}3&3\\1&1\end{pmatrix}=$$

$$=\frac{1}{4}\begin{pmatrix}1+3\cdot5^n&-3+3\cdot5^n\\-1+5^n&3+5^n\end{pmatrix};\begin{cases}b_n=\frac{1}{4}(-3+3\cdot5^n)\\ d_n=\frac{1}{4}(3+5^n)\end{cases}$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to0}\left(\frac{a_nx+b_n}{c_nx+d_n}\right)\right)=\lim_{n\to\infty}\left(\frac{b_n}{d_n}\right)=\lim_{n\to\infty}\frac{3\cdot5^n-3}{5^n+3}=3$$

AN.090. *By Turkeviciu's inequality if* $u, v, w, t \geq 0$ *then:*

$$u^4+v^4+w^4+t^4+2uvwt \geq u^2v^2+u^2w^2+u^2t^2+v^2w^2+v^2t^2+t^2w^2$$

$$\text{Replace: } u=e^{x^2}; v=e^{y^2}; w=e^{z^2}; t=e^{p^2}$$

$$\sum_{cyc} e^{4x^2}+2\prod_{cyc} e^{x^2} \geq \sum_{sym}\left(e^{x^2}\cdot e^{y^2}\right)$$

$$\int_0^a\int_0^a\int_0^a\int_0^a\left(\sum_{cyc} e^{4x^2}\right)dx\,dy\,dz\,dp+2\int_0^a\int_0^a\int_0^a\int_0^a\left(\prod_{cyc} e^{x^2}\right)dx\,dy\,dz\,dp \geq$$

$$\geq \int_0^a\int_0^a\int_0^a\int_0^a\left(\sum_{sym} e^{x^2+y^2}\right)dx\,dy\,dz\,dp$$

$$\sum_{cyc}\left(\left(\int_0^a e^{4x^2}\,dx\right)\cdot\int_0^a dy\cdot\int_0^a dz\cdot\int_0^a dp\right)+$$

$$+2\left(\int_0^a e^{x^2}\,dx\right)\left(\int_0^a e^{y^2}\,dy\right)\left(\int_0^a e^{z^2}\,dz\right)\left(\int_0^a e^{p^2}\,dp\right) \geq$$

$$\geq \sum_{sym}\left(\left(\int_0^a e^{x^2}\,dx\right)\left(\int_0^a e^{y^2}\,dy\right)\left(\int_0^a dz\right)\left(\int_0^a dp\right)\right)$$

$$4a^3\left(\int_0^a e^{4x^2}\,dx\right)+2\left(\int_0^a e^{x^2}\,dx\right)^4 \geq 6a^2\left(\int_0^a e^{x^2}\,dx\right)^2$$

$$2a^3\cdot\int_0^a e^{4x^2}\,dx+\int_0^a e^{x^2}\,dx \geq 3a^2\left(\int_0^a e^{x^2}\,dx\right)$$

Equality holds for $a=0$.

AN.091. *Let be* $f:[0,1]\to\mathbb{R}; f(x)=e^{x^2}; f'(x)=2xe^{x^2}\geq 0; f$ *increasing*

$$\text{If}\left[0,\tfrac{1}{2}\right] \Rightarrow f(x) \leq f\left(\tfrac{1}{2}\right) \Rightarrow f(x)-f\left(\tfrac{1}{2}\right) \leq 0$$

$$If\, x \in \left[0,\tfrac{1}{2}\right] \Rightarrow x+\tfrac{1}{2} \in \left[\tfrac{1}{2},1\right] \Rightarrow f\left(x+\tfrac{1}{2}\right)-f\left(\tfrac{1}{2}\right) \geq 0$$

$$\left(f(x)-f\left(\frac{1}{2}\right)\right)\left(\left(x+\frac{1}{2}\right)-\left(\frac{1}{2}\right)\right) \leq 0$$

$$f(x)\cdot f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\left(f(x)+f\left(x+\frac{1}{2}\right)\right)+f^2\left(\frac{1}{2}\right) \leq 0$$

$$f(x)\cdot f\left(x+\frac{1}{2}\right) \leq f\left(\frac{1}{2}\right)\left(f(x)+f\left(x+\frac{1}{2}\right)\right)-f^2\left(\frac{1}{2}\right)$$

$$\int_0^{\frac{1}{2}} f(x)\, f\left(x+\tfrac{1}{2}\right) dx \leq f\left(\tfrac{1}{2}\right)\left(\int_0^{\frac{1}{2}} f(x)\, dx+\int_0^{\frac{1}{2}} f\left(x+\tfrac{1}{2}\right) dx\right)-\int_0^{\frac{1}{2}} f^2\left(\tfrac{1}{2}\right) dx \quad (1)$$

$$For\, y = x+\tfrac{1}{2} \Rightarrow dx = dy \Rightarrow \int_0^{\frac{1}{2}} f\left(x+\tfrac{1}{2}\right) dx = \int_{\frac{1}{2}}^1 f(x)\, dx$$

Replace in (1):

$$\int_0^{\frac{1}{2}} f(x)\, f\left(x+\frac{1}{2}\right) dx \leq f\left(\frac{1}{2}\right)\left(\int_0^{\frac{1}{2}} f(x)\, dx+\int_{\frac{1}{2}}^{\frac{1}{2}} f(x)\, dx\right)-\frac{1}{2}f^2\left(\frac{1}{2}\right)$$

$$\int_0^1 e^{x^2}\cdot e^{\left(x+\frac{1}{2}\right)^2} dx \leq e^{\frac{1}{4}}\cdot \int_0^1 e^{x^2} dx-\tfrac{1}{2}\left(e^{\frac{1}{4}}\right)^2 \quad (2)$$

$$But{:}\ e^{\frac{1}{4}}\int_0^1 e^{x^2} dx-\tfrac{1}{2}\left(e^{\frac{1}{4}}\right)^2 \leq \tfrac{1}{2}\left(\int_0^1 e^{x^2}\right)^2 \quad (3)\ because$$

$$\frac{1}{2}\left(\int_0^1 e^{x^2} dx\right)^2-e^{\frac{1}{4}}\int_0^1 e^{x^2} dx+\frac{1}{2}\left(e^{\frac{1}{4}}\right)^2 = \frac{1}{2}\left(\int_0^1 e^{x^2} dx-e^{\frac{1}{4}}\right)^2 \geq 0$$

By (2); (3):

$$\int_0^1 e^{x^2}\cdot e^{\left(x+\frac{1}{2}\right)^2} dx < \frac{1}{2}\left(\int_0^1 e^{x^2} dx\right)^2 \Leftrightarrow$$

$$\Leftrightarrow 2\int_0^1 e^{x^2+x^2+x+\frac{1}{4}} dx < \left(\int_0^1 e^{x^2} dx\right)^2 \Leftrightarrow 2\sqrt[4]{e}\int_0^1 \left(e^{2x^2+x}\right) dx < \left(\int_0^1 e^{x^2} dx\right)^2$$

AN.092.

$$\int_1^a \frac{2x\tan^{-1}x-\log(1+x^2)}{(1+x^2)(\tan^{-1}x)^2}dx=\int_1^a\left(\frac{\log(1+x^2)}{\tan^{-1}x}\right)'dx=$$

$$=\frac{\log(1+a^2)}{\tan^{-1}a}-\frac{\log(1+1^2)}{\tan^{-1}1}<\frac{a^2}{\tan^{-1}a}-\frac{4\log 2}{\pi}$$

$$\frac{4\log 2}{\pi}+\int_1^a \frac{2x\tan^{-1}x-\log(1+x^2)}{(1+x^2)(\tan^1 x)^2}dx<\frac{a^2}{\tan^{-1}a}$$

AN.093. *Let be* $f:[k,k+1]\to\mathbb{R}; f(x)=\log x\,; k\in\mathbb{N}^*$

By Lagrange MVT – Theorem $(\exists)c_k\in(k,k+1)$

$$\log(k+1)-\log k=\frac{1}{c_k}(k+1-k)$$

$$k<c_k<k+1\Rightarrow\frac{1}{k+1}<\frac{1}{c_k}<\frac{1}{k}$$

$$\frac{1}{k+1}<\log(k+1)-\log k<\frac{1}{k}; k\in\mathbb{N}^* \quad (1)$$

Replace in (1): $k\in\{1,3,5,\dots,2n-1\}$

$$\frac{1}{2}<\log 2-\log 1<\frac{1}{1}$$

$$\frac{1}{4}<\log 4-\log 3<\frac{1}{3}$$

$$\frac{1}{6}<\log 6-\log 5<\frac{1}{5}$$

$$-----------$$

$$\frac{1}{2n}<\log(2n)-\log(2n-1)<\frac{1}{2n-1}$$

By adding:

$$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}<\log\left(\prod_{k=1}^n\frac{2k}{2k-1}\right)<1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}$$

$$\frac{1}{2}H_n<\log\left(\prod_{k-1}^n\frac{2k}{2k-1}\right)<H_{2n}-\frac{1}{2}H_n$$

$$H_n < \frac{1}{2}H_n + \log\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right) < H_{2n}, \qquad \lim_{n\to\infty}\frac{1}{n}H_n \le \Omega \le \lim_{n\to\infty}\frac{1}{n}H_{2n}$$

$$\textit{By Césaro – Stolz theorem: } \lim_{n\to\infty}\frac{1}{n}H_n = \lim_{n\to\infty}\frac{1}{n}H_{2n} = 0$$

$$\textit{hence } \Omega = 0$$

AN.094. *Let be* $f:[0,\infty)\to\mathbb{R}; f(x) = \arctan x - \int_0^x e^{-t^2}\,dt$

$$f'(x) = \frac{1}{1+x^2} - e^{-x^2} = \frac{1}{1+x^2} - \frac{1}{e^{x^2}} = \frac{e^{x^2}-(1+x^2)}{(1+x^2)e^{x^2}}$$

$$\textit{Let be } g:[0,\infty)\to\mathbb{R}; g(x) = e^{-x^2}-(1+x^2)$$

$$g'(x) = -2xe^{-x^2} - 2x = -2x\left(e^{-x^2}+1\right) \le 0$$

$$g \textit{ decreasing; } \max g(x) = g(0) = 0$$

$$\textit{Hence } g(x) \le 0 \Rightarrow f''(x) \le 0 \Rightarrow f \textit{ decreasing}$$

$$\frac{2ab}{a+b} \le \sqrt{ab} \Rightarrow f\left(\frac{2ab}{a+b}\right) \le f\left(\sqrt{ab}\right)$$

$$\arctan\left(\frac{2ab}{a+b}\right) - \int_0^{\frac{2ab}{a+b}} e^{-t^2}\,dt \le \arctan\left(\sqrt{ab}\right) - \int_0^{\sqrt{ab}} e^{-t^2}\,dt$$

$$\arctan\left(\frac{2ab}{a+b}\right) - \int_0^{\frac{2ab}{a+b}} e^{-t^2}\,dt + \int_0^{\sqrt{ab}} e^{-t^2}\,dt \le \arctan\left(\sqrt{ab}\right)$$

$$\arctan\left(\frac{2ab}{a+b}\right) - \int_0^{\frac{2ab}{a+b}} e^{-t^2}\,dt + \int_0^{\frac{2ab}{a+b}} e^{-t^2}\,dt + \int_{\frac{2ab}{a+b}}^{\sqrt{ab}} e^{-t^2}\,dt \le \arctan\left(\sqrt{ab}\right)$$

$$\arctan\left(\frac{2ab}{a+b}\right) + \int_{\frac{2ab}{a+b}}^{\sqrt{ab}} e^{-t^2}\,dt \le \arctan\left(\sqrt{ab}\right)$$

Equality holds for $a = b$.

AN.095. *Let be* $f:(0,e)\to\mathbb{R}; f(x) = x^{\frac{1}{x}}$

$$f'(x)=\frac{1}{x}\cdot x'\cdot x^{\frac{1}{x}-1}-\frac{1}{x^2}\cdot\log x\cdot x^{\frac{1}{x}}=\frac{1}{x^2}\cdot x^{\frac{1}{x}}(1-\log x)>0$$

$$f\ increasing\ on\ (0,e)$$

$$\frac{2ab}{a+b}\le\sqrt{ab}\le\frac{a+b}{2}\Rightarrow f\left(\frac{2ab}{a+b}\right)\le f\left(\sqrt{ab}\right)\le f\left(\frac{a+b}{2}\right)$$

$$\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}}\le\left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}\le\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}}\ (1)$$

$$x\le y\le z\ \ (2)$$

By (1); (2) and Cebyshev's inequality:

$$x\cdot\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}}+y\cdot\left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}+z\cdot\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}}\ge$$

$$\ge\frac{1}{3}(x+y+z)\cdot\left(\left(\frac{2ab}{a+b}\right)^{\frac{2ab}{2ab}}+\left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}+\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}}\right)=$$

$$=\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}}+\left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}+\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}}$$

$$(x-1)\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}}+(y-1)\left(\sqrt{ab}\right)^{\frac{1}{\sqrt{ab}}}+(z-1)\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}}\ge 0$$

Equality holds for $a=b$ and $x=y=z=1$.

AN.096. Solution (Tran Hong)

$$Let\ \varphi(t)=2\pi sint-3t\sqrt{3},\forall t\in(0,\pi)$$

$$\varphi'(t)=2\pi cost-3\sqrt{3}\Rightarrow\varphi'(t)=0\Leftrightarrow cost=\frac{3\sqrt{3}}{2\pi}\Rightarrow$$

$$t=\alpha=cos^{-1}\left(\frac{3\sqrt{3}}{2\pi}\right)\in(0,\pi)$$

$$\Rightarrow\varphi'(t)<0,\forall t\in(0,\alpha);\ \varphi'(t)>0,\forall t\in(\alpha,\pi)$$

$$\Rightarrow\varphi(t)<\varphi(0)=0;\ \varphi(t)<\varphi(\pi)=-3\pi\sqrt{3}<0$$

Hence

$$For\ x,y\in\left(0,\frac{\pi}{2}\right)\Rightarrow\frac{sinx}{x}\le\frac{3\sqrt{3}}{2\pi}\ and\ \frac{siny}{y}\le\frac{3\sqrt{3}}{2\pi}$$

$$\Rightarrow\frac{\sin(x+y)}{\pi-(x+y)}=\frac{sin[\pi-(x+y)]}{\pi-(x+y)}\le\frac{3\sqrt{3}}{2\pi}\ ;$$

$$\left(\because x,y\in\left(0,\frac{\pi}{2}\right)\Rightarrow x+y<\pi\Rightarrow\pi>\pi-(x+y)>0\right)$$

Hence

$$8\pi^3\int_a^b\int_a^b\left(\frac{sinxsinysin(x+y)}{xy(\pi-x-y)}\right)dxdy\le 8\pi^3\int_a^b\int_a^b\left(\frac{3\sqrt{3}}{2\pi}\right)^3dxdy$$

$$=81\sqrt{3}\cdot(b-a)^2$$

Equality for a=b.

AN.097. Solution (Tran Hong)

Let: $\varphi(a)=\frac{8}{\pi-2}\int_1^a\frac{x-tan^{-1}x}{(1+x^2)^2(tan^{-1}x)^2}dx+\frac{16}{\pi^2}-\frac{1}{(tan^{-1}a)^2}$

$$\varphi'(a)=\frac{8}{\pi-2}\left[\frac{x-tan^{-1}x}{(1+x^2)^2(tan^{-1}x)^2}\right]+\frac{2}{(1+a^2)(tan^{-1}a)^3}$$

$$=\frac{2}{(1+a^2)(tan^{-1}a)^2}\left[\frac{4(a-tan^{-1}a)}{\pi-2}+\frac{1}{tan^{-1}a}\right]\overset{(*)}{>}0;\ \forall a\ge 1$$

(*) is true, because: $\forall a\ge 1\Rightarrow 1+a^2\ge 2a\ge 2>0;\Rightarrow tan^{-1}a\ge\frac{\pi}{4}>0$

$$\pi-2>0$$

$$\forall a\ge 1\Rightarrow tan(a)\ge a\Rightarrow tan^{-1}(tana)>tan^{-1}a\Rightarrow a>tan^{-1}a$$

$$\Rightarrow 4(a-tan^{-1}a)>0$$

So, $\varphi'(a)>0,\forall a\ge 1\Rightarrow\varphi(a)\uparrow[1,\infty)\Rightarrow\varphi(a)\ge\varphi(1)=0$

$$\Rightarrow\frac{8}{\pi-2}\int_1^a\frac{x-tan^{-1}x}{(1+x^2)^2(tan^{-1}x)^2}dx+\frac{16}{\pi^2}-\frac{1}{(tan^{-1}a)^2}\ge 0$$

AN.098. Solution (Rajeev Rastogi)

$$\sum_{k=1}^{i}k\binom{2i}{2k}=\sum_{k=1}^{i}k\cdot\frac{2i}{2k}\binom{2i-1}{2k-1}=\sum_{k=1}^{i}i\cdot\binom{2i-1}{2k-1}$$

$$= i \cdot \left[\binom{2i-1}{1} + \binom{2i-1}{3} + \dots + \binom{2i-1}{2i-1}\right] = i \cdot 2^{2i-2}$$

$$\frac{1}{n!}\prod_{i=1}^{n}\left(\sum_{k=1}^{i} k\binom{2i}{2k}\right) = \frac{1}{n!}\prod_{i=1}^{n}\left(i \cdot 2^{2i-2}\right)$$

$$= \frac{1}{n!}\left[(1 \cdot 2^0) \cdot (2 \cdot 2^2) \cdot (3 \cdot 2^3) \cdot \dots \cdot (n \cdot 2^{2n-2})\right]$$

$$= \frac{1}{n!} \cdot n! \cdot 2^{2+4+6+\dots+(2n-2)} = 2^{n(n-1)}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{3^n} \cdot \sqrt[n]{\frac{1}{n!}\prod_{i=1}^{n}\left(\sum_{k=1}^{i} k\binom{2i}{2k}\right)}\right) = \lim_{n\to\infty}\left(\frac{1}{3^n} \cdot 2^{n(n-1)}\right)$$

$$= \lim_{n\to\infty}\frac{2^{n-1}}{3^n} = \frac{1}{2}\lim_{n\to\infty}\frac{2^n}{3^n} = 0$$

