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2501. Find a closed form:

$$\int_0^1\int_1^\infty \frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1\int_1^\infty \frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy=\int_0^1\frac{Li_3(-y)}{(y+1)^3}dy\,.\,\int_1^\infty\frac{ln^3(x)}{(x+1)^2}dx=A\,.\,B$$

$$A=\int_0^1\frac{Li_3(-y)}{(y+1)^3}dy\ \left\{u=Li_3(-y),\quad \frac{du}{dy}=\frac{Li_2(-y)}{y},\quad v=-\frac{1}{2(y+1)^2},\right.$$

$$\left.dv=\frac{1}{(y+1)^3}\right\}$$

$$A=\frac{-Li_3(-y)}{2(y+1)^2}\Big|_0^1+\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y(y+1)^2}dy=\frac{3\zeta(3)}{32}+\frac{1}{32}+\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y(y+1)^2}dy$$

$$\begin{cases}u=Li_2(-y)\,,\ \dfrac{du}{dy}=\dfrac{-\ln(y+1)}{y}\\ dv=\dfrac{1}{y(y+1)^2},\ \ v=\dfrac{1}{y+1}+\ln(y)-\ln(y+1)\end{cases}$$

$$A=\frac{3\zeta(3)}{32}+\Bigg[-\frac{Li_2(-y)}{y+1}\Big|_0^1+Li_2(-y)\ln(y)\Big|_0^1-Li_2(-y)\ln(y+1)\Big|_0^1$$

$$+\int_0^1\frac{\ln(y+1)}{y(y+1)}dy+\int_0^1\frac{\ln(y)\ln(y+1)}{y}dy-\int_0^1\frac{ln^2(y+1)}{y}dy\Bigg]$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}\Big(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)$$

$$+\int_0^1\frac{\ln(y+1)}{y}dy$$

$$-\int_0^1\frac{\ln(y+1)}{y+1}dy+\int_0^1\frac{\ln(y)\ln(y+1)}{y}dy-\int_0^1\frac{ln^2(y+1)}{y}dy\Big)$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}\Big(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)$$

$$-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 y^{n-1}dy-\frac{1}{2}ln^2(y+1)\Big|_0^1-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 y^{n-1}ln^2(y)dy$$

$$-\int_0^2\frac{ln^2(y)}{y}dy\Big)\ =$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}\Big(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)$$

$$-\sum_{n=1}^\infty\frac{(-1)^n}{n^2}-\frac{1}{2}ln^2(2)+\sum_{n=1}^\infty\frac{(-1)^n}{n^3}+\sum_{n=1}^\infty\int_1^2 y^{n-1}ln^2(y)dy$$

$$A=\frac{3\zeta(3)}{32}+\frac{1}{2}\left(-\frac{\zeta(2)}{4}+\frac{\zeta(2)}{2}\ln(2)+\frac{\zeta(2)}{2}-\frac{1}{2}ln^2(2)-\frac{3}{4}\zeta(3)-\frac{\zeta(3)}{4}\right)$$

$$A=\frac{3\zeta(3)}{32}+\frac{\zeta(2)}{8}+\frac{\zeta(2)}{2}\ln(2)-\frac{1}{4}ln^2(2)-\frac{\zeta(3)}{2}$$

$$B=\int_1^\infty\frac{ln^3(x)}{(x+1)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_1^0\frac{ln^3\left(\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)^2}-\frac{dx}{x^2}=-\int_0^1\frac{x^2ln^3(x)}{(x+1)^2}\frac{dx}{x^2}=-\int_0^1\frac{ln^3(x)}{(x+1)^2}dx$$

$$B=3\int_0^1\frac{ln^2(x)}{1+x}dx=-3\sum_{n=1}^\infty\int_0^1x^{n-1}ln^2(x)dx=-3\sum_{n=1}^\infty(1)^n\left(\frac{2}{n^3}\right)=-6\sum_{n=1}^\infty\frac{1}{n^3}=$$

$$-6.\left(-\frac{3}{4}\zeta(3)\right)=\frac{9}{2}\zeta(3)$$

$$I=A\ .\ B$$

$$I=\left[-\frac{13}{32}\zeta(3)+\frac{\zeta(2)}{8}+\frac{\zeta(2)}{4}\ln(2)-\frac{1}{4}ln^2(2)\right].\ \frac{9}{2}\zeta(3)$$

$$I=-\frac{117}{64}\zeta^3(3)+\frac{9}{16}\zeta(3)\zeta(2)+\frac{9}{8}\zeta(3)\zeta(2)\ln(2)-\frac{9}{8}\zeta(3)ln^2(2)$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1\int_1^\infty\frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy$$

$$H=\left(\left(\int_0^1\frac{Li_3(-y)}{(y+1)^3}dy\right).\left(\int_1^\infty\frac{ln^3(x)}{(x+1)^2}dx\right)\right)$$