AN.099. Solution (Remus Florin Stanca)

$$\Omega(p) = \lim_{n\to\infty}\left(\frac{1}{p}\sum_{k=1}^{n}\sqrt[n]{k}\right)^{np}$$

$$= \lim_{n\to\infty}\left(\frac{\sum_{k=1}^{p}(\sqrt[n]{k}-1)}{p} + 1\right)^{\frac{p}{\sum_{k=1}^{p}(\sqrt[n]{k}-1)} \cdot \frac{\sum_{k=1}^{p}(\sqrt[n]{k}-1)}{p} \cdot np}$$

$$= e^{\sum_{k=1}^{p}\lim_{n\to\infty} n\left(k^{\frac{1}{n}}-1\right)} = e^{\sum_{k=1}^{p}\lim_{n\to\infty}\frac{k^{\frac{1}{n}}-1}{\frac{1}{n}}} = e^{log(p!)} = p! \Rightarrow \Omega = \sum_{k=1}^{\infty}\frac{1}{\Omega(p)}$$

$$= e - 1$$

AN.100. Solution (Remus Florin Stanca)

$$\Omega = \lim_{n\to\infty}\left(n^6 sin\frac{1}{n^3} tan\frac{1}{n^5}\sum_{1\le k<l\le n} sin\left(\frac{k+l}{n}\right)\right)$$

$$= \lim_{n\to\infty}\left(n^3 sin\frac{1}{n^3} \cdot n^5 tan\frac{1}{n^5} \cdot \frac{\sum_{1\le k<l\le n}\left[sin\left(\frac{k}{n}\right)cos\left(\frac{l}{n}\right)+cos\left(\frac{k}{n}\right)sin\left(\frac{l}{n}\right)\right]}{n^2}\right)$$

$$=\lim_{n\to\infty}\frac{\sum_{1\le k<l\le n} sin\left(\frac{k}{n}\right)cos\left(\frac{l}{n}\right)}{n^2}+\lim_{n\to\infty}\frac{\sum_{1\le k<l\le n} cos\left(\frac{k}{n}\right)sin\left(\frac{l}{n}\right)}{n^2}$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{n}\sum_{1\le k<l\le n} sin\left(\frac{k}{n}\right)\right)\left(\frac{1}{n}\sum_{1\le k<l\le n} cos\left(\frac{l}{n}\right)\right)$$

$$+\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{n}\sum_{1\le k<l\le n} sin\left(\frac{l}{n}\right)\right)\left(\frac{1}{n}\sum_{1\le k<l\le n} cos\left(\frac{k}{n}\right)\right)$$

$$=\frac{1}{2}\left(\int_0^1 sinxdx\right)\left(\int_0^1 cosxdx\right)+\frac{1}{2}\left(\int_0^1 cosxdx\right)\left(\int_0^1 sinxdx\right)$$

$$=(1-cos1)sin1$$

AN.101. Solution (Adrian Popa)

$$sin\alpha sin\beta=\frac{cos(\alpha-\beta)-cos(\alpha+\beta)}{2}\Rightarrow$$

$$Lhs=sin\left(\frac{3a+b+2}{4}\right)sin\left(\frac{a+3b+6}{4}\right)=$$

$$=\frac{cos\left(\frac{3a+b+2}{4}-\frac{a+3b+6}{4}\right)-cos\left(\frac{3a+b+2}{4}+\frac{a+3b+6}{4}\right)}{2}=$$

$$=\frac{cos\left(\frac{2a-2b-4}{4}\right)-cos\left(\frac{4a+4b+8}{4}\right)}{2}=$$

$$=\frac{cos\left(\frac{a-b-2}{2}\right)-cos(a+b+2)}{2}$$

$$Rhs=sin\left(\frac{a+3b+2}{4}\right)sin\left(\frac{3a+b+6}{4}\right)=$$

$$=\frac{cos\left(\frac{a+3b+2}{4}-\frac{3a+b+6}{4}\right)-cos\left(\frac{a+3b+2}{4}+\frac{3a+b+6}{4}\right)}{4}=$$

$$=\frac{cos\left(\frac{-2a+2b-4}{4}\right)-cos\left(\frac{4a+4b+8}{4}\right)}{2}=$$

$$=\frac{cos\left(\frac{-a+b-2}{2}\right)-cos(a+b+2)}{2}$$

We must show that:

$$\frac{cos\left(\frac{a-b-2}{2}\right)-cos(a+b+2)}{2}\le\frac{cos\left(\frac{-a+b-2}{2}\right)-cos(a+b+2)}{2}\Leftrightarrow$$

$$cos\left(\frac{a-b-2}{2}\right) \le cos\left(\frac{-a+b-2}{2}\right) \Leftrightarrow cos\left(\frac{a-b}{2}-1\right)$$
$$\le cos\left(-\frac{a-b}{2}-1\right) \Leftrightarrow$$
$$cos\left(-\frac{b-a}{2}-1\right) \le cos\left(-\frac{b-a}{2}+1\right) \Leftrightarrow cos\left(\frac{b-a}{2}+1\right)$$
$$\le cos\left(1-\frac{b-a}{2}\right) \xLeftrightarrow{cosx\downarrow\left(0;\frac{\pi}{2}\right)}$$
$$\frac{b-a}{2}+1 \ge 1-\frac{b-a}{2} \Leftrightarrow b \ge a \text{ true.}$$

AN.102. Solution (Ravi Prakash)

$$\text{Let } f(x) = \frac{(x+1)^{2m}}{x^{2m}+x^{2m-2}+\dots+x^2+1}, m \in \mathbb{N}, x \ge 1$$
$$\text{For } x > 1, f'(x) =$$
$$= \frac{(x^{2m}+x^{2m-2}+\dots+x^2+1)(2m)(x+1)^{2m-1}-(x+1)^{2m}[2mx^{2m-1}+\dots+2x]}{(x^{2m}+x^{2m-2}+\dots+x^2+1)^2}$$
$$\text{Numerator of } f'(x) \text{ is } 2(x+1)^{2m-1}p(x) \text{ when}$$
$$p(x) = m(x^{2m}+x^{2m-2}+\dots+x^2+1)-(x+1)(mx^{2m-1}+\dots+x) =$$
$$= x^{2m-2}+2x^{2m-4}+\dots+(m-1)x^2+m-mx^{2m-1}-(m-1)x^{2m-3}-\dots-x =$$
$$= (x^{2m-2}+x^{2m-4}+\dots+x^2+1-mx^{2m-1})+(x^{2m-4}+x^{2m-6}+\dots.+x^2+1-$$
$$-(m-2)x^{2m-5}+\dots+(1-x) < 0, \forall x > 1 \Rightarrow f(x) < \frac{2^{2m}}{m+1}$$
$$\text{Now, we show that:}$$
$$\frac{2^{2m}}{m+1} \le \frac{(2m)!}{m!} = (2m)(2m-1)\dots(m+1), \forall m \in \mathbb{N}$$
$$\text{For } m=1, Lhs = 2, Rhs = 2.$$
$$\text{For } m=2, Lhs = \frac{16}{3}, Rhs = 4\cdot 3 \Rightarrow Lhs \le Rhs$$
$$\text{For } m \ge 3, (m+1)(m+2)\dots(2m) \ge \underbrace{4\cdot 4\cdot\dots\cdot 4}_{m-times} = 4^m = 2^{2m} > \frac{2^{2m}}{m+1}$$
$$\text{Thus, } f(x) < \frac{2^{2m}}{m+1}, \forall m \in \mathbb{N}, \forall x \ge 1.$$
$$\frac{(x+1)^{2m}}{x^{2m}+x^{2m-2}+\dots+x^2+1} \le \frac{(2m)!}{m!}, \forall x > 1 \Rightarrow$$

$$\frac{(x^2-1)(x+1)^{2m}}{(x^2-1)(x^{2m}+x^{2m-2}+\cdots+x^2+1)}\le\frac{(2m)!}{m!},\forall x>1$$

$$\frac{(x+1)^{2m}(x^2-1)}{x^{2m+2}-1}\le\frac{(2m)!}{m!},\forall x>1$$

Taking $m=p,q,r$ and multiplying, we get

$$\frac{(x+1)^{2p+2q+2r}(x^2-1)^3}{(x^{2p+2}-1)(x^{2q+2}-1)(x^{2r+2}-1)}\le\frac{(2p)!\,(2q)!\,(2r)!}{p!\,q!\,r!}$$

AN.103. Solution (Adrian Popa)

$$\Omega=\lim_{n\to\infty}\left(\frac{(n!)^{ne}}{e^{1+2^n+3^n+\cdots+n^n}}\cdot H_n^{-1}\right)=\lim_{n\to\infty}e^{log\left(\frac{(n!)^{ne}}{e^{1+2^n+3^n+\cdots+n^n}}\cdot H_n^{-1}\right)}$$

$$H_n\cong\gamma+log\,n$$

$$log\left(\frac{(n!)^{ne}}{e^{1+2^n+3^n+\cdots+n^n}}\cdot H_n^{-1}\right)$$

$$=ne\cdot log(n!)-log(\gamma+log\,n)-log\left(e^{1+2^n+3^n+\cdots+n^n}\right)=$$

$$=ne\cdot log(n!)-log(\gamma+log\,n)-(1+2^n+3^n+\cdots+n^n)=$$

$$=ne\cdot(log\,1+log\,2+\cdots+log\,n)-(1+2^n+3^n+\cdots+n^n)$$

$$-\underbrace{log(\gamma+log\,n)}_{\to\infty}$$

For $k\in\{1,2,3,\dots,n\}$

$$\lim_{n\to\infty}(ne\cdot log\,k-k^n)=\lim_{n\to\infty}k^n\left(\underbrace{\frac{ne\cdot log\,k}{k^n}}_{=0}-1\right)=-\infty$$

$$\lim_{n\to\infty}\big(ne\cdot(log\,1+log\,2+\cdots+log\,n)-(1+2^n+3^n+\cdots+n^n)$$

$$-log(\gamma+log\,n)\big)=-\infty\Rightarrow\Omega$$

$$=\lim_{n\to\infty}\left(\frac{(n!)^{ne}}{e^{1+2^n+3^n+\cdots+n^n}}\cdot H_n^{-1}\right)=0$$

AN.104. Solution (Remus Florin Stanca)

We prove that $sin(kx)\le ksinx,\forall k\ge1,k\in\mathbb{N}$ by using Mathematical Induction:

1. The statement $P(1): sinx \leq sinx$ *is true.*

2. We suppose that: $P(k): sin(kx) \leq ksinx$ *is true*

3. We prove that: $P(k+1): sin\big((k+1)x\big) \leq (k+1)sinx$ *by using* $P(k)$:

$$sin\big((k+1)x\big) = sin(kx+x) = sin(kx)cosx + sinxcos(kx); (1)$$

$$sin(kx) \leq ksinx \ |\cdot cosx \Rightarrow sin(kx)cosx \leq ksinxcosx, x \in \left(0, \frac{\pi}{2}\right), cosx > 0$$

$$sin(kx)cosx \leq ksinxcosx \ | \ (+cos(kx)sinx) \Rightarrow$$

$$sin(kx)cosx + cos(kx)sinx \leq ksinxcosx + cos(kx)sinx; (2)$$

$$cosx \leq 1 \ |\cdot ksinx > 0 \left(x \in \left(0, \frac{\pi}{2}\right)\right) \Rightarrow ksinxcosx \leq ksinx; (3)$$

$$cos(kx) \geq 1 \ |\cdot sinx > 0 \Rightarrow cos(kx)sinx \leq sinx; (4)$$

$$\xRightarrow{(3)+(4)} ksinxcosx + cos(kx)sinx \leq (k+1)sinx \xRightarrow{(1)+(2)}$$

$$sin\big((k+1)x\big) \leq (x+1)sinx \Rightarrow sin(kx) \leq ksinx, \forall k \geq 1, k \in \mathbb{N}(*)$$

1. We know that: $sinx \geq -sinx$ *true because* $sinx \geq 0, x \in \left(0, \frac{\pi}{2}\right)$

2. We suppose that $P(k): sin(kx) \geq -ksinx$ *is true*

3. We suppose that $P(k+1): sin\big((k+1)x\big) \geq -(k+1)sinx$ *is true by using* $P(k)$:

$$sin\big((k+1)x\big) = sin(kx+x) = sin(kx)cosx + cos(kx)sinx; (5)$$

$$sin(kx) \geq -ksinx \ |\cdot cosx > 0; \left(x \in \left(0, \frac{\pi}{2}\right)\right) \Rightarrow$$

$$sin(kx)cosx \geq -ksinxcosx \ | \ (+cos(kx)sinx) \Rightarrow$$

$$sin(kx)cosx + cos(kx)sinx \geq -ksinxcosx + cos(kx)sinx; (6)$$

$$cosx \leq 1 \ |\cdot (-ksinx) \Rightarrow -ksinxcosx \geq -ksinx; (7)$$

$$cos(kx) + 1 = 2cos^2\left(\frac{kx}{2}\right) \geq 0 \Rightarrow cos(kx) + 1 \geq 0 \Rightarrow cos(kx) \geq -1 \Rightarrow$$

$$cos(kx)sinx \geq -sinx; (8)$$

$$(7) + (8) \Rightarrow -ksinxcosx + cos(kx)sinx \geq -(k+1)sinx$$

$$\overset{(5)+(6)}{\Longrightarrow} sin\big((k+1)x\big) \geq -(k+1)sinx; (proved)$$

$$sin(kx) \geq -ksinx, \forall k \geq 1, k \in \mathbb{N}; \ (**)$$

From $(*), (**)$ *we get:* $-ksinx \leq sin(kx) \leq ksinx \Rightarrow |sinkx| \leq ksinx \Rightarrow$

$$\frac{|sin(kx)|}{sinx} \leq k \Rightarrow \prod_{k=1}^{2019} \frac{|sin(kx)|}{sinx} \leq \prod_{k=1}^{n} k \Rightarrow$$

$$\frac{|sinx \cdot sin(2) \cdot \ldots \cdot sin(2019x)|}{sin^{2019}x} \leq 2019! \Rightarrow$$

$$\int_a^b \frac{|sinx \cdot sin(2) \cdot \ldots \cdot sin(2019x)|}{sin^{2019}x} dx \leq \int_a^b 2019!\, dx = 2019!$$

AN.105. Solution (Khaled Imouti)

Let be the function $f: (0, \infty) \to \mathbb{R}, f(t) = log^5(t^2 + t + 2),$

$$f'(t) = \frac{5(2t+1)}{t^2+t+2} \cdot log^4(t^2+t+2)$$

$$f'(t) = 0 \Leftrightarrow t = -\frac{1}{2}; t^2 + t + 2 > 0 \Rightarrow f'(t) > 0, \forall t > 0 \Rightarrow f(t) > f(0),$$

$$\forall t > 0 \ \Rightarrow f(t) > log^5 2 > log2; (log2 < 1), \forall t > 0$$

$$\int_0^a \left(\int_0^t log^5(x^2+x+2)dx \right) dt + \int_0^b \left(\int_0^t log^5(x^2+x+2)dx \right) dt \geq$$

$$\geq \int_0^a log2 \int_0^t dx + \int_0^b log2 \int_0^t dx = \frac{t^2}{2}\bigg|_0^a log2 + \frac{t^2}{2}\bigg|_0^b log2$$

$$= \frac{a^2+b^2}{2} \cdot log2 \overset{AM-GM}{\geq} ablog2$$

AN.106. Solution (Kamel Benaicha)

$$\Omega = \int \frac{3x^2 + x}{1 + 6x(1 + e^{3x}) + 2e^{3x} + e^{6x} + 9x^2} dx$$

$$= \int \frac{3x^2 + x}{(3x + e^{3x})^2 + 2(3x + e^{3x}) + 1} dx$$

$$= \int \frac{3x+1}{(3x+e^{3x}+1)^2} \cdot x dx = \int \frac{(3x+1)e^{-3x}}{\left((3x+1)e^{-3x}+1\right)^2} \cdot x e^{-3x} dx$$

$$(t = (3x+1)e^{-3x} \Rightarrow dt = (3e^{-3x} - 3(3x+1)e^{-3x})dx = -9xe^{-3x}dx)$$

$$\Omega = -\frac{1}{9}\int \frac{tdt}{(1+t)^2} = -\frac{1}{9}\left(log(1+t) + \frac{1}{1+t}\right) + C; C \in \mathbb{R}$$

$$\Omega = -\frac{1}{9}\left(log\left(e^{3x} + (3x+1)\right) + \frac{e^{3x}}{e^{3x} + (3x+1)} - 3x\right) + C$$

$$\Omega = \int \frac{3x^2 + x}{1 + 6x(1+e^{3x}) + 2e^{3x} + e^{6x} + 9x^2} dx =$$

$$= \frac{x}{3} - \frac{1}{9}\left(log\left(e^{3x} + (3x+1)\right) + \frac{e^{3x}}{e^{3x} + (3x+1)}\right) + C$$

AN.107. Solution (Adrian Popa)

$$\int_a^b \int_a^b \left(f(x) + f(z)\right) dxdz + \int_a^b \int_a^b \left(f(y) + f(t)\right) dydt = \left(\int_a^b \int_a^b \left(f(x) + f(z)\right) dxdz\right)^2 =$$

$$= \left(\int_a^b \left(f(x) + f(z)\right)(b-a)dz\right)^2 = \left(\int_a^b \left(f(x) + f(x)\right)(b-a)dx\right)^2 =$$

$$= 4(b-a)^2 \left(\int_a^b f(x)dx\right)^2 \overset{(1)}{\geq} 2(b-a)^3 \left(2\int_a^b f(x)dx - 2b + 2a\right)$$

$$(1) \Leftrightarrow \left(\int_a^b f(x)dx\right)^2 \geq (b-a)\int_a^b f(x)dx - (b-a)^2 \Leftrightarrow$$

$$\left(\int_a^b f(x)dx\right)^2 - (b-a)\int_a^b f(x)dx + (b-a)^2 \overset{(2)}{\geq} 0$$

$$\left(\int_a^b f(x)dx\right)^2 - (b-a)\int_a^b f(x)dx + (b-a)^2 \geq$$

$$\geq\left(\int_a^b f(x)dx\right)^2-2(b-a)\int_a^b f(x)dx+(b-a)^2$$

$$=\left(\int_a^b f(x)dx-(b-a)\right)^2\geq 0$$

$(0<a\leq b<1\Rightarrow b-a\geq 0)\Rightarrow(2)$ true $\Rightarrow(1)$ true.