$$I=\int_1^\infty\frac{ln^3(x)}{(x+1)^2}dx\overset{x\to\frac{1}{x}}{\cong}-\int_0^1\frac{ln^3(x)}{(x+1)^2}dx=\sum_{k\in N}(-1)^kk\int_0^1x^{k-1}ln^3(x)dx=6\sum_{k\in N}\frac{(-1)^{k-1}}{k^3}=\frac{9}{2}\zeta(3)$$

$$J=\int_1^\infty\frac{ln^3(x)}{(x+1)^2}dx\overset{I.B.P}{\cong}-\frac{Li_3(-y)}{2(y+1)^2}\bigg|_0^1+\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y(y+1)^2}dy$$

$$=-\frac{Li_3(-1)}{8}-\frac{1}{2}\int_0^1\frac{Li_2(-y)}{y+1}dy+$$

$$\int_0^1\frac{Li_2(-y)}{y}dy-\frac{1}{2}\int_0^1\frac{Li_2(-y)}{(y+1)^2}dy=\frac{3\zeta(3)}{32}-\frac{1}{2}(J_1-J_2+J_3)$$

$\{\therefore Li_s(-1)=-\eta(s);\eta(s)=(1-2^{1-s})\zeta s),R[s]\geq 1;Li_s(1)=\zeta s),R[s]>1\}$

$$J_1=\int_0^1\frac{Li_2(-y)}{y+1}dy\overset{I.B.P}{\cong}Li_2(-1)\ln(2)$$

$$+\int_0^1\frac{ln^2(y+1)}{y}dy=\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}+\int_{\frac{1}{2}}^1\frac{ln^2(y)}{y}dy+\int_{\frac{1}{2}}^1\frac{ln^2(y)}{1-y}dy=$$

$$\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}-\frac{2ln^3(2)}{3}-2\int_{\frac{1}{2}}^1Li_2(x)\ln(x)\,dx=\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}-\frac{2ln^3(2)}{3}+2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right)+2\int_0^1dLi_3(y)=$$

$$\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}-\frac{2ln^3(2)}{3}+2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right)+2\left(Li_3(1)-Li_3\left(\frac{1}{2}\right)\right)=\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}+\frac{\zeta(3)}{4}$$

$$J_2=\int_0^1\frac{Li_2(-y)}{y}dy=\int_0^1 dLi_3(-x)=-\frac{3\zeta(3)}{4}$$

$$J_3=\int_0^1\frac{Li_2(-y)}{(y+1)^2}dy\overset{I.B.P}{=}-\frac{Li_2(-y)}{y+1}\bigg|_0^1$$

$$-\int_0^1\frac{\ln(y+1)}{y(y+1)}dy=\frac{Li_2(-1)}{2}+\int_0^1\frac{\ln(y+1)}{y+1}dy-\int_0^1\frac{\ln(y+1)}{y}dy=$$

$$\frac{1}{2}\cdot\frac{\pi^2}{12}+\int_0^1 d\left(\frac{ln^2(x+1)}{2}\right)-\int_0^1 d(-Li_2(-1))=\frac{\pi^2}{24}+\frac{ln^2(2)}{2}+Li_2(-1)$$

$$=\frac{ln^2(2)}{2}-\frac{\pi^2}{24}$$

$$H=\left[\frac{3\zeta(3)}{32}-\frac{1}{2}\left(\frac{\pi^2\ln\left(\frac{1}{2}\right)}{12}+\frac{\zeta(3)}{4}\right)-\frac{3\zeta(3)}{8}-\frac{1}{2}\left(\frac{ln^2(2)}{2}-\frac{\pi^2}{24}\right)\right]\cdot\frac{9}{2}\zeta(3)$$

$$\int_0^1\int_1^\infty\frac{Li_3(-y)ln^3(x)}{(x+1)^2(y+1)^3}dxdy=\frac{9}{2}\zeta(3)\left(\frac{\pi^2\ln(2)}{24}-\frac{13}{32}\zeta(3)+\frac{\pi^2}{48}-\frac{ln^2(2)}{2}\right)$$

2502. Solve the differential equation:

$$u_{tt}+u_{xx}=0\quad u(x,0)=\sec^a x+cosec^b x-1\qquad u_t(x,0)=0$$

where: b+1=a=tg2z-2 and $\frac{2}{tg2x}=sinzcosz(2ctgz-1), 0<x<\frac{\pi}{2}$

Proposed by Samir Cabiyev -Azerbaijan

Solution by proposer

Firstly, solve the trigonometric equation: $0<x<\frac{\pi}{2}$

$$\frac{2}{tg2x}=\frac{2sinxcosxcosx}{sinx}-sinxcosx$$

and accepted that ,

$$tg2x=\frac{2tgx}{1-tg^2x}\quad sinx=cosxtgx\quad cosx=\frac{1}{1+tg^2x}.Then:$$

$$\frac{1-tg^2x}{tgx}=\frac{2-tgx}{1+tg^2x},\qquad 1-tg^4x=2tgx-tg^2x$$

$$tg^4x-tg^2x+2tgx-1=0,\qquad (tg^2x)^2-(tgx-1)^2=0$$

$$tg^2x-tgx+1=0\ here\ D<0\quad but\quad tg^2x+tgx-1=0$$

$tgx=\frac{\sqrt{5}-1}{2}$. According to the formula above: $tg2x=2\,.And\ a=0\,,\ b=-1$