AN.108. Solution (Abner Chinga Bazo)

$$\Omega=\int_0^{\frac{\pi}{4}} sin\left(x-\frac{\pi}{4}\right)\sqrt{(1+sinx)(1+cosx)}dx=$$

$$=\int_0^{\frac{\pi}{4}} sin\left(x-\frac{\pi}{4}\right)\sqrt{4sin^2\left(\frac{x}{2}+\frac{\pi}{4}\right)cos^2\frac{x}{2}}dx=\int_0^{\frac{\pi}{4}} sin\left(x-\frac{\pi}{4}\right)2sin\left(\frac{x}{2}+\frac{\pi}{4}\right)cos\frac{x}{2}dx=$$

$$=\int_0^{\frac{\pi}{4}} sin\left(x-\frac{\pi}{4}\right)\left(sin\left(x+\frac{\pi}{4}\right)+sin\frac{\pi}{4}\right)dx=$$

$$=\int_0^{\frac{\pi}{4}}\left[sin\left(x-\frac{\pi}{4}\right)sin\left(x+\frac{\pi}{4}\right)+sin\left(x-\frac{\pi}{4}\right)sin\frac{\pi}{4}\right]dx=$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{4}}\left[cos2x-cos\frac{\pi}{2}+cosx-cos\left(\frac{\pi}{2}-x\right)\right]dx=$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{4}}(cos2x+cosx-sinx)dx=-\frac{1}{2}\left[\frac{1}{2}sin2x+sinx+cosx\right]_0^{\frac{\pi}{4}}$$

$$=-\frac{1}{4}\left(2\sqrt{2}-1\right)$$

AN.109. Solution (Khanh Hung Vu)

$$f(x)-log_3 x=4-f\left(5^{log_3 x}\right);\quad(1)$$

Put $f(x)=g(x)+2+log_{15}x$*, we have*

$$f\left(5^{\log_3 x}\right) = g\left(5^{\log_3 x}\right) + 2 + \log_{15}\left(5^{\log_3 x}\right)$$

$$= g\left(5^{\log_3 x}\right) + 2 + \log_3 x \cdot \log_{15} 5 =$$

$$= g\left(5^{\log_3 x}\right) + 2 + \log_3 x\,(1 - \log_{15} 3)$$

So, the equation (1) is equivalent to:

$$g(x) + 2 + \log_{15} x - \log_3 x = 4 - g\left(5^{\log_3 x}\right) - 2 - \log_3 x\,(1 - \log_{15} 3)$$

Or: $g(x) + 2 + \log_3 x\,(\log_{15} 3 - 1) = 4 - g\left(5^{\log_3 x}\right) - 2 + \log_3 x\,(\log_{15} 3 - 1) \Rightarrow$

$$g(x) = -g\left(5^{\log_3 x}\right);\quad (2)$$

Substitute $x \to 3^{\log_5 x}$*, we have the functional equation (2) equivalent to:*

$$g\left(3^{\log_5 x}\right) = -g(x) \Rightarrow g(x) = -g\left(3^{\log_5 x}\right);\quad (3)$$

Substitute $x \to 3^{\log_5 x}$*, we have the functional equation (3) equivalent to:*

$$g\left(3^{\log_5 x}\right) = -g\left(3^{\log_5 3 \cdot \log_5 x}\right);\quad (4)$$

From (3),(4) we have: $g(x) = (-1)^2\, g\left(3^{\log_5 3 \cdot \log_5 x}\right)$

Similarly, we have:

$$g(x) = (-1)^1 g\left(3^{\log_5 x}\right) = (-1)^2 g\left(3^{\log_5 3 \cdot \log_5 x}\right) = (-1)^3 g\left(3^{\log_5^2 3 \cdot \log_5 x}\right)$$

$$= \cdots =$$

$$= (-1)^{2n} g\left(3^{\log_5^{2n-1} 3 \cdot \log_5 x}\right); \forall n \geq 1$$

So, we have: $g(x) = g\left(3^{\log_5^{2n-1} 3 \cdot \log_5 x}\right)$

Given $n \to \infty$*, we have* $g(x) = g(1)$

Substitute $x = 1$ *in (2), we have* $g(1) = -g(1) \Rightarrow g(1) = 0 \Rightarrow f(x) = 2 + \log_{15} x$

$$\Omega = \int_2^3 (f(x) - 2) \cdot \log_x 15\, dx = \int_2^3 (2 + \log_{15} x - 2) \cdot \log_x 15\, dx =$$

$$= \int_2^3 \log_{15} x \cdot \log_x 15\, dx = 1$$

AN.110. Solution (George Florin Şerban)

$$u\cdot v\le\frac{u^2+v^2}{2},\forall u,v\in\mathbb{R}\Leftrightarrow(u-v)^2\ge 0,\forall u,v\in\mathbb{R}$$

$$\int_{-a}^{a}\int_{-a}^{a}|(x+y)(1-xy)|\,dxdy=\int_{-a}^{a}\int_{-a}^{a}|x+y|\cdot|1-xy|\,dxdy\le$$

$$\le\int_{-a}^{a}\int_{-a}^{a}\frac{|x+y|^2+|1-xy|^2}{2}dxdy$$

$$=\int_{-a}^{a}\int_{-a}^{a}\frac{x^2+y^2+2xy+1-2xy+x^2y^2}{2}dxdy=$$

$$=\int_{-a}^{a}\int_{-a}^{a}\frac{x^2+y^2+1+x^2y^2}{2}dxdy=\int_{-a}^{a}\int_{-a}^{a}\frac{(x^2+1)(y^2+1)}{2}dxdy=$$

$$=\int_{-a}^{a}2\int_{0}^{a}\frac{x^2+y^2+1+x^2y^2}{2}dxdy=\int_{-a}^{a}\left(\frac{a^3}{3}y^2+\frac{a^3}{3}+ay^2+a\right)dy=$$

$$=2\int_{0}^{a}\left(\frac{a^3}{3}y^2+\frac{a^3}{3}+ay^2+a\right)dy=2\left(\frac{a^6}{9}+\frac{2a^4}{3}+a^2\right)=\frac{2}{9}(3a+a^3)^2$$

AN.111. Solution (Florentin Vişescu)

$$\textit{Let be } P=(1+a^3+(a^3)^2)\left(1+a^{3^2}+\left(a^{3^2}\right)^2\right)\cdot\ldots\cdot\left(1+a^{3^n}+\left(a^{3^n}\right)^2\right)$$

$$(1-a^3)P=(1-a^3)(1+a^3+(a^3)^2)\left(1+a^{3^2}+\left(a^{3^2}\right)^2\right)\cdot\ldots$$

$$\cdot\left(1+a^{3^n}+\left(a^{3^n}\right)^2\right)=$$

$$=\left(1-a^{3^3}\right)\left(1+a^{3^2}+\left(a^{3^2}\right)^2\right)\cdot\ldots\cdot\left(1+a^{3^n}+\left(a^{3^n}\right)^2\right)=$$

$$\vdots$$

$$=\left(1-a^{3^n}\right)\left(1+a^{3^n}+\left(a^{3^n}\right)^2\right)=1-a^{3^{n+1}}\Rightarrow P=\frac{1-a^{3^{n+1}}}{1-a^3}$$

$$\Omega=\lim_{n\to\infty}\prod_{n=1}^{\infty}\left(1+\left(\frac{1}{\pi}\right)^{3^n}+\left(\frac{1}{\pi^2}\right)^{3^n}\right)=\lim_{n\to\infty}\frac{1-\left(\frac{1}{\pi}\right)^{3^{n+1}}}{1-\left(\frac{1}{\pi}\right)^3}=\frac{\pi^3}{\pi^3-1}$$

AN.112. Solution (Mokhtar Khassani)

$$\Omega = \lim_{n\to\infty}\left(n\left(\frac{\log\left(1+\frac{\sqrt[n]{e}}{n}\right)^{n+1}}{\log\left(1+\frac{\sqrt[n+1]{e}}{n+1}\right)^{n}}-1\right)\right)$$

$$\overset{\log(1+x)\overset{0}{\sim}x}{\cong} \lim_{n\to\infty} n\left(\frac{(n+1)^2\cdot\sqrt[n]{e}}{n^2\cdot\sqrt[n+1]{e}}-1\right) = \lim_{n\to\infty}\frac{1}{n}\left((n+1)^2\cdot\sqrt[n]{e}-n^2\cdot\sqrt[n+1]{e}\right)$$

$$= 2+\lim_{n\to\infty} n\left(\sqrt[n]{e}-\sqrt[n+1]{e}\right) = 2+\lim_{n\to\infty} n\left(\frac{\frac{\sqrt[n]{e}-1}{\frac{n}{n}}}{}-\frac{\frac{\sqrt[n+1]{e}-1}{\frac{n+1}{n+1}}}{}\right) = 2$$

AN.113. Solution (Kamel Benaicha)

$$\Omega_1 = \lim_{(x,y)\to(0,0)}\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y}\sqrt{tanx}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{tanx}\,dx = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{\sqrt{y}}{1+y^2}dy = 2\int_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}}\frac{u^2du}{1+u^4}$$

$$= 2\int_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}}\frac{dv}{1+v^4} \Rightarrow 2\Omega_1 = 2\int_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}}\frac{\left(1+\frac{1}{u^2}\right)du}{u^2+\frac{1}{u^2}} = 2\int_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}}\frac{d\left(1-\frac{1}{u}\right)}{\left(u-\frac{1}{u}\right)^2+2}$$

$$\Omega_1 = \frac{1}{\sqrt{2}}\left[tan^{-1}\left(\frac{1}{\sqrt{2}}\left(u-\frac{1}{u}\right)\right)\right]_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}} = \sqrt{2}tan^{-1}\left(\frac{1}{\sqrt{2}}\left(\sqrt[4]{3}-\frac{1}{\sqrt[4]{3}}\right)\right) =$$

$$= \sqrt{2}tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}\right)$$

$$\Omega_2 = \lim_{(x,y)\to(0,0)}\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y}\sqrt{cotx}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{cotx} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{tanx}\,dx = \Omega_1$$

$$\Omega = \Omega_1\Omega_2 = 2\left(tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}\right)\right)^2$$

AN.114. Solution (Ali Jaffal)

$$\text{Let: } x \in \left[0,\frac{\pi}{2}\right] \to sinx \in [0,1]$$

$$\text{Consider the function } f(t) = t^{sinx}, t \in [0,\infty)$$

$$f''(t) = sinx(sinx-1)t^{sinx-2} < 0 \Rightarrow f -\text{concave on } [0,\infty)$$

$$\text{Let: } a,b,c > 0, a+b+c = 3$$

$$f\left(\frac{a+b+c}{3}\right) \ge \frac{f(a)+f(b)+f(c)}{3} \Rightarrow f(a)+f(b)+f(c) \le 3f(1) \le 3$$

$$\text{So, } a^{sinx}+b^{sinx}+c^{sinx} \le 3, \forall x \in \left[0,\frac{\pi}{2}\right]$$

$$\int_0^{\frac{\pi}{2}} a^{sinx}dx + \int_0^{\frac{\pi}{2}} b^{sinx}dx + \int_0^{\frac{\pi}{2}} c^{sinx}dx \le \frac{3\pi}{2}$$

AN.115. Solution (Mokhtar Khassani)

$$\text{Since: } \lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = \lim_{n\to 0}\sqrt[n]{\frac{a^n+b^n}{2}} = exp\left(\lim_{n\to 0}\frac{log\left(\frac{a^n+b^n}{2}\right)}{n}\right)$$

$$\overset{hopital}{=} exp\left(\lim_{n\to 0}\frac{a^n loga+b^n logb}{a^n+b^n}\right) = \sqrt{ab}$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{k+1}+\sqrt[n]{k+3}}{\sqrt[n]{k+2}+\sqrt[n]{k+4}}\right)^n = \lim_{n\to\infty}\left(\frac{\frac{\sqrt[n]{k+1}+\sqrt[n]{k+3}}{2}}{\frac{\sqrt[n]{k+2}+\sqrt[n]{k+4}}{2}}\right)^n$$

$$= \sqrt{\frac{(k+1)(k+3)}{(k+2)(k+4)}}$$

$$\Omega = \prod_{k=1}^{\infty}\left(\lim_{n\to\infty}\left(\frac{\sqrt[n]{k+1}+\sqrt[n]{k+3}}{\sqrt[n]{k+2}+\sqrt[n]{k+4}}\right)^n\right)^2 = \lim_{m\to\infty}\frac{\prod_{k=1}^m(k+1)\prod_{k=1}^m(k+3)}{\prod_{k=1}^m(k+2)\prod_{k=1}^m(k+4)}$$

$$= \lim_{m\to\infty}\frac{8}{(m+2)(m+4)} = 0$$

AN.116. Solution (Kamel Benaicha)

$$\Omega = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{k\cdot(2(n-k)-1)!!}{(k+1)!\cdot(2(n-k)+2)!!}\right) \overset{p=n-k}{\triangleq}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{p=0}^{n}\frac{(n-p)(2p-1)!!}{(n-p+1)!\cdot(2p+2)!!}\right)$$

We have: $\frac{(2p-1)!!}{(2p+2)!!}=\frac{(2p)!}{2^p p! 2^{p+1}(p+1)!}$

$$\Omega=\sum_{n=0}^{+\infty}\sum_{p=0}^{n}\frac{(n-p)(2p)!}{(n-p+1)!\,2^{2p+1}(p+1)!\,p!}$$

$$=\sum_{n=0}^{+\infty}\sum_{p=0}^{n}\frac{(2p)!}{(n-p)!\,(p!)^2 2^{2p+1}(p+1)}-\sum_{n=0}^{+\infty}\sum_{p=0}^{n-1}\frac{(2p)!}{(n-p)!\,(p!)^2 2^{2p+1}(p+1)}$$

$$=\frac{1}{2}+\sum_{n=1}^{+\infty}\sum_{p=0}^{n}\frac{(2p)!}{(n-p)!\,(p!)^2 2^{2p+1}(p+1)}$$

$$-\sum_{n=1}^{+\infty}\sum_{p=0}^{n-1}\frac{(2p)!}{(n-p)!\,(p!)^2 2^{2p+1}(p+1)}$$

$$=\frac{1}{2}+\sum_{n=1}^{+\infty}\frac{(2n)!}{(n!)^2 2^{2n+1}(n+1)}+\sum_{n=1}^{+\infty}\sum_{p=0}^{n-1}\frac{(2p)!}{(n-p)!\,(p!)^2 2^{2p+1}(p+1)}$$

$$-\sum_{n=1}^{+\infty}\sum_{p=0}^{n-1}\frac{(2p)!}{(n-p)!\,(p!)^2 2^{2p+1}(p+1)}$$

$$=\sum_{n=0}^{+\infty}\frac{(2n)!}{(n!)^2 2^{2n+1}(n+1)}$$

Put $f(x)=\sum_{n=0}^{+\infty}\frac{(2n)!x^{n+1}}{(n!)^2 2^{2n+1}(n+1)}$

$$f'(x)=\sum_{n=0}^{+\infty}\frac{(2n)!\,x^n}{(n!)^2 2^{2n+1}}=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(2n)!\left(\sqrt{x}\right)^{2n}}{(n!)^2 2^{2n}}=\frac{1}{2}\cdot\frac{d}{dt}sin^{-1}t\Big|_{t=\sqrt{x}}$$

$$=\frac{1}{2\sqrt{1-x}}\Rightarrow\int_0^1 f(x)dx=\Omega=-\sqrt{1-x}\Big|_0^1=1$$

$$\Omega=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{k\cdot(2n-2k-1)!!}{(k+1)!\cdot(2+2n-2k)!!}\right)=1$$

AN.117. Solution (Khaled Abd Imouti)

Let be the function $f(x)=\frac{1}{(x-1)^2}, x\in(0,1)$ *−increasing and concave.*

x	0	1
f′(x)	++++++++++++++++++++++++++++++++	
f(x)	1 ↗↗↗↗↗↗↗↗↗↗↗↗↗↗	∞

$$\text{So: } \frac{1}{(x-1)^2}>1, \forall x\in(0,1)$$

$$(x^3+x^2+1)^2 \overset{?}{\gtrless} 4x^3, \quad x^6+2x^5+x^4+2x^3+2x^2+1 \overset{?}{\gtrless} 4x^3$$

$$x^6+2x^5+x^4-2x^3+2x^2+1 \overset{?}{\gtrless} 0; \ (*)$$

$$x^6+2x^5+x^4-2x^3+2x^2 \overset{?}{\gtrless} 0, \quad x^4+2x^3+(x^2-2x+2) \overset{?}{\gtrless} 0; (**)$$

$$x^4+2x^3+x^2-2x+1 \overset{?}{\gtrless} 0, \ \underbrace{(x^4+2x^3)}_{>0}+\underbrace{(x-1)^2}_{>0} \overset{?}{\gtrless} 0 \text{ it's true.}$$

$$\text{So: } \left(\frac{x^3+x^2+1}{x-1}\right)^2 > 4x^3, \forall x\in(0,1)$$

$$\int_0^{\sqrt[4]{a}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx+\int_0^{\sqrt[4]{b}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx+\int_0^{\sqrt[4]{c}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx$$

$$>\int_0^{\sqrt[4]{a}}4x^3dx+\int_0^{\sqrt[4]{b}}4x^3dx+\int_0^{\sqrt[4]{c}}4x^3dx$$

$$=x^4\Big|_0^{\sqrt[4]{a}}+x^4\Big|_0^{\sqrt[4]{b}}+x^4\Big|_0^{\sqrt[4]{c}}=a+b+c=1$$

AN.118. Solution (Khaled Abd Imouti)

$$\int_a^b sinf(x)dx+\frac{1}{2}\int_a^b tanf(y)dy+\int_a^b cosf(t)dt+\frac{1}{2}\int_a^b cotf(z)dz \overset{?}{\gtreqless} (\sqrt{2}+1)(b-a)$$

$$\int_a^b [sinf(x)+cosf(x)]dx+\frac{1}{2}\int_a^b [tanf(y)+cotf(y)]dy \overset{?}{\gtreqless} (\sqrt{2}+1)(b-a)$$

Let be the function: $g(\theta) = sin\theta + cos\theta + \frac{1}{2}(tan\theta + cot\theta), \theta \in \left(0, \frac{\pi}{2}\right)$

$$\lim_{\theta \to 0_+}[g(\theta)] = +\infty;\ \lim_{\theta \to \frac{\pi}{2}}[g(\theta)] = +\infty$$

$$g'(\theta) = cos\theta - sin\theta + \frac{1}{2}(tan^2\theta - cot^2\theta)$$

$$g'(\theta) = 0 \Leftrightarrow \theta = \frac{\pi}{4}; g\left(\frac{\pi}{4}\right) = \sqrt{2} + 1$$

θ	0 $\frac{\pi}{4}$ $\frac{\pi}{2}$
$g'(\theta)$	$- - - - - - - - - 0 + + + + + + +$
$g(\theta$	$+\infty$ ↘↘↘↘↘ $\sqrt{2}+1$ ↗↗↗↗↗ $+\infty$