We consider in wave equation: $u_{tt}+u_{xx}=0\quad u(x,0)=sinx\qquad u_t(x,0)=0$

$$u(x,t)=\frac{\sin(x-it)+\sin(x+it)}{2}=\frac{\frac{e^{i(x-it)}-e^{-i(x-it)}}{2i}+\frac{e^{i(x+it)}-e^{-i(x+it)}}{2i}}{2}=\frac{e^{ix}-e^{-ix}}{2i}\frac{e^{t}+e^{-t}}{2}=\sin x\cosh t$$

2503. Prove the below closed form

$$\Omega=\int_0^{\frac{\pi}{2}}\log\left(1+\frac{\sin(x)}{\sin(2x)}\right)dx=\frac{4G}{3}$$

where, G is the *Catalan's constant*.

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{2}}\log\left(1+\frac{\sin(x)}{\sin(2x)}\right)dx=\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin(2x)+\sin(x)}{\sin(2x)}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log\left(\frac{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)}{\sin(2x)}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log(2)\,dx+\int_0^{\frac{\pi}{2}}\log\left(\sin\left(\frac{3x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}}\log\left(\cos\left(\frac{x}{2}\right)\right)dx-\int_0^{\frac{\pi}{2}}\log(\sin(2x))\,dx$$

$$=\frac{\pi}{2}\log(2)+\frac{2}{3}\int_0^{\frac{3\pi}{4}}\log(\sin(x))\,dx+2\int_0^{\frac{\pi}{4}}\log(\cos(x))\,dx-\frac{1}{2}\int_0^{\pi}\log(\sin(x))\,dx$$

$$=\frac{\pi}{2}\log(2)+\frac{2}{3}\left\{-\frac{3\pi}{4}\log(2)+\frac{1}{2}G\right\}+2\left\{\frac{1}{2}G-\frac{\pi}{4}\log(2)\right\}-\frac{1}{2}\{-\pi\log(2)\}=\frac{4G}{3}$$

$$\text{Note: }\int_0^{\frac{3\pi}{4}}\log(\sin(x))\,dx=\int_0^{\frac{3\pi}{4}}\left\{-\log(2)-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}\right\}dx$$

$$=-\frac{3\pi}{4}\log(2)-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{3\pi}{4}}\cos(2kx)\,dx=-\frac{3\pi}{4}\log(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{3\pi}{2}k\right)}{k^2}$$

$$=-\frac{3\pi}{4}\log(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}=-\frac{3\pi}{4}\log(2)+\frac{1}{2}G$$

2504. Let be $a_1 = a \in \mathbb{R}$, $a \neq 2$, $a_2 = \frac{a}{a-2}$, and:

$a_{n+1} = \frac{a_n^2}{a_n^2 - 2a_n + 2}$, $\forall n \geq 2$. Prove that:

$$\sum_{k=1}^{n} 2^{k-1} a_k = \prod_{k=1}^{n} a_k$$

Determine a_n and compute:

$$\lim_{n\to\infty} a_n$$

Proposed by Bela Kovacs-Romania

Solution by Khaled Abd Imouti-Syria, Omar Alhafeez-Syria

Let us prove the following issue by mathematical induction:

$$E(n) \ : \quad \sum_{k=1}^{n} 2^{k-1} a_k = \prod_{k=1}^{n} a_k$$

At $= 1$: $a_1 = a_1$, so the issue is valid for $n = 1$.

$$\text{At} = 2 : a_1 + 2a_2 \overset{?}{=} a_1 . a_2$$

Firstly, $l_1 := a_1 + 2a_2 = a_1 + \frac{2a_1}{a_1 - 2} = \frac{a_1^2 - 2a_1 + 2a_1}{a_1 - 2} = \frac{a_1^2}{a_1 - 2} = a_1 . \frac{a_1}{a_1 - 2} = a_1 . a_2 := l_2$

Now, we assume that the issue is valid at n=p

$$E(p) \ : \quad \sum_{k=1}^{p} 2^{k-1} a_k = \prod_{k=1}^{p} a_k$$

And let's prove its validity at n=p+1:

$$E(p+1) \ : \quad \sum_{k=1}^{p+1} 2^{k-1} a_k = \prod_{k=1}^{p+1} a_k$$

$$l_1 := \sum_{k=1}^{p+1} 2^{k-1} a_k = \sum_{k=1}^{p} 2^{k-1} a_k + 2^p a_{p+1} = \prod_{k=1}^{p} a_k + \frac{2^p a_p^2}{a_p^2 - 2a_p + 2}$$

$$= \frac{a_p^2 \prod_{k=1}^{p} a_k - 2a_p \prod_{k=1}^{p} a_k + 2 \prod_{k=1}^{p} a_k + 2^p a_p^2}{a_p^2 - 2a_p + 2}$$

Let's look at:

$$\prod_{k=1}^{p-1} a_k + 2^{p-1} a_p = \sum_{k=1}^{p-1} 2^{k-1} a_k + 2^{p-1} a_p = \sum_{k=1}^{p} 2^{k-1} a_k = \prod_{k=1}^{p} a_k$$

$$\Rightarrow -\prod_{k=1}^{p} a_k + \prod_{k=1}^{p-1} a_k + 2^{p-1} a_p = -\prod_{k=1}^{p} a_k + \prod_{k=1}^{p} a_k = 0$$

$$\Rightarrow -\prod_{k=1}^{p} a_k + \prod_{k=1}^{p-1} a_k + 2^{p-1} a_p = 0$$

We multiply both sides by a_p :

$$\Rightarrow\ -2a_p\prod_{k=1}^{p}a_k+2a_p\prod_{k=1}^{p-1}a_k+2^pa_p^2=2a_p(0)=0$$

$$\Rightarrow\ -2a_p\prod_{k=1}^{p}a_k+2\prod_{k=1}^{p}a_k+2^pa_p^2=0$$

Hence,

$$l_1=\frac{a_p^2\prod_{k=1}^{p}a_k-2a_p\prod_{k=1}^{p}a_k+2\prod_{k=1}^{p}a_k+2^pa_p^2}{a_p^2-2a_p+2}=\prod_{k=1}^{p}a_k\frac{a_p^2}{a_p^2-2a_p+2}=$$

$$=a^{p+1}\prod_{k=1}^{p}a_k=\prod_{k=1}^{p+1}a_k:=l_2$$

So, $E(p+1)$ is valid and the issue is valid for any $n\geq 1$.

Now, let's consider a function f defined on $\mathbb{R}$ by :

$$f:x\mapsto f(x):=\frac{x^2}{x^2-2x+2}$$

This function is differentiable on $\mathbb{R}$ and :

$$f'(x)=-\frac{2x(x-2)}{(x^2-2x+2)^2}$$

So, f is decreasing on $I_1:=]-\infty,0]$ and $0=f(0)\leq f(x)<\lim\limits_{x\to-\infty}f(x)=1$ on I_1.

On $I_2:=[0,2]$ the function is increasing and $0=f(0)\leq f(x)\leq f(2)=2$ on I_2.

On $I_3:=[2,+\infty[$ the function is decreasing and $1=\lim\limits_{x\to+\infty}f(x)<f(x)\leq 2$ on I_3

Hence, For all $\in\mathbb{R}$, $0\leq f(x)\leq 2$.

Since $f([0,2])=[0,2]$ and $f(x)\in[0,2]$ for any $x\in\mathbb{R}$ then $(a_n)_{n\geq1}$ is bounded and $a_n\in[0,2]$ when $n\geq 3$.

On the other hand,

$$a_{n+1}-a_n=-\frac{a_n(a_n-2)(a_n-1)}{a_n^2-2a_n+2}\ ,\qquad n\geq 2$$

$if\ a_1=a>2\ \Rightarrow\ a_2=\frac{a}{a-2}>1\ \Rightarrow\ a_3-a_2>0$ and while $0\leq a_n\leq 2$, $n\geq 3$ then a_3 stays in the interval $]1,2]$, hence $a_4-a_3>0$, so a_4 stays in the same interval, hence the same is true for $a_5,a_6,\dots$ .We deduce that $(a_n)_{n\geq2}$ is increasing.

So, $(a_n)_{n\geq1}$ is increasing, then it's convergent to a number such as x

$$x=\frac{x^2}{x^2-2x+2}\ \Leftrightarrow\ x(x^2-3x+2)=0\ \Leftrightarrow\ x(x-2)(x-1)=0$$

Since the sequence is increasing then $x=2$ and $\lim\limits_{n\to\infty}a_n=2$.

$if\ a_1=a<2\ \Rightarrow\ a_2=\frac{a}{a-2}<1\ \Rightarrow\ a_3-a_2<0$ and while $0\leq a_n\leq 2$, $n\geq 3$ then a_3 stays in the interval $[0,1[$, hence $a_4-a_3<0$, so a_4 stays in the same interval too, hence the same is true for $a_5,a_6,\dots$ .We deduce that $(a_n)_{n\geq2}$ is decreasing.

So, $(a_n)_{n\geq1}$ is decreasing, then it's convergent to a number such as x

$$x=\frac{x^2}{x^2-2x+2}\ \Leftrightarrow\ x(x^2-3x+2)=0\ \Leftrightarrow\ x(x-2)(x-1)=0$$

Since the sequence is decreasing then $x=0$ and $\lim\limits_{n\to\infty}a_n=0$.