So, $g(\theta) \geq \sqrt{2} + 1, \forall\theta \in \left(0, \frac{\pi}{2}\right)$

$$\int_a^b sinf(x)dx + \frac{1}{2}\int_a^b tanf(y)dy + \int_a^b cosf(t)dt + \frac{1}{2}\int_a^b cotf(z)dz$$
$$\geq \left(\sqrt{2} + 1\right)(b - a)$$

AN.119. Solution (Sergio Esteban)

$$\Omega = \lim_{n\to\infty}\left(tan\left(\alpha + \frac{\pi}{4}\right)\right)^{\frac{1}{sin\alpha}}, \textit{where } \alpha = \gamma - H_n + logn$$

$$\Omega = \lim_{n\to\infty}\left(tan\left(\alpha + \frac{\pi}{4}\right)\right)^{\frac{1}{sin\alpha}} = \lim_{\alpha\to 0}\left(1 + tan\left(\alpha + \frac{\pi}{4}\right) - 1\right)^{\frac{1}{sin\alpha}}$$

$$= \lim_{\alpha\to 0}\left(1 + \frac{1}{\frac{1}{tan\left(\alpha + \frac{\pi}{4}\right) - 1}}\right)^{\frac{1}{tan\left(\alpha+\frac{\pi}{4}\right)-1}\cdot\frac{tan\left(\alpha+\frac{\pi}{4}\right)-1}{sin\alpha}}$$

$$= e^{\lim\limits_{\alpha\to 0}\frac{tan\left(\alpha+\frac{\pi}{4}\right)-1}{sin\alpha}} \overset{L'H}{=} e^{\lim\limits_{\alpha\to\infty}\frac{sec^2\left(\alpha+\frac{\pi}{4}\right)}{cos\alpha}} = e^2, \qquad \Omega = e^2$$

AN.120. Solution (Adrian Popa)

$$(*):\ \frac{log(a+b)-log2}{log(a+b-2)-log2}\le\frac{log(ab)}{2log\left(\sqrt{ab}-1\right)}\Leftrightarrow$$

$$\frac{log\left(\frac{a+b}{2}\right)}{log\left(\frac{a+b}{2}-1\right)}\le\frac{log\sqrt{ab}}{log\left(\sqrt{ab}-1\right)}$$

Let be the function: $f(x)=\frac{logx}{log(x-1)},x>2\ ,\ f'(x)=\frac{(x-1)log(x-1)-xlogx}{x(x-1)log^2(x-1)}<0$

$$\begin{cases}m=\dfrac{a+b}{2},n=\sqrt{ab}\\ m\ge n\ (Am-Gm)\end{cases}\Rightarrow f\left(\frac{a+b}{2}\right)<f\left(\sqrt{ab}\right)\Leftrightarrow(*)$$

AN.121. Solution (Adrian Popa)

$$\left(\frac{3\pi}{2\left(cos\frac{\pi}{2m}+cos\frac{\pi}{2n}+cos\frac{\pi}{2p}\right)}\right)^3\overset{Am-Gm}{\lesseqgtr}\left(\frac{3\pi}{2\left(3\cdot\sqrt[3]{cos\frac{\pi}{2m}\cdot cos\frac{\pi}{2n}\cdot cos\frac{\pi}{2p}}\right)}\right)^3$$

$$=\frac{\pi^3}{2^3\cdot cos\frac{\pi}{2m}\cdot cos\frac{\pi}{2n}\cdot cos\frac{\pi}{2p}}$$

We must show that: $\int_0^{\pi/2}\sqrt[m]{tanx}dx\ge\frac{\pi}{2cos\frac{\pi}{2m}}$

$$I=\int_0^{\frac{\pi}{2}}\sqrt[m]{tanx}dx=\int_0^{\frac{\pi}{2}}(tanx)^{\frac{1}{m}}dx=\int_0^{\frac{\pi}{2}}(tanx)^n dx$$

$$\underbrace{\overset{\frac{1}{m}=n\in\left(0,\frac{1}{2}\right)}{\cong}}_{tanx=t}\int_0^{\infty}\frac{t^n}{1+t^2}dt\overset{t^2=y}{\cong}\frac{1}{2}\int_0^{\infty}\frac{y^{\frac{n-1}{2}}}{1+y}dy=\frac{1}{2}\int_0^{\infty}\frac{y^{\frac{n+1}{2}-1}}{(1+y)^{\frac{n+1}{2}-\frac{1-n}{2}}}dy$$

$$=B\left(\frac{n+1}{2},\frac{1-n}{2}\right)=\frac{\pi}{2sin\frac{\pi(n+1)}{2}}=\frac{\pi}{2cos\frac{\pi}{2m}}$$

So: $I=\int_0^{\frac{\pi}{2}}\sqrt[m]{tanx}dx=\frac{\pi}{2cos\frac{\pi}{2m}}$ *then:*

$$\Omega(m)\cdot\Omega(n)\cdot\Omega(p)\ge\left(\frac{3\pi}{2\left(cos\frac{\pi}{2m}+cos\frac{\pi}{2n}+cos\frac{\pi}{2p}\right)}\right)^3$$

AN.122. Solution (Rahim Shahbazov)

$$(x+y)^4 \geq 16x^2y^2$$

$$\int_a^b\int_a^b \frac{dxdy}{(x+y)^4} \leq \int_a^b\int_a^b \frac{dxdy}{16x^2y^2} = \frac{1}{16}\int_a^b \frac{dx}{x^2}\cdot\int_a^b \frac{dy}{y^2} = \frac{1}{16}\cdot\frac{(b-a)^2}{a^2b^2}$$

$$\frac{1}{16}\cdot\frac{(b-a)^2}{a^2b^2} \leq \frac{(b-a)^2(a^2+ab+b^2)}{48a^3b^3} \Leftrightarrow a^2+ab+b^2 \geq 3ab \Leftrightarrow (a-b)^2 \geq 0$$

AN.123. Solution (Soumitra Mandal)

$$\frac{1^3}{(3x+2y)^2}+\frac{1^3}{(2x+3y)^2} \overset{Radon}{\geq} \frac{(1+1)^3}{(2x+3y+3x+2y)^2} = \frac{8}{25(x+y)^2}$$

$$\int_a^b\int_a^b \frac{dxdy}{(3x+2y)^2}+\int_a^b\int_a^b \frac{dxdy}{(2x+3y)^2} \geq \frac{8}{25}\int_a^b\int_a^b \frac{dxdy}{(x+y)^2}$$

$$= \frac{8}{25}\int_a^b \left(-\frac{1}{x+y}\right)\Big|_a^b dy = \frac{8}{25}log\frac{\left(1+\frac{b}{a}\right)^2}{4\frac{a}{b}}$$

We need to prove: $\frac{8}{25}log\frac{\left(1+\frac{b}{a}\right)^2}{4\frac{a}{b}} \geq \frac{8}{25}\left(\frac{b-a}{b+a}\right)^2 = \frac{8}{25}\left(\frac{\frac{b}{a}-1}{\frac{b}{a}+1}\right)^2$

$$\Rightarrow log\frac{(1+x)^2}{4x} \geq \left(\frac{x-1}{x+1}\right)^2 \qquad (1)$$

Let $f(x) = 2log(1+x) - log(4x) - \left(\frac{x-1}{x+1}\right)^2; x \geq 1$

$$f'(x) = \frac{(x-1)^2(x+2)}{x(x+1)^3} \geq 0, x \geq 0$$

Hence f is increasing fuction $f(x) \geq f(1) = 0 \Rightarrow \quad (1)$

$$\int_a^b\int_a^b \frac{dxdy}{(3x+2y)^2}+\int_a^b\int_a^b \frac{dxdy}{(2x+3y)^2} \geq \frac{8}{25}\left(\frac{b-a}{b+a}\right)^2$$

AN.124. Solution (Igor Soposki)

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(log\left(1+\frac{1}{n+1}\right)\right)^2}{log\left(1+\frac{1}{n+2}\right)}\right)$$

$$= \lim_{n\to\infty}\frac{2log\left(1+\frac{1}{n+1}\right)\cdot\frac{1}{1+\frac{1}{n+1}}\cdot\left(-\frac{1}{(n+1)^2}\right)}{\frac{1}{1+\frac{1}{n+2}}\cdot\left(-\frac{1}{(n+2)^2}\right)}$$

$$= \lim_{n\to\infty}\frac{2log\left(1+\frac{1}{n+1}\right)\cdot\frac{n+1}{n+2}\cdot\frac{1}{(n+1)^2}}{\frac{n+2}{n+3}\cdot\frac{2}{(n+2)^2}}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{(n+1)(n+2)}\cdot log\left(1+\frac{1}{n+1}\right)}{\frac{1}{(n+2)(n+3)}} = 2\lim_{n\to\infty}\frac{(n+3)log\left(1+\frac{1}{n+1}\right)}{n+1}$$

$$= 2\lim_{n\to\infty}\frac{(n+1)log\left(1+\frac{1}{n+1}\right)}{n+1} + 4\lim_{n\to\infty}\frac{log\left(1+\frac{1}{n+1}\right)}{n+1} = 0$$

AN.125. Solution (Khaled Abd Imouti)

$$l = \frac{2\sqrt{2}}{5}\int_a^b sinf(x)dx + \frac{1}{10}\int_a^b tanf(x)dx + \frac{2\sqrt{2}}{5}\int_a^b cosf(x)dx +$$

$$\frac{1}{10}\int_a^b cotf(x)dx \overset{?}{\gtreqless} b-a$$

$$\int_a^b\left[\frac{2\sqrt{2}}{5}\left(sinf(x)+cosf(x)\right)+\frac{1}{10}\left(tanf(x)+cotf(x)\right)\right]dx \overset{?}{\gtreqless} b-a$$

$$g(\theta) = \frac{2\sqrt{2}}{5}(sin\theta+cos\theta)+\frac{1}{10}(tan\theta+cot\theta)$$

$$\lim_{\theta\to 0} g(\theta) = 0;\ \lim_{\theta\to\infty} g(\theta) = +\infty$$

$$g'(\theta) = \frac{2\sqrt{2}}{5}(cos\theta-sin\theta)+\frac{1}{10}\cdot\frac{cos^2\theta-sin^2\theta}{sin\theta\cdot cos\theta}$$

$$g'(\theta)=0\Rightarrow\theta=\frac{\pi}{4},\qquad g\left(\frac{\pi}{4}\right)=1\Rightarrow g(\theta)\geq 1,\forall\theta\in\left(0,\frac{\pi}{2}\right)$$

$$So:\ l\geq\int_a^b 1dx=b-a$$

θ	0 $\qquad\frac{\pi}{4}\qquad$ $\frac{\pi}{2}$
$g'(\theta)$	$-----0+++++$
$g(\theta)$	$+\infty$ ↘↘↘↘ 1 ↗↗↗↗ $+\infty$

AN.126. Solution (Rahim Shahbazov)

$$x+y=k\Rightarrow(x+y+z)^3=(k+z)^3=\left(\frac{k}{2}+\frac{k}{2}+2\right)^3\geq 27\cdot\frac{k^2}{4}\cdot z\Rightarrow$$

$$(x+y+z)^3\geq\frac{27}{4}\cdot(x+y)^2\cdot z\Rightarrow\frac{(x+y+z)^3}{(x+y)^2}\geq\frac{27}{4}\cdot z\ then$$

$$\int_0^a\int_0^a\int_0^a\frac{(x+y+z)^3}{(x+y)^2}dxdydz\geq\frac{27}{4}\int_0^a dx\int_0^a dy\int_0^a zdz=\frac{27a^3}{8}$$

AN.127. Solution (Adrian Popa)

$$\sum_{cyc}\frac{\Gamma'(a)}{\Gamma(a)}<log(abc)-\frac{1}{2}\quad(1),\qquad log(abc)-\frac{1}{2}<\frac{1}{2}+\sum_{cyc}\frac{\Gamma'(a)}{\Gamma(a)}\quad(2)$$

$$\frac{\Gamma'(a)}{\Gamma(a)}=\psi(a)$$

We must show that: $\sum_{cyc}\psi(a)<log(abc)-\frac{1}{2}$

$$\psi(a)=log(a)-\frac{1}{2a}-\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\cdot e^{-at}dt\Rightarrow$$

$$\begin{cases}\psi(a)<log(a)-\frac{1}{2a}\\ \psi(b)<log(b)-\frac{1}{2b}\\ \psi(c)<log(c)-\frac{1}{2c}\end{cases}\Rightarrow$$

$$\sum_{cyc}\psi(a)<log(a)+log(b)+log(c)-\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=log(abc)-\frac{1}{2}$$

$$e^t-1>t\Rightarrow\frac{1}{e^t-1}<\frac{1}{t}\Rightarrow\frac{1}{e^t-1}-\frac{1}{t}<0\Rightarrow-\left(\frac{1}{e^t-1}-\frac{1}{t}\right)>0$$

$$\psi(a)=log(a)-\frac{1}{2a}-\int_0^\infty\frac{1}{2}e^{-at}dt-\underbrace{\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt}_{>0}\Rightarrow$$

$$\psi(a)=log(a)-\frac{1}{2a}-\int_0^\infty\frac{1}{2}e^{-at}dt=log(a)-\frac{1}{2a}+\frac{1}{2a}e^{-at}\Big|_0^\infty=log(a)-\frac{1}{a}$$

$$\text{So: }\begin{cases}\psi(a)>log(a)-\frac{1}{a}\\\psi(b)>log(b)-\frac{1}{b}\\\psi(c)>log(c)-\frac{1}{c}\end{cases}$$

$$\sum_{cyc}\psi(a)>log(abc)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=log(abc)-1$$

AN.128. Solution (Ali Jaffal)

$$x_0=7,y_0=5$$

$$2x_n=x_{n-1}+y_{n-1}\Rightarrow x_n=\frac{x_{n-1}+y_{n-1}}{2}\text{ and }y_n=\sqrt{x_{n-1}\cdot y_{n-1}}$$

We will prove that by induction $x_n>0$ *and* $y_n>0$ *for all* $n\in\mathbb{N}$

For $n=0$ *we have* $x_0=7,y_0=5$

Suppose that $x_n>0$ *and* $y_n>0$ *for all* $n\in\mathbb{N}$ *then*

$$x_{n+1}=\frac{x_n+y_n}{2}>0\text{ and }y_{n+1}=\sqrt{x_n\cdot y_n}>0$$

So: $x_n>0$ *and* $y_n>0$ *for all* $n\in\mathbb{N}$

We know that by inequality Gm-Am for all $n\in\mathbb{N}$ $x_n>y_n$.

So $x_n<x_{n-1}$ *then* (x_n) *–is strictly decreasing but* $x_n>0$ *then* (x_n) *is convergent to* l.

We have: $y_n=\sqrt{x_{n-1}\cdot y_{n-1}}>\sqrt{y_{n-1}\cdot y_{n-1}}\geq y_{n-1}$ *then* (y_n) *–is increasing but* $y_n<x_n<7$ *then* (y_n) *–is convergent to* l'.

We have $5<l'<l<7$ *but* $l=\frac{l+l'}{2}$ *then* $l=l'$ *and* $\lim_{n\to\infty}\frac{x_n}{y_n}=1$

$$\text{So }\lim_{n\to\infty}\left(\frac{5x_n}{7y_n}\right)^{\frac{H_n}{n}}=?$$

Let: $u_n=\left(\frac{5x_n}{7y_n}\right)^{\frac{H_n}{n}}\Rightarrow log(u_n)=\frac{H_n}{n}log\left(\frac{5x_n}{7y_n}\right)=\frac{log(n)+\lambda+\varphi(n)}{n}\cdot\left[log\left(\frac{5}{7}\right)+log(\frac{x_n}{y_n})\right]$

Where $\lim_{n\to\infty}\varphi(n)=0$. *So,* $\lim_{n\to\infty}\big(log(u_n)\big)=0$ *therefore* $\lim_{n\to\infty}\left(\frac{5x_n}{7y_n}\right)^{\frac{H_n}{n}}=1$

AN.129. Solution (Adrian Popa)

$$\Omega_k(m) = 2\lim_{x\to 0}\frac{-\frac{1}{k^{m+2}}\cdot (coskx)^{\frac{1}{k^{m+2}}-1}\cdot(-k)\cdot sinkx}{2x} = \frac{k^2}{k^{m+2}} = \frac{1}{k^m}$$

$$\Omega = \left(\sum_{k=1}^{\infty}\Omega_k(2)\right)\left(\sum_{k=1}^{\infty}\Omega_k(3)\right) = \sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(2)\zeta(3)$$

AN.130. Solution (Soumitra Mandal)

$$\underbrace{\int_a^b\int_a^b\dots\int_a^b\dots}_{for\ "2n"times}\prod_{i=1}^{n}\frac{dx_i dy_i}{x_i+y_i} \overset{Am-Gm}{\leq} \frac{1}{2^n}\left(\prod_{i=1}^{n}\int_a^b\frac{dx_i}{\sqrt{x_i}}\right)\left(\prod_{i=1}^{n}\int_a^b\frac{dy_i}{\sqrt{y_i}}\right)$$

$$= \frac{1}{2^n}\left(\prod_{i=1}^{n}\frac{\sqrt{b}-\sqrt{a}}{1-\frac{1}{2}}\right)\left(\prod_{i=1}^{n}\frac{\sqrt{b}-\sqrt{a}}{1-\frac{1}{2}}\right) = 2^n\left(\sqrt{b}-\sqrt{a}\right)^{2n}$$

We need to prove:

$$\left(\frac{b-a}{2}log\frac{b}{a}\right)^n \geq 2^n\left(\sqrt{b}-\sqrt{a}\right)^{2n} \Leftrightarrow \frac{b-a}{2}log\frac{b}{a} \geq 2\left(\sqrt{b}-\sqrt{a}\right)^2$$

$$\Leftrightarrow\ log\frac{b}{a} \geq \frac{4\left(\sqrt{b}-\sqrt{a}\right)^2}{b-a} = \frac{4\left(\sqrt{b}-\sqrt{a}\right)}{\sqrt{b}+\sqrt{a}}$$

Let: $f(x) = logx - \frac{4(\sqrt{x}-1)}{\sqrt{x}+1}$ *for all* $x \geq 1$

$$f'(x) = \frac{1}{x} - \frac{1}{x(\sqrt{x}+1)^2} > 0\ for\ all\ x \geq 1.$$

Hence f is an increasing function $\therefore\ f(x) \geq f(1) = 0$ *then:*

$logx - \frac{4(\sqrt{x}-1)}{\sqrt{x}+1}$ *for all* $x \geq 1$

Let: $x = \frac{b}{a}$ *hence* $log\frac{b}{a} \geq \frac{4(\sqrt{b}-\sqrt{a})}{\sqrt{b}+\sqrt{a}}$ *is proved.*