2505. Find:

$$\Omega=\lim_{x\to0}\frac{ln(2-cos^2x)}{x(e^x-1)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega=\lim_{x\to0}\frac{ln(2-cos^2x)}{x(e^x-1)}=\lim_{x\to0}\frac{ln(2-(1-sin^2x))}{x(e^x-1)}=\lim_{x\to0}\frac{ln(1+sin^2x)}{x(e^x-1)}=$$

$$=\lim_{x\to0}\frac{x}{e^x-1}\cdot\frac{ln(1+sin^2x)}{x^2}=1\cdot\lim_{x\to0}\frac{ln(1+sin^2x)}{x^2}=$$

$$=\lim_{x\to0}\frac{1}{2x}\cdot\frac{2sinxcosx}{1+sin^2x}=\lim_{x\to0}\frac{cosx}{1+sin^2x}\cdot\frac{sinx}{x}=\lim_{x\to0}\frac{cosx}{1+sin^2x}\cdot\lim_{x\to0}\frac{sinx}{x}=$$

$$=\frac{cos0}{1+sin^20}\cdot1=1$$

2506. Find:

$$\Omega=\int_0^1\left(\frac{Li_2(-x)}{1+x}+xarctan^2\left(\frac{1}{x^2}\right)\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$*$ ***Results will be used*** :

1. $\int_0^1\frac{\ln(x)}{1-x}dx=-\frac{\pi^2}{6}$, ***easy to evaluate***

2. $\int_0^1\frac{\ln(x)\ln(1+x)}{x}dx=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1x^{n-1}\ln(x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=-\frac{3}{4}\zeta(3)$

3. $\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}dx\overset{I.B.P}{=}$

$$=-\zeta(2)\ln(2)+\int_0^1\frac{Li_2(x)}{1+x}dx=-\zeta(2)\ln(2)-\int_0^1\int_0^1\frac{xln(t)}{(1+x)(1-xt)}dxdt=$$

$$=-\zeta(2)\ln(2)+\int_0^1\ln(t)\left(\frac{\ln(2)}{1+t}+\frac{\ln(1-t)}{t}-\frac{\ln(1-t)}{1+t}\right)dt=-\zeta(2)\ln(2)+\ln(2)\int_0^1\frac{\ln(t)}{1+t}dt+$$

$$+\underbrace{\int_0^1\frac{\ln(t)\ln(1-t)}{t}dt}_{I.B.P}-\underbrace{\int_0^1\frac{\ln(t)\ln(1-t)}{1+t}dt}_{I.B.P}=-\frac{3}{2}\zeta(2)\ln(2)+\zeta(3)-\int_0^1\frac{\ln(t)\ln(1+t)}{1-t}dt+$$

$$\int_0^1 \frac{\ln(1+t)\ln(1-t)}{t}dt \to \int_0^1 \frac{\ln(t)\ln(1+t)}{1-t}dt = -\frac{3}{2}\zeta(2)\ln(2)+\zeta(3)$$

4. $\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx = -2\int_0^{\frac{\pi}{4}} \ln(cos(x))\,dx = \frac{\pi}{2}\ln(2) - G \;\therefore\; \int_0^{\frac{\pi}{4}} \ln(cos(x))\,dx = \frac{G}{2} - \frac{\pi}{4}\ln(2)$

$* I = \int_0^1 \frac{Li_2(-x)}{1+x}dx = \int_0^1\int_0^1 \frac{x\ln(t)}{(1+x)(1+xt)}dxdt = \int_0^1 \ln(t)\left(\int_0^1 \frac{x}{(1+x)(1+xt)}dx\right)dt =$

$$\int_0^1 \ln(t)\left(\frac{\ln(1+t)}{t(1-t)} - \frac{\ln(2)}{1-t}\right)dt = \zeta(2)\ln(2) + \int_0^1 \frac{\ln(t)\ln(1+t)}{t}dt + \int_0^1 \frac{\ln(t)\ln(1+t)}{1-t}dt =$$

$$\zeta(2)\ln(2) + \left(-\frac{3}{4}\zeta(3)\right) + \left(-\frac{3}{2}\zeta(2)\ln(2)+\zeta(3)\right) = \frac{1}{4}\zeta(3) - \frac{1}{2}\zeta(2)\ln(2)$$

$* J = \int_0^1 x arctan^2\left(\frac{1}{x^2}\right)dx = \frac{1}{2}\int_0^1 arctan^2\left(\frac{1}{x^2}\right)(x^2) = \frac{1}{2}\int_0^1 arctan^2\left(\frac{1}{x}\right)dx =$

$$-\frac{\pi^2}{8}\int_0^1 dx -$$

$$\frac{\pi}{2}\int_0^1 \arctan(x)\,dx$$

$$+\frac{1}{2}\int_0^1 arctan^2(x)dx = \frac{\pi^2}{8} - \frac{\pi}{2}\left(x\arctan(x) - \frac{1}{2}\ln(1+x^2)\Big|_0^1\right) + \frac{1}{2}\int_0^1 arctan^2(x)dx =$$

$$= \frac{\pi}{4}\ln(2) + \frac{1}{2}\int_0^1 arctan^2(x)dx$$

$And: \int_0^1 arctan^2(x)dx \overset{I.B.P}{\triangleq} x\,arctan^2(x)\Big|_0^1 - 2\underbrace{\int_0^1 \frac{x\arctan(x)}{1+x^2}dx}_{I.B.P}$

$$= \frac{\pi^2}{16} - \left(\frac{1}{2}\ln(1+x^2)\,arctan^2(x)\Big|_0^1 -\right.$$

$\left.\frac{1}{2}\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx\right) = \frac{\pi^2}{16} - \frac{\pi}{4}\ln(2) + \int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx) = \frac{\pi^2}{16} - \frac{\pi}{4}\ln(2) + \frac{\pi}{2}\ln(2) - G = \frac{\pi^2}{16} + \frac{\pi}{4}\ln(2)$
$- G$

$J = \int_0^1 x arctan^2\left(\frac{1}{x^2}\right)dx = \frac{\pi}{4}\ln(2) + \frac{1}{2}\left(\frac{\pi^2}{16} + \frac{\pi}{4}\ln(2) - G\right) = \frac{\pi^2}{32} + \frac{3\pi}{8}\ln(2) - \frac{G}{2}$

Combine all results :

$$\Omega = \int_0^1 \left(\frac{Li_2(-x)}{1+x} + x arctan^2\left(\frac{1}{x^2}\right)\right)dx = \frac{1}{4}\zeta(3) - \frac{1}{2}\zeta(2)\ln(2) + \frac{\pi^2}{32} + \frac{3\pi}{8}\ln(2) - \frac{G}{2}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1 \left(\frac{Li_2(-x)}{1+x} + x arctan^2\left(\frac{1}{x^2}\right)\right)dx$$

$U = \int_0^1 \frac{Li_2(-x)}{1+x}dx = \left(Li_2(-x)\int d ln((1+x))\right|_0^1 + \int_0^1 \frac{ln^2(1+x)}{x}dx =$

$Li_2(-1)\ln(2) + \int_{\frac{1}{2}}^1 \frac{ln^2(x)}{x}dx + \int_{\frac{1}{2}}^1 \frac{ln^2(x)}{1-x}dx = \frac{\pi^2}{12}\ln\left(\frac{1}{2}\right) - \frac{2}{3}ln^3(2) + 2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right) +$

$$\int_{\frac{1}{2}}^{1}\frac{Li_2(x)}{x}dx=\frac{\pi^2}{12}\ln\left(\frac{1}{2}\right)-\frac{2}{3}ln^3(2)+2Li_2\left(\frac{1}{2}\right)\ln\left(\frac{1}{2}\right)+2\int_{\frac{1}{2}}^{1}d\,Li_3(x)=\frac{\pi^2}{12}\ln\left(\frac{1}{2}\right)-\frac{2}{3}ln^3(2)+$$
$$2\ln\left(\frac{1}{2}\right)\left(\frac{\pi^2}{12}-\frac{ln^2(2)}{2}\right)+2\zeta(3)-2\left(\frac{7}{8}\zeta(3)+\frac{ln^3(2)}{6}-\frac{\pi^2}{12}\ln\left(\frac{1}{2}\right)\right)$$
$$U=\frac{1}{4}\zeta(3)-\frac{\pi^2}{12}\ln(2)$$
$$V=\int_0^1 x\arctan^2\left(\frac{1}{x^2}\right)dx=\frac{1}{2}\int_0^1\arctan^2\left(\frac{1}{x}\right)dx=\frac{1}{2}\int_0^1\left(\frac{\pi}{2}-\arctan(x)\right)^2dx=$$
$$-\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}x^2csc^2(x)dx=\frac{1}{2}\left(x^2\int d\cot(x)\right)\Bigg|_{\frac{\pi}{2}}^{\frac{\pi}{4}}-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\frac{x}{\tan(x)}dx=\frac{\pi^2}{32}\cot\left(\frac{\pi}{4}\right)-\left(x\int d\ln(\sin(x))\right)\Bigg|_{\frac{\pi}{2}}^{\frac{\pi}{4}}+$$
$$+\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\ln(\sin(x)\,dx$$
$$\left\{\therefore\int_0^{\theta}\ln(\sin(z)\,dz=-\frac{1}{2}Cl_2(2\theta)-\theta\ln(2);\;Cl_2\left(\frac{\pi}{2}\right)=G;\;Cl_2(\pi)=0\right\}$$
$$V=\frac{\pi^2}{32}+\frac{\pi}{8}\ln(2)+\frac{\pi}{4}\ln(2)-\frac{G}{2}=\frac{\pi^2}{32}+\frac{3\pi}{8}\ln(2)-\frac{G}{2}$$
$$U+V=\frac{1}{4}\zeta(3)-\frac{\pi^2}{12}\ln(2)+\frac{\pi^2}{32}+\frac{3\pi}{8}\ln(2)-\frac{G}{2}$$