So:

$$\underbrace{\int_a^b \int_a^b \ldots \int_a^b \ldots}_{for\ "2n"times} \prod_{i=1}^{n} \frac{dx_i dy_i}{x_i + y_i} \le \left(\frac{b-a}{2} log \frac{b}{a}\right)^n$$

AN.131. Solution (Tran Hong)

$$g(x) = e^{\varphi(x)} f(x) \Rightarrow g'(x) = e^{\varphi(x)}[\varphi'(x)f(x) + f'(x)]$$

So, $(1 + 2x^2 log2)f(x) + xf'(x) = 1 \Leftrightarrow \frac{1+2x^2 log2}{x} \cdot f(x) + f'(x) = \frac{1}{x}$

$$\Rightarrow \varphi'(x) = \frac{1}{x} + 2xlog2 \Rightarrow \varphi(x) = logx + x^2 log2$$

$$\Rightarrow g'(x) = \left(f(x) \cdot e^{logx + x^2 log2}\right)' = \frac{1}{x} = (logx)';\ x > 0$$

$$\Rightarrow f(x) \cdot e^{logx + x^2} = logx + C; C = const.$$

$$f(1) = 2 \Rightarrow C = 4$$

$$\Rightarrow f(x) = \frac{logx + 4}{e^{logx + x^2 log2}} = \frac{logx + 4}{x \cdot 2^{x^2}} \Rightarrow f(n) = \frac{logn + 4}{n \cdot 2^{n^2}}$$

$$\Rightarrow \lim_{n\to\infty} nf(n) = \lim_{n\to\infty} \frac{logn + 4}{n \cdot 2^{n^2}} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2n \cdot 2^{n^2} log2} = \lim_{n\to\infty} \frac{1}{n^2 \cdot 2^{n^2+1} log2}$$

$$= 0$$

AN.132. Solution (Ali Jaffal)

$$\Omega(x) = \int_0^x \frac{sinht \cdot cost}{(sint + cosht)(sinht + cost)} dt$$

$$So: \lim_{x\to 0} \Omega(x) = 0$$

$$\lim_{x\to 0} xlog(2\Omega(x)) = \lim_{x\to 0}\left(xlog2 + xlog\Omega(x)\right) = 0 + \lim_{x\to 0}\left(xlog\Omega(x)\right)$$

But $\lim_{x\to 0}\left(xlog\Omega(x)\right) = \lim_{x\to 0} \frac{log\Omega(x)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0} \frac{-x^2}{\Omega(x)} \cdot \Omega'(x)$

We know that: $\Omega'(x) = \frac{sinhx \cdot cosx}{(sinx + coshx)(sinhx + cosx)}$ *then* $\lim_{x\to 0} \Omega'(x) = 0$ *and*

$$\lim_{x\to 0} \frac{-x^2}{\Omega(x)} = \lim_{x\to 0} \frac{-2x}{\Omega'(x)} = \lim_{x\to 0} \frac{-2}{\Omega''(x)} = -2$$

$$\text{Then: } \lim_{x\to 0}\left(x\log\Omega(x)\right)=0 \text{ therefore } \Omega=\lim_{x\to 0}(2\Omega(x))^x=1$$

AN.133. Solution (Avishek Mitra)

$$erf^3(x)+36erf(x)\overset{Am-Gm}{\geq}2\sqrt{36erf^4(x)}=12erf^2(x)$$

$$\Rightarrow\int_0^1 erf^3(x)dx+36\int_0^1 erf(x)dx\geq 12\int_0^1 erf^2(x)dx$$

$$\Rightarrow\int_0^1 erf^3(x)dx+36\int_0^1 erf(x)dx-12\left(\int_0^1 erf(x)dx\right)^2$$

$$\geq 12\left[\int_0^1 erf^2(x)dx-\left(\int_0^1 erf(x)dx\right)^2\right]\geq 0$$

$$\therefore [\int_0^1 1\cdot erf(x)dx\overset{Holder}{\leq}\sqrt{\int_0^1 1^2dx\cdot\int_0^1 erf^2(x)dx}$$

$$\Rightarrow\int_0^1 erf^2(x)dx\geq\left(\int_0^1 erf(x)dx\right)^2]$$

AN.134. Solution (Tran Hong) *Let:*

$$\varphi(x)=log(tanx)+4\sqrt{2}(sinb-sina)-10x; x\in\left(0,\frac{\pi}{2}\right)$$

$$\varphi'(x)=4\sqrt{2}(sinx+cosx)+\frac{1}{sinxcosx}\overset{t=sinx+cosx>1}{\triangleq}4\sqrt{2}t+\frac{2}{t^2-1}-10$$

$$=\frac{2\left[2\sqrt{2}t^3-5t^2-2\sqrt{2}+6\right]}{t^2-1}=\frac{2\left(\sqrt{2}-t\right)^2\left(2\sqrt{2}t+3\right)}{t^2-1}\geq 0,\forall t\geq 1$$

$$\Rightarrow\varphi(t)\nearrow on\left(0,\frac{\pi}{2}\right) then\ 0\leq a\leq b\leq\frac{\pi}{2}\Rightarrow\varphi(a)\leq\varphi(b)$$

$$log(tana)+4\sqrt{2}(sina-cosa)-10a$$

$$\leq log(tanb)+4\sqrt{2}(sinb-cosb)-10b$$

$$\Leftrightarrow log\left(\frac{tanb}{tana}\right)+4\sqrt{2}(cosa-cosb)+4\sqrt{2}(sinb-sina)\geq 10(b-a)$$

AN.135. Solution (Ravi Prakash)

$$Let: m=2020, 0\leq k\leq m$$

$$\frac{\binom{2020}{k}}{\binom{n+2020}{k}}=\frac{m!}{k!\,(m-k)!}\cdot\frac{(n+k)!\,(m-k)!}{(n+m)!}=\frac{m!\,n!}{(n+m)!}\binom{n+k}{k}$$

Then

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+k}}=\frac{m!n!}{(n+m)!}\sum_{k=0}^{m}\binom{n+k}{k}. Let: S=\sum_{k=0}^{m}\binom{n+k}{k}$$

$$=\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+m}{m}$$

$$=\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+m}{m}$$

$$=\binom{n+2}{0}+\binom{n+2}{1}+\binom{n+2}{2}+\cdots+\binom{n+m}{m}$$

$$=\binom{n+3}{0}+\binom{n+3}{1}+\binom{n+3}{2}+\cdots+\binom{n+m}{m}$$

$$=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$=\binom{n+m}{m-1}+\binom{n+m}{m}$$

$$=\binom{n+m+1}{m}$$

Thus,

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+k}}=\frac{m!\,n!}{(n+m)!}\cdot\frac{(n+m+1)!}{m!\,(n+1)!}=\frac{n+m+1}{n+1}=1+\frac{m}{n+1}$$

$$\left(\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+k}}\right)^{n}=\left(1+\frac{m}{n+1}\right)^{n}=\left(\left[\left(1+\frac{m}{n+1}\right)^{\frac{n+1}{m}}\right]^{\frac{n}{n+1}}\right)^{n}$$

$$\lim_{n\to\infty}\left(\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+m}{n+k}}\right)^{n}=e^{m}=e^{2020}$$

AN.136.Solution (Tran Hong)

$$\text{For } x \in \left[0,\frac{\pi}{2}\right] \text{ we have: } \frac{2}{\pi}x \le \sin x \le x,\ \frac{2}{\pi} \overset{(1)}{\le} \cos\frac{x}{2} \le 1$$

$$\text{Let: } \varphi(x) = \cos\frac{x}{2} - \frac{2}{\pi}, \forall x \in \left[0,\frac{\pi}{2}\right]$$

$$\varphi'(x) = -\frac{1}{2}\sin\frac{x}{2} < 0, \forall \in \left[0,\frac{\pi}{2}\right] \text{ then } \varphi(x) \searrow \text{ on } \left[0,\frac{\pi}{2}\right]$$

$$\varphi(x) \ge \varphi\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{4} - \frac{2}{\pi} = \frac{\sqrt{2}}{2} - \frac{2}{\pi} > 0 \Rightarrow (1)\text{true}.$$

Now,

$$\pi xy + \pi(1-y)\sin x \overset{0\le y\le 1}{\ge} \pi xy \cdot \frac{2}{\pi} + \pi(1-y)\cdot\frac{2}{\pi}\cdot x = 2x$$

$$\pi xy\cos\frac{x}{2} + \pi(1-y)\sin x \le \pi xy + \pi(1-y)x = \pi x$$

AN.137. Solution (Naren Bhandari)

We note that

$$\forall k \ge 1, \frac{k}{k+1} < 1 \Rightarrow \sqrt{\frac{k}{k+1}} < 1 \Rightarrow 0 < \sqrt{\frac{k}{k+1}} < 1$$

and hence we have

$$0 < \sum_{k=1}^{n}\sqrt{\frac{k}{k+1}}\cdot\binom{n}{k} < \sum_{k=1}^{n}\binom{n}{k} = 2^n - 1$$

and hence we have

$$0 < \Omega < \lim_{n\to\infty}(2^n-1)\sqrt{\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}}$$

$$= \lim_{n\to\infty}\left(\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}\right)^{\frac{1}{2}} = \lim_{n\to\infty}\left(\frac{2(2^n-1)^2}{n^2\cdot 2^{2n+1}}\right)^{\frac{1}{2}} = 0$$

AN.138. Solution (Tran Hong)

$$\text{For } 0 \le x \le 1 \text{ let } u = \sin^{-1}x, \left(0 \le u \le \frac{\pi}{2}\right) \Rightarrow x = \sin u$$

$$\text{We have: } \frac{sin^{-1}x}{1+sin^{-1}x} \le x \Leftrightarrow sin^{-1}x = x + x \cdot sin^{-1}x \Leftrightarrow$$

$$u \le sinu + u \cdot sinu \Leftrightarrow (1+u)sinu - u \ge 0 \ldots (1)$$

$$\text{Let } f(u) = (1+u)sinu - u, \left(0 \le u \le \frac{\pi}{2}\right)$$

$$f'(u) = sinu + (1+u)cosu - 1$$

$$f''(u) = sinu + cosu + ucosu - 1 \ge 0, 0 \le u \le \frac{\pi}{2} \ldots (*)$$

$$\therefore sinx + cosx \ge 1 \Leftrightarrow (sinx + cosx)^2 \ge 1 \Leftrightarrow$$

$$2sinxcosx \ge 0 \text{ true for } x \in \left[0, \frac{\pi}{2}\right] \text{ and } xcosx \ge 0$$

$$\text{So, } (*)true \Rightarrow f'(u) \uparrow on \left[0, \frac{\pi}{2}\right] \Rightarrow f'(u) \ge f'(0) \Rightarrow f'(u) \ge 0$$

$$\Rightarrow f(u) \uparrow on \left[0, \frac{\pi}{2}\right] \Rightarrow f(u) \ge f(0) = 0$$

$$\Rightarrow (1)true \Rightarrow \frac{sin^{-1}x}{1+sin^{-1}x} \le x$$

$$LHS = \sum_{cyc} \left[\frac{9yzsin^{-1}x}{1+sin^{-1}x}\right] \le 27xyz \overset{Am-Gm}{\leqq} (x+y+z)^3$$

$$Equality \Leftrightarrow x = y = z = 0$$

AN.139. Solution (Ravi Prakash)

$$\text{For } k \in \mathbb{N}, k \ge 1 \Rightarrow k! \ge 2^k - 1 > 2^{k-1} \Rightarrow$$

$$\left(2^{k-1}\right)^2 < \left(2^k - 1\right)^2 \le \left(2^k - 1\right)k! \le (k!)^2 \Rightarrow$$

$$\frac{1}{k!} \le \frac{1}{\sqrt{(2^k-1)k!}} < \frac{1}{2^{k-1}} \Rightarrow$$

$$e - 1 < \sum_{k=1}^{n} \frac{1}{k!} \le \sum_{k=1}^{n} \frac{1}{\sqrt{(2^k-1)k!}} < \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 2, \forall n \in \mathbb{N}$$

$$\frac{e-1}{\sqrt{2n-1}} < \frac{1}{\sqrt{2n-1}} \sum_{k=1}^{n} \frac{1}{\sqrt{(2^k-1)k!}} < \frac{2}{\sqrt{2n-1}}, \forall n \in \mathbb{N}$$

$$Since \lim_{n\to\infty} \frac{e-1}{\sqrt{2n-1}} = \lim_{n\to\infty} \frac{2}{\sqrt{2n-1}} = 0$$

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{2n-1}}\sum_{k=1}^{n}\frac{1}{\sqrt{(2^k-1)k!}}\right) = 0$$

AN.140. Solution (Florentin Vişescu)

$$3abc - a^3 - b^3 - c^3 = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

$$(3abc - a^3 - b^3 - c^3)^2 = (a+b+c)^2(ab+bc+ca-a^2-b^2-c^2)^2$$

We must show:

$$(a+b+c)^2(ab+bc+ca-a^2-b^2-c^2)^2 \le (a^2+b^2+c^2)^3$$

$$(a+b+c)^2(3(ab+bc+ca)-(a+b+c)^2)^2 \le ((a+b+c)^2 - 2(ab+bc+ca))^3$$

$$Let\ a+b+c = p; ab+bc+ca = q$$

$$p^2(3q-p^2)^2 \le (p^2-2q)^3, \qquad 8q^3 \le 3p^2q^2 \Leftrightarrow 8q \le 3p^2$$

$$8(ab+bc+ca) \le 3(a+b+c)^2$$

$$8(ab+bc+ca) \le 3(a^2+b^2+c^2) + 6(ab+bc+ca)$$

$$2(ab+bc+ca) \le 3(a^2+b^2+c^2)\ \textit{true from:}$$

$$\overset{Am-Gm}{\Longrightarrow}\begin{cases} a^2+b^2 \ge 2ab \\ b^2+c^2 \ge 2bc \Rightarrow \\ c^2+a^2 \ge 2ca \end{cases}$$

$$3(a^2+b^2+c^2) \ge 2(a^2+b^2+c^2) \ge 2(ab+bc+ca)$$

$$(3abc - a^3 - b^3 - c^3)^2 \le (a^2+b^2+c^2)^3$$

$$\int_0^1\int_0^1\int_0^1 \sqrt[3]{(3f(x)f(y)f(z) - f^3(x) - f^3(y) - f^3(z))^2}\,dxdydz$$

$$\le \int_0^1\int_0^1\int_0^1 (f^2(x)+f^2(y)+f^2(z))dxdydz \le 3\int_0^1 f^2(x)dx$$

AN.141. Solution (Cao Mai Tanh Tam)

$$\int_a^b\int_a^b\int_a^b \left(\frac{(x+y+z)(xy+yz+zx)}{xyz}\right)dxdydz$$

$$= \int_a^b\int_a^b\int_a^b \left(\frac{x^2y + x^2z + xy^2 + 3xyz + xz^2 + y^2z + yz^2}{xyz}\right)dxdydz$$

$$= \int_a^b\int_a^b\int_a^b \left[\frac{x}{z} + \frac{x}{y} + \frac{y}{z} + 3 + \frac{z}{y} + \frac{y}{x} + \frac{z}{x}\right]dxdydz$$

$$= \int_a^b\int_a^b \left[\frac{x^2}{2z} + \frac{x^2}{2y} + \frac{xy}{z} + 3x + \frac{xz}{y} + (y+z)log|x|\right]\Big|_a^b dydz$$

$$=\int_a^b\int_a^b\left[\frac{b^2-a^2}{2}\left(\frac{1}{y}+\frac{1}{z}\right)+(b-a)\left(\frac{y}{z}+3+\frac{y}{z}\right)+(y+z)log\left(\frac{b}{a}\right)\right]dydz$$

$$=\int_a^b\left[\frac{b^2-a^2}{2}\left(\frac{y}{z}+log|y|\right)+(b-a)\left(\frac{y^2}{2z}+3y+\frac{y^2}{2z}\right)+\left(zy+\frac{y^2}{2}\right)log\left(\frac{b}{a}\right)\right]\Big|_a^b$$

$$=\int_a^b\left[\frac{b^2-a^2}{2}\left(\frac{b-a}{z}+log\left(\frac{b}{a}\right)\right)+(b-a)\left(\frac{b^2-a^2}{2z}+3(b-a)+\frac{b^2-a^2}{2z}\right)\right.$$

$$\left.+\left((b-a)z+\frac{b^2-a^2}{2}\right)log\left(\frac{b}{a}\right)\right]dz$$

$$=\left[\frac{b^2-a^2}{2}\left((b-a)log|z|+zlog\left(\frac{b}{a}\right)\right)\right.$$

$$+(b-a)\left(\frac{b^2-a^2}{2}log|z|+3z(b-a)+\frac{b^2-a^2}{2}log|z|\right)$$

$$\left.+\left((b-a)\frac{z^2}{2}+\frac{b^2-a^2}{2}z\right)log\left(\frac{b}{a}\right)\right]\Big|_a^b=3(b-a)^2(b+a)\left[log\left(\frac{b}{a}\right)+1\right]$$

Let prove:

$$3(b-a)^2(b+a)\left[log\left(\frac{b}{a}\right)+1\right]\le\frac{(2a^2+5ab+2b^2)(b-a)^3}{ab}$$

$$\Leftrightarrow 3ab(b+a)\left[log\left(\frac{b}{a}\right)+1\right]\le(2a^2+5ab+2b^2)(b-a)^3$$

$$\Leftrightarrow [2(a+b)^2+ab](b-a)-3ab(b+a)\left[log\left(\frac{b}{a}\right)+1\right]\ge 0$$

$$\Leftrightarrow 2(a+b)^2(b-a)+ab(b-a)-2ab(b+a)\left[log\left(\frac{b}{a}\right)+1\right]-ab(b+a)\left[log\left(\frac{b}{a}\right)+1\right]\ge 0$$

$$\Leftrightarrow 2(a+b)\left[b^2-a^2-ablog\left(\frac{b}{a}\right)-1\right]+ab\left(b-a-blog\left(\frac{b}{a}\right)-alog\left(\frac{b}{a}\right)-b-a\right)$$

AN.142. Solution (Abner Chinga Bazo)

$$\Omega=\int_{\frac{\pi^5}{1024}}^{\frac{\pi^5}{243}}\frac{sin\left(\sqrt[5]{x}\right)\cdot sin\left(5\sqrt[5]{x}\right)\cdot sin\left(5\sqrt[5]{x}\right)}{\sqrt[5]{x^4}}dx\overset{\sqrt[5]{x}=t}{=}5\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}sint\cdot sin3t\cdot sin5t\,dt$$

$$=\frac{5}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(cos2t-cos4t)sin5t\,dt$$

$$=\frac{5}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(sin7t+sin3t-sin9t-sint)\,dt=$$

$$=\frac{5}{4}\left[-\frac{cos7t}{7}-\frac{cos3t}{3}+\frac{cos9t}{9}+cost\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{5}{252}\left(41-41\sqrt{2}\right)=\frac{205}{252}\left(1-\sqrt{2}\right)$$