2507. Find a closed form:

$$\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx$$

Proposed by Shirvan Tahirov, Gulkhanim Pirieyeva-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

$$We\ have:\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx\overset{Partial\ fraction}{\Rightarrow}\frac{1}{5}\int_0^1\left(\frac{1+2x}{1+x^2}-\frac{2}{2+x}\right)(ln^2(x)+2\ln(x)+1)dx=$$
$$=-\frac{2}{5}\int_0^1\frac{dx}{2+x}+\frac{2}{5}\int_0^1\frac{x}{1+x^2}dx+\frac{1}{5}\int_0^1\frac{dx}{1+x^2}-\underbrace{\frac{4}{5}\int_0^1\frac{\ln(x)}{2+x}dx}_{I.B.P}+\frac{4}{5}\int_0^1\frac{x\ln(x)}{1+x^2}dx+\frac{2}{5}\int_0^1\frac{\ln(x)}{1+x^2}dx-$$
$$-\underbrace{\frac{2}{5}\int_0^1\frac{ln^2(x)}{2+x}dx}_{I.B.P}+\frac{2}{5}\int_0^1\frac{xln^2(x)}{1+x^2}dx+\frac{1}{5}\int_0^1\frac{ln^2(x)}{1+x^2}dx\underset{|x^2|<1,x\in(0,1)}{=}-\frac{2}{5}\int_0^1 d\ln(2+x)+\int_0^1\frac{d\ln(1+x^2)}{5}$$
$$+\int_0^1\frac{d\arctan(x)}{5}-\frac{4}{5}\underbrace{\left(\ln(x)\int d\ln\left(1+\frac{x}{2}\right)\right)\Bigg|_0^1}_{0}+\frac{4}{5}\int_0^1 ln\left(1+\frac{x}{2}\right)\frac{dx}{x}+\frac{4}{5}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n-1}\ln(x)\,dx$$
$$+\frac{2}{5}\sum_{n\in N}(-1)^n\int_0^1 x^{2n}\ln(x)dx-\frac{2}{5}\underbrace{\left(ln^2(x)\int d\ln\left(1+\frac{x}{2}\right)\right)\Bigg|_0^1}_{0}+\underbrace{\frac{4}{5}\int_0^1\frac{\ln(x)}{x}\left(1+\frac{x}{2}\right)dx}_{I.B.P}+$$

$$+\frac{2}{5}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n-1}ln^2(x)dx+\frac{1}{5}\sum_{n\in N_0}(-1)^n\int_0^1 x^{2n}\ln(x)dx$$

$$\left(\therefore \int_0^1 t^m ln^n(t)dt=\frac{(-1)^n.n!}{(m+1)^{n+1}},\ n>-1\ \bigwedge m\neq 1\right)$$

$$\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx$$

$$=\frac{\pi}{20}-\frac{2}{5}\ln(3)+\frac{3}{5}\ln(2)-\frac{4}{5}\int_0^1 dLi_2\left(-\frac{x}{2}\right)-\frac{2}{5}\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^2}}_{=G(Catalan's\ constant)}-$$

$$-\frac{1}{5}\sum_{n\in N}\frac{(-1)^{n-1}}{n^2}-\frac{4}{5}\underbrace{(\ln(x)\int dLi_2\left(-\frac{x}{2}\right))\Big|_0^1}_{0}+\frac{4}{5}\int_0^1 d\,Li_3\left(-\frac{x}{2}\right)$$

$$+\frac{1}{10}\sum_{n\in N}\frac{(-1)^{n-1}}{n^3}+\frac{2}{5}\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^3}$$

$$\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx$$

$$=\frac{4}{5}Li_3\left(-\frac{1}{2}\right)-\frac{4}{5}Li_2\left(-\frac{1}{2}\right)-\frac{2G}{5}+\frac{2}{5}\beta(3)+\frac{\eta(3)}{10}-\frac{\eta(2)}{10}+\frac{\pi}{20}-\frac{2}{5}\ln(3)+\frac{3}{5}\ln(2)$$

$$\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx=\frac{4}{5}\left(Li_3\left(-\frac{1}{2}\right)-Li_2\left(-\frac{1}{2}\right)\right)-\frac{2G}{5}+\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}-\frac{\pi^2}{60}+\frac{\pi}{20}-\frac{2}{5}\ln(3)+\frac{3}{5}\ln(2)$$