AN.143. Solution (Adrian Popa)

$$\Omega(n,r)=\sum_{k=0}^{n}\frac{(-1)^k}{3r+3k-2}\binom{n}{k}=\sum_{k=0}^{n}\left(\int_0^1 x^{3r+3k-3}dx\,(-1)^k\binom{n}{k}\right)=$$

$$=\int_0^1\sum_{k=0}^{n}x^{3r-3}\cdot x^{3k}\cdot(-1)^k\binom{n}{k}dx=\int_0^1 x^{3r-3}\cdot\sum_{k=0}^{n}x^{3k}\cdot(-1)^k\binom{n}{k}dx=$$

$$=\int_0^1 x^{3r-3}\cdot(1-x^3)^n dx\overset{\substack{x^3=t\\ dx=\frac{1}{3}\cdot t^{-\frac{2}{3}}dt}}{=}\frac{1}{3}\int_0^1 t^{r-1}\cdot(1-t)^n\cdot t^{-\frac{2}{3}}dt=$$

$$=\frac{1}{3}\int_0^1 t^{r-\frac{2}{3}-1}\cdot(1-t)^{n+1-1}dt=\frac{1}{3}B\left(r-\frac{2}{3},n+1\right)=\frac{1}{3}\cdot\frac{\Gamma\left(r-\frac{2}{3}\right)\Gamma(n+1)}{\Gamma\left(r+n+\frac{1}{3}\right)}$$

$$\omega(r)=\lim_{n\to\infty}\sqrt[n]{\Omega(n,r)}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{\Omega(n+1,r)}{\Omega(n,r)}=$$

$$=\lim_{n\to\infty}\frac{\Gamma\left(r-\frac{2}{3}\right)\Gamma(n+2)}{\Gamma\left(r+n+\frac{4}{3}\right)}\cdot\frac{\Gamma\left(r+n+\frac{1}{3}\right)}{\Gamma\left(r-\frac{2}{3}\right)\Gamma(n+1)}=$$

$$=\lim_{n\to\infty}\frac{(n+1)!\,\Gamma\left(r+n+\frac{1}{3}\right)}{\left(r+n+\frac{1}{3}\right)\Gamma\left(r+n+\frac{1}{3}\right)\cdot n!}=\lim_{n\to\infty}\frac{n+1}{r+n+\frac{1}{3}}=1$$

AN.144. Solution (Adrian Popa)

$$(sin^3x+tany)^2(cos^3x+tany)\overset{Holder}{\geq}\left(sin^2x\cdot cosx+\sqrt[3]{tan^2y}\cdot\sqrt[3]{tany}\right)^3$$

$$(sin^3x+tany)^2(cos^3x+tany)\geq(sin^2x\cdot cosx+tany)^3\Leftrightarrow$$

$$\frac{(sin^3x+tany)^2(cos^3x+tany)}{(sin^2x\cdot cosx+tany)^3}\geq 1\Leftrightarrow$$

$$\int_a^b\int_a^b\frac{(sin^3x+tany)^2(cos^3x+tany)}{(sin^2x\cdot cosx+tany)^3}dxdy\geq\int_a^b\int_a^b 1\,dxdy=(b-a)^2$$

AN.145. Solution (Rovsen Pirguliev)

From Chebyshev's integral inequality:

$$\left(\int_a^b f(x)dx\right)\cdot\left(\int_a^b g(x)dx\right)\le(b-a)\cdot\int_a^b f(x)g(x)dx$$

$$Lhs=\int_0^1 e^{x^2}dx\cdot\int_0^1 e^{-x^2}dx\le(1-0)\cdot\int_0^1 e^{x^2}\cdot e^{-x^2}dx=x|_0^1=1$$

It remains to prove that:

$$1<\left(\frac{1+e}{2\sqrt{e}}\right)^2\Leftrightarrow 1<\frac{1+e}{2\sqrt{e}}\Leftrightarrow 2\sqrt{e}<1+e\Leftrightarrow\sqrt{e}\overset{AM-GM}{\le}\frac{e+1}{2}\ (true)$$

AN.146. Solution (Adrian Popa)

$$\int_a^b(f^8(x)+f^2(x))dx+b-a\ge\int_a^b(f^5(x)+f(x))dx\Leftrightarrow$$

$$\int_a^b\left(f^8(x)+f^2(x)\right)dx+\int_a^b f(x)dx\ge\int_a^b\left(f^5(x)+f(x)\right)dx\Leftrightarrow$$

$$\int_a^b(f^8(x)+f^2(x)+1)dx\ge\int_a^b\left(f^5(x)+f(x)\right)dx\Leftrightarrow$$

$$(f^8(x)+f^2(x)+1)\ge f^5(x)+f(x)$$

Let be the function: $g(y)=y^8-y^5+y^2-y+1;\ \left(y^8\ge 0, y^2\ge 0\Rightarrow y^8+y^2\ge 2\left|y^5\right|\right)$

We must show that: $2\left|y^5\right|+1-y^5-y\ge 0$

$\therefore$ *If* $y<0\Rightarrow 2\left|y^5\right|+1-y^5-y\ge 0$ *is clearly true.*

$\therefore$ *If* $y>0\Rightarrow 2\left|y^5\right|+1-y^5-y=2y^5-y^5-y+1=y^5-y+1$

Let be the function: $h(y)=y^5-y+1; h'(y)=5y^4-1$

$$h'(y)=0\Leftrightarrow y^4=\frac{1}{5}\Rightarrow y^2=\frac{1}{\sqrt{5}}\Rightarrow y_{1,2}=-\frac{1}{\sqrt[4]{5}}<0$$

(contradiction with $y > 0$)

and $y_{3,4} = \frac{1}{\sqrt[4]{5}}$

y	0	$\frac{1}{\sqrt[4]{5}}$	∞
$h'(y)$	$+ + + + +0 + + + + + + + + + + + +$		
$h(y)$	1	$\nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow$	∞

Then $h(y) > 0$. *So,*

$$2|y^5| + 1 - y^5 - y \geq 0, \forall y \in \mathbb{R} \Rightarrow g(y) = y^8 - y^5 + y^2 - y + 1 > 0 \Rightarrow$$

$$(f^8(x) + f^2(x) + 1) \geq f^5(x) + f(x) \Leftrightarrow$$

$$\int_a^b (f^8(x) + f^2(x))dx + b - a \geq \int_a^b (f^5(x) + f(x))dx$$

AN.147. Solution (Khaled Abd Imouti)

$$\Omega = \int (4cot^3x - 5cot^2x + 7cotx)e^x dx$$

Let be the function: $f(x) = (4cot^3x - 5cot^2x + 7cotx)e^x$ and

$$F(x) = (Acot^2x + Bcotx + C)e^x$$

$$F'(x) = [-2Acot^3x + (A - B)cot^2x + (-2A + B)cotx + (-B + C)]e^x$$
$$= f(x) \Rightarrow$$

$$-2A = 4 \Rightarrow A = -2; A - B = -5 \Rightarrow B = 3; -B + C = 0 \Rightarrow C = 3.$$

So, $F(x) = (-2cot^3x + 3cotx + 3)e^x$

$$\Omega = \int (4cot^3x - 5cot^2x + 7cotx)e^x dx = (-2cot^3x + 3cotx + 3)e^x + C$$

AN.148. Solution (Ravi Prakash)

$$For\ x \geq 1$$

$$\frac{2xtan^{-1}x - log(1 + x^2)}{(1 + x^2)(tan^{-1}x)^2} = \frac{\frac{2x}{1 + x^2} tan^{-1} - (log(1 + x^2))\frac{1}{1 + x^2}}{(tan^{-1}x)^2}$$

$$= \frac{\left(\frac{d}{dx}(log(1 + x^2))(tan^{-1}x) - (log(1 + x^2))\frac{d}{dx}(tan^{-1}x)\right)}{(tan^{-1}x)^2}$$

$$=\frac{d}{dx}\left(\frac{log(1+x^2)}{tan^{-1}x}\right)$$

$$\int_1^a \frac{2xtan^{-1}x-log(1+x^2)}{(1+x^2)(tan^{-1}x)^2}dx=\int_1^a \frac{d}{dx}\left(\frac{log(1+x^2)}{tan^{-1}x}\right)$$

$$=\frac{log(1+x^2)}{tan^{-1}x}\Big|_1^a=\frac{log(1+a^2)}{tan^{-1}a}-\frac{log2}{\pi/4}$$

$$\Rightarrow \frac{4log2}{\pi}+\int_1^a \frac{2xtan^{-1}x-log(1+x^2)}{(1+x^2)(tan^{-1}x)^2}dx=\frac{log(1+a^2)}{tan^{-1}a}<\frac{a^2}{tan^{-1}a}$$

$$as\ log(1+x)<x,\forall x>0$$

AN.149. Solution (Ali Jaffal)

$$Let\ u_n=n\ and\ v_n=\frac{1}{2}H_n+log\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)$$

We know that:

$$v_{n+1}-v_n=\frac{1}{2}H_{n+1}+log\left(\prod_{k=1}^{n+1}\frac{2k}{2k-1}\right)-\frac{1}{2}H_n-log\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)=$$

$$=\frac{1}{2(n+1)}+log\left(\frac{2(n+1)}{2n}\right)=\frac{1}{2(n+1)}+log\left(1+\frac{1}{n}\right)$$

$$Then:\ \lim_{n\to\infty}\frac{v_{n+1}-v_n}{u_{n+1}-u_n}=\lim_{n\to\infty}\left(\frac{1}{2(n+1)}+log\left(1+\frac{1}{n}\right)\right)$$

So, by Cesaro-Stolz we have:

$$\lim_{n\to\infty}\frac{v_n}{u_n}=0\ then\ \Omega=0.$$

AN.150. Solution (Ravi Prakash)

$$Let\ f(x)=\frac{x}{x^2+1},x\in[0,2]$$

$$Now,let\ a_n=n\sum_{k=1}^{n}\frac{k(k+1)}{(k^2+n^2)(k^2+2k+1+n^2)}$$

$$=\frac{1}{n}\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)\cdot\left(\frac{k+1}{n}\right)}{\left(\left(\frac{k}{n}\right)^2+1\right)\left(\left(\frac{k+1}{n}\right)^2+1\right)}=\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\cdot f\left(\frac{k+1}{n}\right)$$

As f is uniforly continuous on $[0,2]$ given $\varepsilon>0$ there exists $\delta>0$ such that:

$$|f(x+h)-f(x)|<\varepsilon,\forall x\in[0,2]\ whenever\ |h|<\delta\ and\ x+h\in[0,2]$$

Choose n sufficiently large so that $n\delta>1$,

$$\left|f\left(\frac{k}{n}+\frac{1}{n}\right)-f\left(\frac{k}{n}\right)\right|<\varepsilon$$

$$f\left(\frac{k}{n}\right)-\varepsilon<f\left(\frac{k+1}{n}\right)<f\left(\frac{k}{n}\right)+\varepsilon$$

$$\frac{1}{n}\sum_{k=1}^{n}\left(f\left(\frac{k}{n}\right)\right)^2-\varepsilon\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\right)<\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)f\left(\frac{k+1}{n}\right)$$

$$<\frac{1}{n}\sum_{k=1}^{n}\left(f\left(\frac{k}{n}\right)\right)^2+\ \varepsilon\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\right)$$

Taking limit as $n\to\infty$, we get

$$\int_0^1 f^2(x)dx-\varepsilon\int_0^1 f(x)dx\le\Omega\le\int_0^1 f^2(x)dx+\varepsilon\int_0^1 f(x)dx$$

Make $\varepsilon\to 0_+$, so that

$$\Omega=\int_0^1 f^2(x)dx=\int_0^1\frac{x^2}{(x^2+1)^2}dx\overset{x=tan\theta}{\cong}\int_0^{\pi/4}\frac{tan^2\theta}{sec^4\theta}\cdot sec^2\theta d\theta$$

$$=\int_0^{\pi/4}sin^2\theta d\theta=\frac{1}{2}\int_0^{\pi/4}(1-cos2\theta)d\theta=\frac{1}{2}\left[\theta-\frac{1}{2}sin(2\theta)\right]\Big|_0^{\pi/4}=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

AN.151. Solution (Ali Jaffal)

$$\Omega_n=\frac{1}{n^3}\sum_{k=1}^{n}H_kH_{n+k}$$

We know that $(H_n)_{n\ge1}$ is increasing for

So,

$$H_k \leq H_n \text{ and } H_{n+k} \leq H_{2n} \text{ for all } 1 \leq k \leq n \text{ then:}$$

$$\Omega_n \leq \frac{1}{n^3}\sum_{k=1}^{n} H_k H_{2n} \leq \frac{1}{n^2} H_n H_{2n}$$

But: $H_n \leq H_{2n}$ *then* $0 < \Omega \leq \frac{(H_{2n})^2}{n^2}$

We have: $H_{2n} = \gamma + log(2n) + \zeta(2n)$

$$Where: \lim_{n\to\infty} \zeta(2n) = 0 \Rightarrow \frac{H_{2n}}{n} = \frac{\gamma}{n} + \frac{log(2n)}{n} + \frac{\zeta(2n)}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{H_{2n}}{n} = 0 \text{ since } \lim_{n\to\infty} \frac{log(2n)}{n} = 0$$

$$So, 0 \leq \lim_{n\to\infty} \Omega_n \leq 0 \Rightarrow \lim_{n\to\infty} \Omega_n = 0. \ Then:$$

$$\lim_{n\to\infty} \sqrt{\Omega_n} = 0 \text{ and } \Omega = 0$$

AN.152. Solution (Michael Sterghiou)

$$\int_a^{2a}\int_b^{3b}\int_c^{4c}\left(\sqrt[6]{\frac{x+1}{y+1}} + \sqrt[8]{\frac{y+1}{z+1}} + \sqrt[10]{\frac{z+1}{x+1}}\right)dxdydz \geq 15abc; (1)$$

Let $\frac{x+1}{y+1} = u^2, \frac{y+1}{z+1} = v^2, \frac{z+1}{x+1} = t^2$ then $T = \sqrt[3]{u} + \sqrt[4]{v} + \sqrt[5]{t}$ with $uvt =$

$1; (c); u, v, t > 0$ as $x, y, z > 0$

We will minimize $T(u, v, t)$ by $uvt = 1$.

Consider the Lagrangian $L(u, v, t, \lambda) = T(u, v, t) - \lambda(uvt - 1)$

For the extreme of T we need to look into the points that make the vector

$\nabla L(u, v, t, \lambda) = 0$ or $\frac{\partial L}{\partial u} = \frac{\partial L}{\partial v} = \frac{\partial L}{\partial t} = \frac{\partial L}{\partial \lambda} = 0$ or

$$\begin{cases} \frac{1}{3\sqrt[3]{u^2}} = \lambda vt = \lambda \cdot \frac{1}{u} \\ \frac{1}{4\sqrt[4]{v^3}} = \lambda ut = \lambda \cdot \frac{1}{v} \\ \frac{1}{5\sqrt[5]{t^4}} = \lambda uv = \lambda \cdot \frac{1}{t} \end{cases} \Rightarrow \begin{cases} u = 27\lambda^3; \ (2) \\ v = 256\lambda^4; \ (3) \\ t = 3125\lambda^5; \ (4) \end{cases}, uvt = 1 \Rightarrow$$

$uvt = 1 = 216 \cdot 10^5 \cdot \lambda^{12} \Rightarrow \lambda = \sqrt[12]{\frac{1}{216 \cdot 10^5}}$ $as\ \lambda > 0$ (from (2) for example)

Now, $(u_0, v_0, t_0) = (27\lambda^3, 256\lambda^4, 3125\lambda^5)$ and $T(u_0, v_0, t_0) = 12\lambda \cong$

$2.937 > \frac{5}{2}$

Hence $T > \frac{5}{2}$ and

$$\int_a^{2a}\int_b^{3b}\int_c^{4c} T dxdydz \geq \int_a^{2a}\int_b^{3b}\int_c^{4c} \frac{5}{2} dxdydz = \frac{5}{2}(4c - c)(3b - b)(2a - a)$$

$$= 15abc.$$

Equality for $a = b = c = 0$. Done!

AN.153. Solution (Adrian Popa)

$$1) \int_a^b \frac{x^2 dx}{1 + x^2} = \int_a^b \frac{(x^2 + 1 - 1)dx}{1 + x^2} = \int_a^b \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= (b - a) - (tan^{-1}b - tan^{-1}a)$$

$$2) \int_a^b\int_a^b \frac{y^2 dxdy}{(1 + x^2)(1 + y^2)} = \int_a^b \frac{dx}{1 + x^2}\int_a^b \frac{y^2 dy}{1 + y^2}$$

$$= \left(tan^{-1}x|_a^b\right) \cdot \left(y|_a^b - tan^{-1}y|_a^b\right) =$$

$$= (b - a)(tan^{-1}b - tan^{-1}a) - (tan^{-1}b - tan^{-1}a)^2$$

$$3) \int_a^b\int_a^b\int_a^b \frac{dxdydz}{(1 + x^2)(1 + y^2)(1 + z^2)} = \int_a^b \frac{dx}{1 + x^2}\int_a^b \frac{y^2 dy}{1 + y^2}\int_a^b \frac{z^2 dz}{1 + z^2} =$$

$$= (b - a)(tan^{-1}b - tan^{-1}a)^2 - (tan^{-1}b - tan^{-1}a)^3$$

$$S = (b - a)^3 - (b - a)^2(tan^{-1}b - tan^{-1}a) + (b - a)^2(tan^{-1}b - tan^{-1}a)$$

$$- (b - a)(tan^{-1}b - tan^{-1}a)^2$$

$$+ (b - a)(tan^{-1}b - tan^{-1}a)^2 -$$

$$-(tan^{-1}b - tan^{-1}a)^3 + log^3\left(\sqrt{\frac{b}{a}}\right) \geq (b - a)^3 \Leftrightarrow$$

$$log^3\left(\sqrt{\frac{b}{a}}\right) \ge (tan^{-1}b - tan^{-1}a)^3 \Leftrightarrow \frac{1}{2}log\left(\frac{b}{a}\right) \ge tan^{-1}b - tan^{-1}a \Leftrightarrow$$

$$\frac{logb - loga}{tan^{-1}b - tan^{-1}a} \overset{(*)}{\ge} 2$$

Let be the functions: $f(x) = logx; g(x) = tan^{-1}x, x \in [a,b]$ *and applying Cauchy Theorem:* $\exists c \in [a,b]$ *such that:*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} = \frac{\frac{1}{c}}{\frac{1}{1+c^2}} = \frac{1+c^2}{c} \overset{AM-GM}{\ge} \frac{2c}{c} = 2 \Rightarrow (*)true.$$

AN.154. Solution (Remus Florin Stanca)

$$3f(3x) = 3x + f(x) \Rightarrow f(3x) = x + \frac{f(x)}{3} = x + \frac{1}{3}\left(\frac{x}{3} + \frac{f\left(\frac{x}{3}\right)}{3}\right)$$

$$= x + \frac{x}{3^2} + \frac{f\left(\frac{x}{3}\right)}{3^2} = \cdots = \frac{x}{3^0} + \frac{x}{3^2} + \cdots + \frac{x}{3^{2n}} + \frac{f\left(\frac{x}{3^n}\right)}{3^{n+1}}$$

We prove by using the mathematical induction that:

$$f(3x) = \frac{x}{3^0} + \frac{x}{3^2} + \cdots + \frac{x}{3^{2n}} + \frac{f\left(\frac{x}{3^n}\right)}{3^{n+1}}$$

1. We prove that: $P(1): f(3x) = \frac{x}{3^0} + \frac{x}{3^2} + \frac{f\left(\frac{x}{3^1}\right)}{3^2}$ *is true (proved)*

2. Suppose that: $P(n): f(3x) = \frac{x}{3^0} + \frac{x}{3^2} + \cdots + \frac{x}{3^{2n}} + \frac{f\left(\frac{x}{3^n}\right)}{3^{n+1}}$ *is true.*

3. We prove by using the fact that $P(n)$ *is true that*

$P(n+1): f(3x) = \frac{x}{3^0} + \frac{x}{3^2} + \cdots + \frac{x}{3^{2n+2}} + \frac{f\left(\frac{x}{3^{n+1}}\right)}{3^{n+2}}$ *is true.*

$$f(3x) = \frac{x}{3^0} + \frac{x}{3^2} + \cdots + \frac{x}{3^{2n}} + \frac{f\left(\frac{x}{3^n}\right)}{3^{n+1}}$$

$$= \frac{x}{3^0} + \frac{x}{3^2} + \cdots + \frac{x}{3^{2n}} + \frac{1}{3^{n+1}}\left(\frac{x}{3^{n+1}} + \frac{f\left(\frac{x}{3^{n+1}}\right)}{3}\right)$$

$$=\frac{x}{3^0}+\frac{x}{3^2}+\cdots+\frac{x}{3^{2n+2}}+\frac{f\left(\frac{x}{3^{n+1}}\right)}{3^{n+2}}\Rightarrow P(n+1) \text{ is true(proved).}$$

$$\Rightarrow f(3x)=\frac{x}{3^0}+\frac{x}{3^2}+\cdots+\frac{x}{3^{2n}}+\frac{f\left(\frac{x}{3^n}\right)}{3^{n+1}}$$

$$\lim_{n\to\infty} x\left(\left(\left(\frac{1}{3}\right)^2\right)^0+\cdots+\left(\left(\frac{1}{3}\right)^2\right)^n\right)+\lim_{n\to\infty}\frac{f\left(\frac{x}{3^n}\right)}{3^{n+1}};\quad (1)$$

$$f \text{ −continuous in } x=0\Rightarrow \lim_{x\to 0} f(x)=f(0)$$

$$f(3x)=x+\frac{f(x)}{3}\Rightarrow f(0)=\frac{f(0)}{3}\Rightarrow 3f(0)=f(0)\Rightarrow f(0)=0\overset{(1)}{\Rightarrow}$$

$$f(3x)=\lim_{n\to\infty} x\cdot\frac{\left(\left(\frac{1}{3}\right)^2\right)^{n+1}-1}{\frac{1}{9}-1}+0\Rightarrow f(x)=\frac{x}{1-\frac{1}{9}}=\frac{9x}{8}\Rightarrow f(x)=\frac{3x}{8}$$

The inequality can be written as:

$$\frac{3}{8}\cdot\left(\frac{8}{3}\cdot\frac{3}{8}x\right)^{sin^2t}\cdot\left(\frac{8}{3}\cdot\frac{3}{8}y\right)^{cos^2t}\le\frac{3}{8}(xsin^2t+ycos^2t)\Leftrightarrow$$

$$x^{sin^2t}\cdot y^{cos^2t}\le\ xsin^2t+ycos^2t;\quad (2)$$

Case I: $x=y=0\Rightarrow 0=0\ (true)$. *Case II:* $x=0; y\neq 0\Leftrightarrow ycos^2t$, *but* $y\ge 0\ (true)$. *Case III:* $x\neq 0, y\neq 0$

Let $g(x)=logx, g:\mathbb{R}_+^*\to\mathbb{R}, \frac{\partial g}{\partial x}=\frac{1}{x}; \frac{\partial^2 g}{\partial x^2}=-\frac{1}{x^2}<0\Rightarrow g$ *−concave, by Jensen inequality, we get:* $\forall x_1,x_2\in(0,1)$ *such that* $x_1+x_2=1$ *and* $x,y\in I\Rightarrow x_1g(x)+x_2g(y)\le g(x\cdot x_1+y\cdot x_2)$

Let $x_1=sin^2t, x_2=cos^2t\Rightarrow$

$$sin^2tlogx+cos^2tlogy\le log(xsin^2t+ycos^2t)\Leftrightarrow$$

$$log\left(x^{sin^2t}\cdot y^{cos^2t}\right)\le log(xsin^2t+ycos^2t)\Leftrightarrow$$

$$x^{sin^2t}\cdot y^{cos^2t}\le xsin^2t+ycos^2t$$

AN.155. Solution (Remus Florin Stanca)

$$\Omega(n)=\lim_{x\to 0}\left(\frac{5^x-1}{x^{n+1}}-\frac{log5}{x^n}-\frac{log^2 5}{2x^{n-2}}-\cdots-\frac{log^n 5}{n!\cdot x}\right)=$$

$$=\lim_{x\to 0}\frac{5^x-1-\frac{xlog5}{1!}-\frac{(xlog5)^2}{2!}-\cdots-\frac{(xlog5)^n}{n!}}{x^{n+1}}=$$

$$= \lim_{x\to 0}\frac{5^x - \left(1 + \frac{xlog5}{1!} + \frac{(xlog5)^2}{2!} + \cdots + \frac{(xlog5)^n}{n!}\right)}{x^{n+1}};\ (1)$$

Let $f_n(x) = 1 + \frac{xlog5}{1!} + \frac{(xlog5)^2}{2!} + \cdots + \frac{(xlog5)^n}{n!}$ then,

$$\Omega(n) = \lim_{x\to 0}\frac{5^x - f_n(x)}{x^{n+1}} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{5^x log5 - f_n'(x)}{(n+1)x^n} \overset{f_n'(x)=f_{n-1}(x)log5}{=}$$

$$= \lim_{x\to 0}\frac{\left(5^x - f_{n-1}(x)\right)log5}{(n+1)x^n}$$

Let $a_n = \lim_{x\to 0}\frac{5^x - f_{n-1}(x)}{x^{n+1}} \Rightarrow a_n = \frac{a_{n-1}}{n+1}log5 \Rightarrow \frac{a_n}{a_{n-1}} = \frac{log5}{n+1} \Rightarrow$

$$\prod_{k=1}^{n}\frac{a_k}{a_{k-1}} = \frac{log^n 5}{(n+1)!} \Rightarrow \frac{a_n}{a_0} = \frac{log^n 5}{(n+1)!}$$

$$\lim_{x\to 0}\frac{5^x - f_{n-1}(x)}{x^{n+1}} = \lim_{x\to 0}\frac{5^x - 1}{x}\frac{log^n 5}{(n+1)!}$$

Therefore,

$$\Omega(n) = \lim_{x\to 0}\left(\frac{5^x - 1}{x^{n+1}} - \frac{log5}{x^n} - \frac{log^2 5}{2x^{n-2}} - \cdots - \frac{log^n 5}{n!\cdot x}\right) = \frac{log^{n+1}5}{(n+1)!}$$

AN.156. Solution (Adrian Popa)

Applying Steiner's theorem for ΔABC we have:

$$c^2\cdot CM_n + b^2\cdot BM_n - AM_n^2\cdot a = BM_n\cdot CM_n\cdot a$$

$$\begin{cases}\frac{BM_n}{CM_n} = n\\ BM_n + CM_n = a\end{cases} \Rightarrow \frac{BM_n + CM_n}{CM_n} = \frac{n+1}{1} \Rightarrow CM_n = \frac{a}{n+1} \Rightarrow$$

$$BM_n = a - \frac{a}{n+1} = \frac{an}{n+1}$$

So, $c^2\cdot CM_n + b^2\cdot BM_n - AM_n^2\cdot a = a\cdot\frac{a}{n+1}\cdot\frac{an}{n+1}\Big| : a \Rightarrow$

$$AM_n^2 = \frac{c^2}{n+1} + \frac{nb^2}{n+1} - \frac{a^2}{(n+1)^2} \Rightarrow$$

$$\sum_{cyc} AM_n^2 = \frac{1}{n+1}\sum_{cyc} a^2 + \frac{n}{n+1}\sum_{cyc} b^2 - \frac{n}{n+1}\sum_{cyc} a^2 \Rightarrow$$

$$\Omega=\frac{1}{a^2+b^2+c^2}\cdot\lim_{n\to\infty}(AM_n^2+BN_n^2+CP_n^2)$$

$$=\frac{1}{a^2+b^2+c^2}\cdot\lim_{n\to\infty}\left(\sum_{cyc}\frac{1}{n+1}\sum_{cyc}a^2+\frac{n}{n+1}\sum_{cyc}b^2-\frac{n}{(n+1)^2}\sum_{cyc}a^2\right)$$

$$=\frac{1}{a^2+b^2+c^2}\cdot\lim_{n\to\infty}\left(\frac{n}{(n+1)^2}\sum_{cyc}a^2\right)=1$$

AN.157. Solution (Ahmed Yackoube Chach)

$$\Omega_n=\frac{\sqrt{(1+n!)^{n!}}}{n\cdot(n!)!}=\frac{\sqrt{(n!)^{n!}}\sqrt{\left(1+\frac{1}{n!}\right)^{n!}}}{n\sqrt{2\pi n!}\left(\frac{n!}{e}\right)^{n!}}=\frac{\sqrt{\left(1+\frac{1}{n!}\right)^{n!}}\sqrt{n!^{n!}}\,e^{n!}}{n\sqrt{2\pi}\sqrt{n!^{n!+1}}}=$$

$$=\frac{\sqrt{\left(1+\frac{1}{n!}\right)^{n!}}}{n\sqrt{2\pi}}e^{n!-\frac{n!+1}{2}log(n!)},\qquad \lim_{n\to\infty}\sqrt{\left(1+\frac{1}{n!}\right)^{n!}}=\sqrt{e}$$

$$\lim_{n\to\infty}e^{n!-\frac{n!+1}{2}log(n!)}=e^{-\infty}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{\sqrt{(1+n!)^{n!}}}{n\cdot(n!)!}=\lim_{n\to\infty}\frac{1}{n}\sqrt{\frac{e}{2\pi}}e^{-\infty}=0$$

AN.158. Solution (Rovsen Pirguliyev)

$$cos^2xcos^2(1+tanxtany)|tanx-tany|=$$

$$=cos^2xcos^2y\left(1+\frac{sinxsiny}{cosxcosy}\right)\left|\frac{sinx}{cosx}-\frac{siny}{cosy}\right|=$$

$$=cos(x-y)|sin(x-y)|sin(x-y)cos(x-y)=\frac{1}{2}sin2(x-y)$$

$$2\int_a^b\int_a^b cos^2xcos^2y(1+tanxtany)|tanx-tany|dxdy\le$$

$$\le 2\int_a^b\int_a^b\frac{1}{2}sin2(x-y)dxdy\le\int_a^b\int_a^b dxdy=(b-a)^2$$

AN.159. Solution (Remus Florin Stanca)

Let's prove that:

$$\frac{a+b}{\sqrt{1+ab}} \le \frac{a}{\sqrt{1+b^2}} + \frac{b}{\sqrt{1+a^2}}, \forall a,b>0$$

The inequality can be written as:

$$\frac{1}{\sqrt{1+ab}} \le \frac{a}{a+b}\cdot\frac{1}{\sqrt{1+b^2}} + \frac{b}{a+b}\cdot\frac{1}{\sqrt{1+a^2}}$$

$$\text{Let } g:(0,\infty)\to(0,\infty), g(x)=\frac{1}{\sqrt{x+1}}; \frac{\partial g}{\partial x} = -\frac{1}{2}(x+1)^{-\frac{3}{2}};$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{3}{2}\cdot\frac{1}{2}(x+1)^{-\frac{5}{2}} \ge 0$$

$\Rightarrow g$ *−convexe, then for any* $t_1, t_2 \in (0,1), t_1+t_2=1$ *and for any* $x_1, x_2 \in I$

we have: $t_1 f(x_1) + t_2 f(x_2) \ge f(t_1 x_1 + t_2 x_2)$

Let $t_1 = \frac{a}{a+b}; t_2 = \frac{b}{a+b}$ *and* $x_1 = b^2, x_2 = a^2$

$$\frac{a}{a+b}\cdot\frac{1}{\sqrt{1+b^2}} + \frac{b}{a+b}\cdot\frac{1}{\sqrt{1+a^2}} \ge \frac{1}{\sqrt{1+\frac{a^2b+ab^2}{a+b}}} = \frac{1}{\sqrt{1+\frac{ab(a+b)}{a+b}}}$$

$$= \frac{1}{\sqrt{1+ab}} \Rightarrow \frac{a+b}{\sqrt{1+ab}} \le \frac{a}{\sqrt{1+b^2}} + \frac{b}{\sqrt{1+a^2}}, \forall a,b>0$$

$$\frac{f(x)+f(y)}{\sqrt{1+f(x)f(y)}} \le \frac{f(x)}{\sqrt{1+f^2(y)}} + \frac{f(y)}{\sqrt{1+f^2(x)}}$$

$$\frac{(f(x)+f(y))f'(x)f'(y)}{\sqrt{1+f(x)f(y)}} \le \frac{f(x)f'(x)f'(y)}{\sqrt{1+f^2(y)}} + \frac{f(y)f'(x)f'(y)}{\sqrt{1+f^2(x)}}; \quad (3)$$

$$\frac{1}{2}\int_a^b\int_a^b \frac{2f(x)f'(x)f'(y)}{\sqrt{1+f^2(y)}}dxdy = \frac{1}{2}\int_a^b \frac{f^2(x)f'(y)}{\sqrt{1+f^2(y)}}\Bigg|_a^b dy =$$

$$= \frac{1}{2}\left(f^2(b)-f^2(a)\right)\left(log\left(f(b)+\sqrt{1+f^2(b)}\right) - log\left(f(a)+\sqrt{1+f^2(a)}\right)\right) =$$

$$= \frac{1}{2}\left(f^2(b)-f^2(a)\right)log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right); \quad (1)$$

$$\frac{1}{2}\int_a^b\int_a^b \frac{2f(y)f'(y)f'(x)}{\sqrt{1+f^2(x)}}dxdy = \frac{1}{2}\int_a^b \frac{f^2(y)f'(x)}{\sqrt{1+f^2(x)}}\Bigg|_a^b dy =$$

$$= \frac{1}{2}\left(f^2(b)-f^2(a)\right)\left(log\left(f(b)+\sqrt{1+f^2(b)}\right) - log\left(f(a)+\sqrt{1+f^2(a)}\right)\right) =$$

$$= \frac{1}{2}\left(f^2(b)-f^2(a)\right)log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right); \ (2)$$

From $(1),(2),(3)$ *it follows that:*

$$\int_a^b\int_a^b \frac{\left(f(x)+f(y)\right)f'(x)f'(y)}{\sqrt{1+f(x)f(y)}}dxdy \le log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right)^{f^2(b)-f^2(a)}$$

AN.160. Solution (Adrian Popa)

$$P = \prod_{k=1}^{n-1} sin\left(\frac{k\pi}{n}\right) = sin\frac{\pi}{n}\cdot sin\frac{2\pi}{n}\cdot \ldots \cdot sin\frac{(n-1)\pi}{n}$$

Let be the equation: $x^n - 1 = 0 \Leftrightarrow x^n = cos0 + isin0 \Rightarrow$

$$x_k = cos\frac{2k\pi}{n} + isin\frac{2k\pi}{n}, k \in \{0,1,\ldots,(n-1)\}$$

$$x^n - 1 = (x-1)(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n) =$$

$$= (x-1)\left(x - cos\frac{2\pi}{n} - isin\frac{2\pi}{n}\right)\cdot\ldots$$

$$\cdot\left(x - cos\frac{2(n-1)\pi}{n} - isin\frac{2(n-1)\pi}{n}\right); (1)$$

But $\frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \cdots + x + 1; \ (2)$

From (1),(2) and $x = 1$*, we get:*

$$n = \prod_{k=1}^{n-1}\left(1 - cos\frac{2k\pi}{n} - isin\frac{2k\pi}{n}\right) = \prod_{k=1}^{n-1}\left(2sin^2\frac{k\pi}{n} - 2isin\frac{k\pi}{n}cos\frac{k\pi}{n}\right) =$$

$$= \prod_{k=1}^{n-1} 2sin\frac{k\pi}{n}\left(sin\frac{k\pi}{n} - icos\frac{k\pi}{n}\right) =$$

$$=\frac{2^{n-1}}{i^{n-1}}\cdot sin\frac{\pi}{n}\cdot sin\frac{2\pi}{n}\cdot\ldots\cdot sin\frac{(n-1)\pi}{n}\cdot\prod_{k=1}^{n-1}\left(cos\frac{\pi}{n}+isin\frac{\pi}{n}\right)^k=$$

$$=\frac{2^{n-1}}{i^{n-1}}\cdot P\cdot\left(cos\frac{\pi}{n}+isin\frac{\pi}{n}\right)^{1+2+\cdots+n-1}=$$

$$=\frac{2^{n-1}}{i^{n-1}}\cdot P\cdot\left(cos\frac{n(n-1)\pi}{2n}+isin\frac{n(n-1)\pi}{2n}\right)=$$

$$=\frac{cos0+isin0}{\left(cos\frac{\pi}{2}+isin\frac{\pi}{2}\right)^{n-1}}\cdot 2^{n-1}\cdot P\cdot\left(cos\frac{(n-1)\pi}{2n}+isin\frac{(n-1)\pi}{2n}\right)=$$

$$=\left(cos\frac{(n-1)\pi}{2n}-isin\frac{(n-1)\pi}{2n}\right)\cdot 2^{n-1}\cdot P$$

$$\cdot\left(cos\frac{(n-1)\pi}{2n}+isin\frac{(n-1)\pi}{2n}\right)=2^{n-1}\cdot P$$

Hence,

$$P=\prod_{k=1}^{n-1}sin\left(\frac{k\pi}{n}\right)=sin\frac{\pi}{n}\cdot sin\frac{2\pi}{n}\cdot\ldots\cdot sin\frac{(n-1)\pi}{n}=\frac{n}{2^{n-1}}$$

Therefore,

$$\prod_{k=1}^{n}sin\left(\frac{k\pi}{n+1}\right)=\frac{n+1}{2^n}\Rightarrow\prod_{k=1}^{n}sin^2\frac{k\pi}{n+1}=\left(\prod_{k=1}^{n}sin\left(\frac{k\pi}{n+1}\right)\right)^2=\frac{(n+1)^2}{2^{2n}}$$

$$\prod_{k=n+1}^{2n}sin\left(\frac{k\pi}{2n+1}\right)=\frac{\prod_{k=1}^{2n}sin\left(\frac{k\pi}{2n+1}\right)}{\prod_{k=1}^{n}sin\left(\frac{k\pi}{2n+1}\right)}=\frac{\frac{2n+1}{2^{2n}}}{P_2}=\frac{\frac{2n+1}{2^{2n}}}{\frac{\sqrt{2n+1}}{2^n}}=\frac{\sqrt{2n+1}}{2^n}$$

$$P_2=\prod_{k=1}^{n}sin\left(\frac{k\pi}{2n+1}\right)=\prod_{k=1}^{n}sin\left(\pi-\frac{k\pi}{2n+1}\right)=\frac{\sqrt{2n+1}}{2^n}$$

$$\Omega=\lim_{n\to\infty}\left(\frac{8^n}{n(2n+1)^2}\cdot\prod_{k=1}^{n-1}sin\left(\frac{k\pi}{n}\right)\cdot\prod_{k=1}^{n}sin^2\left(\frac{k\pi}{2n+1}\right)\cdot\prod_{k=n+1}^{2n}sin\left(\frac{k\pi}{2n+1}\right)\right)=$$

$$=\lim_{n\to\infty}\frac{8^n}{n(2n+1)^2}\cdot\frac{2n}{2^n}\cdot\frac{(n+1)^2}{2^{2n}}\cdot\frac{\sqrt{2n+1}}{2^n}=\lim_{n\to\infty}\frac{2(n+1)^2}{(2n+1)^2}\cdot\frac{\sqrt{2n+1}}{2^n}=0$$

AN.161. Solution (Adrian Popa)

$$\prod_{k=1}^{n}\frac{k^2+k+1+i}{\sqrt{(k^2+1)(k^2+2k+2)}}=\prod_{k=1}^{n}\frac{(k+i)(k+1-i)}{\sqrt{(k+i)(k-i)(k+1+1)(k+1-i)}}=$$

$$=\prod_{k=1}^{n}\sqrt{\frac{(k+i)(k+1-i)}{(k-i)(k+1+i)}}=\sqrt{\frac{(1+i)(n+1-i)}{(1-i)(n+1+i)}}$$

$$\Omega=\lim_{n\to\infty}\left((1-i)\prod_{k=1}^{n}\frac{k^2+k+1+i}{\sqrt{(k^2+1)(k^2+2k+2)}}\right)=$$

$$=\lim_{n\to\infty}\sqrt{\frac{(1+i)(n+1-i)}{(1-i)(n+1+i)}}=\lim_{n\to\infty}\sqrt{\frac{2(n+1-i)}{n+1+i}}=\sqrt{2}$$

AN.162. Solution (Rovsen Pirguliyev)

Lemma: If $x>2$, *then* $\sin\frac{\pi}{x}>\frac{3}{\sqrt{x^2+9}}$

$$x>2\Rightarrow \pi x>2\pi\Rightarrow\frac{\pi}{x}>\frac{\pi}{2}\ and\ x<\tan x\Rightarrow\tan\frac{\pi}{x}>\frac{\pi}{x}>\frac{3}{x};\ \ (1)$$

Using $1+\tan^2x=\frac{1}{\cos^2x}$ *we have:*

$$\cos^2x=\frac{1}{1+\tan^2x}\overset{(1)}{<}\frac{1}{1+\left(\frac{\pi}{x}\right)^2}\overset{(1)}{<}\frac{1}{1+\left(\frac{3}{x}\right)^2}=\frac{x^2}{x^2+9}$$

$$\sin^2x=1-\cos^2x>1-\frac{x^2}{x^2+9}=\frac{9}{x^2+9}$$

Now, take $x\to 3$, *then* $\sin\frac{\pi}{3x}>\frac{3}{\sqrt{9x^2+9}}=\frac{1}{\sqrt{x^2+1}};\ \ (2)$

Hence,

$$\int_a^b x\cdot\sin\frac{\pi}{3x}dx\overset{(2)}{\geq}\int_a^b x\cdot\frac{1}{\sqrt{x^2+1}}dx$$

$$\int_a^b x\cdot\frac{1}{\sqrt{x^2+1}}dx=\frac{1}{2}\int_a^b\frac{d(x^2+1)}{\sqrt{x^2+1}}=\frac{1}{2}\cdot 2\sqrt{x^2+1}\Big|_a^b=\sqrt{1+b^2}-\sqrt{1+a^2}$$

$$\int_a^b x\cdot sin\frac{\pi}{3x}dx \ge \sqrt{1+b^2}-\sqrt{1+a^2}$$

AN.163. Solution (Abdallah El Farissi)

We have for $k\in\{1,2,\dots,n\}, k^2\binom{n}{k}^2 \le k^3\binom{n}{k}^2 \le nk^2\binom{n}{k}^2$ *then*

$$\sum_{k=1}^{n} k^2\binom{n}{k}^2 \le \sum_{k=1}^{n} k^3\binom{n}{k}^2 \le \sum_{k=1}^{n} nk^2\binom{n}{k}^2 \Leftrightarrow$$

$$1 \le \sqrt[n]{\left(\sum_{k=1}^{n} k^3\binom{n}{k}^2\right)\left(\sum_{k=1}^{n} k^2\binom{n}{k}^2\right)^{-1}} \le n^{\frac{1}{n}}$$

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\left(\sum_{k=1}^{n} k^3\binom{n}{k}^2\right)\left(\sum_{k=1}^{n} k^2\binom{n}{k}^2\right)^{-1}} = 1$$

AN.164. Solution (Adrian Popa)

$$\sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\frac{3^{i-j}}{4^i} = \sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\left(\frac{3}{4}\right)^i\cdot\frac{1}{3^j}$$

$$= \left(\frac{3}{4}\right)^i\cdot\sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\left(\frac{1}{3}\right)^j =$$

$$= \left(\frac{3}{4}\right)^i\cdot\left(1-\frac{1}{3}\right)^i = \left(\frac{3}{4}\right)^i\cdot\left(\frac{2}{3}\right)^i = \left(\frac{1}{2}\right)^i$$

$$\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\frac{3^{i-j}}{4^i} = \sum_{i=0}^{k}\left(\frac{1}{2}\right)^i = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\dots+\left(\frac{1}{2}\right)^k$$

$$= 2\left[1-\left(\frac{1}{2}\right)^k\right]$$

$$\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\frac{3^{i-j}}{4^i} = 2\sum_{k=0}^{n}\left[1-\left(\frac{1}{2}\right)^k\right] = 2n-4\left[1-\left(\frac{1}{2}\right)^n\right]$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^j\cdot\binom{i}{j}\cdot\frac{3^{i-j}}{4^i}\right)=\lim_{n\to\infty}\frac{1}{n}\left(2n-4\left[1-\left(\frac{1}{2}\right)^n\right]\right)=$$

$$=\lim_{n\to\infty}\left(2-\frac{4}{n}\right)=2$$

AN.165.Solution (Remus Florin Stanca)

$$\therefore (ab+1)^2\overset{CBS}{\leq}(a^2+1)(b^2+1)$$

$$(tanxtany+1)^2\leq(tan^2x+1)(tan^2y+1)$$

$$(tanytanz+1)^2\leq(tan^2y+1)(tan^2z+1)$$

$$(tanztanx+1)^2\leq(tan^2z+1)(tan^2x+1)$$

$$\overset{(\cdot)}{\Rightarrow}\left(\prod_{cyc}(tanxtany+1)\right)^2\leq\left(\prod_{cyc}(tan^2x+1)\right)^2\Rightarrow$$

$$\int_a^b\int_a^b\int_a^b\prod_{cyc}(tanxtany+1)\,dxdydz\leq\int_a^b\int_a^b\int_a^b\prod_{cyc}(tan^2x+1)\,dxdydz=$$

$$=\left(tanx|_a^b\right)^3=(tanb-tana)^3$$

AN.166. Solution (Pavlos Trifon)

Let $x=\frac{a}{b}\in(0,1]\Leftrightarrow a=bx$.

$$\frac{b^b}{a^a}\geq\left(e\sqrt{ab}\right)^{b-a}\Leftrightarrow log\frac{b^b}{a^a}\geq log\left(e\sqrt{ab}\right)^{b-a}$$

$$\Leftrightarrow blogb-aloga\geq(b-a)\left(1+\frac{1}{2}log(ab)\right)$$

$$blogb-(bx)log(bx)\geq(b-bx)\left(1+\frac{log(bx)+logb}{2}\right)$$

$$blogb-bxlogb-bxlogx\geq b(1-x)\left(1+logb+\frac{logx}{2}\right)$$

$$logx\leq\frac{2x-2}{x+1},\forall x\in(0,1];\quad(1)$$

Let $f(x)=logx-\frac{2x-2}{x+1},x\in(0,1]$

$$f'(x)=\frac{(x-1)^2}{x(x+1)^2}>0;\ \forall x\in(0,1]\Rightarrow f\uparrow(0,1]$$

Therefore, for $0<x\leq 1$ we have $f(x)\leq f(1)\Rightarrow(1)$ is true.

AN.167. Solution (Abdul Hannan)

Let $g(x)=\sqrt[3]{f(x)}$. Then the desired inequality is equivalent to

$$(b-a)\left(\int_a^b g^9(x)\,dx\right)\left(\int_a^b\frac{dx}{g^6(x)}\right)\geq\left(\int_a^b g^5(x)\,dx\right)\left(\int_a^b\frac{dx}{g(x)}\right)^2$$

This is true, because

$$(b-a)\left(\int_a^b g^9(x)\,dx\right)\left(\int_a^b\frac{dx}{g^6(x)}\right)\overset{Chebyshev}{\geq}\left(\int_a^b g^5(x)dx\right)\left(\int_a^b g^4(x)dx\right)\left(\int_a^b\frac{dx}{g^6(x)}\right)$$

$$\overset{CBS}{\geq}\left(\int_a^b g^5(x)\,dx\right)\left(\int_a^b\frac{dx}{g(x)}\right)^2$$

AN.168. Solution (Adrian Popa)

$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$ –concave and increasing for all $z>0$.

Denote $f(z)=erf(z)$ we have:

$$f\left(\frac{3a+b}{4}\right)=f\left(\frac{a+a+a+b}{4}\right)\overset{Jensen}{\geq}\frac{3f(a)+f(b)}{4}$$

Hence,

$$f\left(\frac{3a+b}{4}\right)-f(a)=\frac{3f(a)+f(b)}{4}-f(a)=\frac{f(b)-f(a)}{4};\ \ (1)$$

$$f\left(\frac{a+3b}{4}\right)=f\left(\frac{a+b+b+b}{4}\right)\overset{Jensen}{\geq}\frac{f(a)+3f(b)}{4}$$

Hence,

$$f\left(\frac{a+3b}{4}\right)-f(a)\geq\frac{f(a)+3f(b)}{4}-f(a)=\frac{3\big(f(b)-f(a)\big)}{4};\ \ (2)$$

From (1),(2) we have:

$$\left(erf\left(\frac{3a+b}{4}\right)-erf(a)\right)\left(erf\left(\frac{a+3b}{4}\right)-erf(a)\right)$$

$$\geq \frac{f(b)-f(a)}{4}\cdot\frac{3\big(f(b)-f(a)\big)}{4}=\frac{3}{16}\big(erf(b)-erf(a)\big)$$

Therefore,

$$\big(erf(b)-erf(a)\big)^2\leq\frac{16}{3}\left(erf\left(\frac{3a+b}{4}\right)-erf(a)\right)\left(erf\left(\frac{a+3b}{4}\right)-erf(a)\right)$$

AN.169. Solution (Ravi Prakash)

Consider $\frac{-sin^2(7x)+sin^2(10x)}{sin^2x}=\frac{sin(17x)sin(3x)}{sin^2x}=\frac{sin(17x)}{sinx}\cdot\frac{sin(3x)}{sinx}$

But $\frac{sin(17x)}{sinx}=\frac{sin(17x)-sin(15x)+sin(15x)-sin(13x)+\cdots+sin(3x)-sinx+sinx}{sinx}=$

$$=2cos(16x)+2cos(14x)+\cdots+2cos(2x)+1$$

$$\frac{sin(3x)}{sinx}=2cos(2x)+1$$

$$\frac{sin(17x)}{sinx}\cdot\frac{sin(3x)}{sinx}=2cos(18x)+4cos(16x)+6\sum_{k=1}^{7}cos(2kx)+3$$

Thus,

$$\Omega=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{sin^2(7x)+cos^2(10x)}{sin^2x}dx$$

$$=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left[csc^2x-3-2cos(18x)-4cos(16x)-6\sum_{k=1}^{7}cos(2kx)\right]dx$$

$$=\left[-cotx-3x-\frac{1}{9}sin(18x)-\frac{1}{4}sin(16x)-3\sum_{k=1}^{7}\frac{1}{k}sin(2kx)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$=-\frac{3\pi}{2}-\frac{1}{\sqrt{3}}+\pi+\frac{1}{9sin(6\pi)}+\frac{1}{4}sin\left(\frac{16\pi}{3}\right)+3\sum_{k=1}^{7}\frac{1}{k}sin\left(\frac{2k\pi}{3}\right)$$

$$=-\frac{\pi}{2}+\frac{1}{\sqrt{3}}-\frac{\sqrt{3}}{8}+3\left[\frac{1}{1}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+0+\frac{1}{4}\cdot\frac{\sqrt{3}}{2}-\frac{1}{5}\cdot\frac{\sqrt{3}}{2}+\frac{1}{7}\cdot\frac{\sqrt{3}}{2}\right]$$

$$=-\frac{\pi}{2}-\sqrt{3}\left(-\frac{1}{3}+\frac{1}{8}\right)+\frac{3\sqrt{3}}{2}\cdot\frac{97}{140}=\frac{131\sqrt{3}}{105}-\frac{\pi}{2}$$

AN.170. Solution (Asmat Qatea)

We know: $\lim_{n\to\infty} n^{\frac{1}{n}}=1$ *and* $\lim_{n\to 0}\frac{a^n-1}{n}=\log a$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2}\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right)^{n}=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{2}\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}-1\right)^{\frac{\frac{1}{2}\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}-1}{\frac{1}{2}\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}-1}\cdot n}=$$

$$=e^{\lim\limits_{n\to\infty} n\left(\frac{1}{2}\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}-1\right)}\overset{\frac{1}{n}=n}{=}e^{\lim\limits_{n\to 0}\frac{1}{n}\left[\frac{1}{2}(1+n)^n+\frac{1}{2}(1-n)^n-1\right]}=$$

$$=e^{\frac{1}{2}\lim\limits_{n\to 0}\frac{(1+n)^n-1}{n}+\frac{1}{2}\lim\limits_{n\to 0}\frac{(1-n)^n-1}{n}}=e^{\frac{1}{2}\lim\limits_{n\to 0}\left(\log(1+n)+\log(1-n)\right)}=e^0=1$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{2}\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right)^{n}=1$$

AN.171 Solution (Kamel Benaicha)

We have: $\forall k\geq 2,\frac{k}{2k-1}-\frac{2}{3}=\frac{1}{3}\left(\frac{-k+2}{2k-1}\right)=-\frac{1}{3}\left(\frac{k-2}{2k-1}\right)\leq 0$

$$\therefore \frac{k}{2k-1} \le \frac{2}{3} \leftrightarrow \left(\frac{3}{2}\right)^k \cdot \left(\frac{k}{2k-1}\right)^k \le 1 \to \left(\frac{k}{2k-1}\right)^k \le \left(\frac{2}{3}\right)^k$$

$$\therefore n^{n-2} \cdot \left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{5}\right)^3 \cdot \ldots \cdot \left(\frac{n}{2n-1}\right)^n \le n^{n-2} \prod_{k=2}^{n} \left(\frac{2}{3}\right)^k = n^{n-2} \left(\frac{2}{3}\right)^{\sum_{k=2}^{n} k}$$

$$= \frac{3}{2} n^{n-2} \left(\frac{2}{3}\right)^{\frac{n(n+1)}{2}}$$

$$\lim_{n\to\infty} n^{n-2} \left(\frac{2}{3}\right)^{\frac{n(n+1)}{2}} = \lim_{n\to\infty} \frac{1}{n^2} e^{nlogn+\frac{n(n+1)}{2}log\left(\frac{2}{3}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} e^{n\left(logn+\frac{n+1}{2}log\left(\frac{2}{3}\right)\right)}$$

$$\lim_{n\to\infty} \left(logn + \frac{n+1}{2} log\left(\frac{2}{3}\right)\right) \overset{x=\frac{1}{n}}{=} \lim_{x\to 0_+} \left(\frac{-2xlogx - (1+x)log\left(\frac{3}{2}\right)}{2x}\right) = -\infty$$

$$\therefore \lim_{x\to 0_+} xlogx = 0;\ \lim_{x\to 0_+} (1+x)log\left(\frac{3}{2}\right) = log\left(\frac{3}{2}\right) > 0$$

$$\lim_{n\to\infty} n^{n-2} \left(\frac{2}{3}\right)^{\frac{n(n+1)}{2}} = \left(\lim_{n\to\infty} \frac{1}{n^2}\right) e^{\lim_{n\to\infty} n\left(logn+\frac{n+1}{2}log\left(\frac{2}{3}\right)\right)} = 0$$

$$\Omega = \lim_{n\to\infty} \left(n^{n-2} \cdot \left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{5}\right)^3 \cdot \ldots \cdot \left(\frac{n}{2n-1}\right)^n\right) = 0$$

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