Note section :

$$\eta(s)=(1-2^{1-s}).\zeta(s)$$

$$\beta(3)=\frac{\pi^2}{32},\qquad \zeta(2)=\frac{\pi^2}{6}$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx=\int_0^1\frac{x}{(1+x^2)(2+x)}dx+2\int_0^1\frac{xln(x)}{(1+x^2)(2+x)}dx+\int_0^1\frac{x^2}{(1+x^2)(2+x)}dx=$$

$$=A+2B+C$$

$$A=\int_0^1\frac{x}{(1+x^2)(2+x)}dx=\int_0^1\left(\frac{2x}{5(1+x^2)}-\frac{1}{5(1+x^2)}-\frac{2}{5(x+2)}\right)dx=\frac{1}{5}\ln(1+x^2)\Big|_0^1-$$

$$-\frac{2}{5}\ln(2+x)\Big|_0^1+\frac{1}{5}\arctan(x)\Big|_0^1=\frac{3}{5}\ln(2)+\frac{\pi}{20}-\frac{2}{5}\ln(3)=\frac{\pi}{20}+\frac{1}{5}\ln\left(\frac{8}{9}\right)$$

$$B=\int_0^1\frac{xln(x)}{(1+x^2)(2+x)}dx$$

$$=\frac{2}{5}\int_0^1\frac{xln(x)}{1+x^2}dx+\frac{1}{5}\int_0^1\frac{\ln(x)}{1+x^2}dx-\frac{2}{5}\int_0^1\frac{\ln(x)}{x+2}dx=\frac{1}{5}(2B_1+B_2-2B_3)$$

$$B_1=\int_0^1\frac{x\ln(x)}{1+x^2}dx$$
$$=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln(x)\,dx=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2}=-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=-\frac{1}{4}\eta(2)=-\frac{\pi^2}{48}$$
$B_2=-G$, B_3
$$=\int_0^1\frac{\ln(x)}{x+2}dx=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\int_0^1 x^n\ln(x)\,dx=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{(n+1)^2}=-Li_2\left(-\frac{1}{2}\right)$$
$$B=\frac{1}{5}(2B_1+B_2-2B_3)=\frac{1}{5}\left\{-\frac{\pi^2}{24}-G-2Li_2\left(-\frac{1}{2}\right)\right\}\quad C=\int_0^1\frac{x\ln^2(x)}{(1+x^2)(2+x)}dx=\frac{2}{5}\int_0^1\frac{x\ln^2(x)}{1+x^2}dx+$$
$$+\frac{1}{5}\int_0^1\frac{\ln^2(x)}{1+x^2}dx-\frac{2}{5}\int_0^1\frac{\ln^2(x)}{x+2}dx=\frac{1}{5}(2C_1+C_2-2C_3)\qquad C_1=\int_0^1\frac{x\ln^2(x)}{1+x^2}dx=$$
$$\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln^2(x)dx=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^3}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{1}{4}\eta(3)=\frac{3}{16}\zeta(3)\ ,$$
$$C_2=\int_0^1\frac{\ln^2(x)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln^2(x)dx=2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{16}\ ,$$
$$C_3=\int_0^1\frac{\ln^2(x)}{x+2}dx=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\int_0^1 x^n\ln^2(x)dx=$$
$$\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{(n+1)^3}=-2\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{n^3}=2Li_3\left(-\frac{1}{2}\right)\ ,\quad \frac{1}{5}(2C_1+C_2-2C_3)=\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}+\frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$
$$\Omega=A+2B+C=\frac{\pi}{20}+\frac{1}{5}\ln\left(\frac{8}{9}\right)+\frac{2}{5}\left\{-\frac{\pi^2}{24}-G-2Li_2\left(-\frac{1}{2}\right)\right\}+\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}+\frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$
$$\int_0^1\frac{x(1+\ln(x))^2}{(1+x^2)(2+x)}dx=\frac{4}{5}\left(Li_3\left(-\frac{1}{2}\right)-Li_2\left(-\frac{1}{2}\right)\right)-\frac{2G}{5}+\frac{3}{40}\zeta(3)+\frac{\pi^3}{80}-\frac{\pi^2}{60}+\frac{\pi}{20}-\frac{2}{5}\ln(3)+\frac{3}{5}\ln(2)$$

2508. Find:

$$\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$Note\ \ section:\ \therefore\ \ln(\cos(t))=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(2nt),\qquad |t|<\frac{\pi}{2}$$
$$Let:\ -2\Omega=-2\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx=\int_0^1 x\arctan(x)\ln(1$$
$$+x^2)\left(\frac{-2x}{(1+x^2)^2}\right)=$$
$$\int_0^1 x\arctan(x)\ln(1+x^2)d\left(\frac{1}{1+x^2}\right)\overset{I.B.P}{=}\frac{x\arctan(x)\ln(1+x^2)}{1+x^2}\Bigg|_0^1-$$

$$\int_0^1 \frac{1}{1+x^2}(\arctan(x)\ln(1+x^2)+\frac{x\ln(1+x^2)}{1+x^2}+2\frac{x\arctan(x)}{1+x^2})dx$$

$$=\frac{\pi}{8}\ln(2)-\underbrace{\int_0^1 \frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx}_{A}-$$

$$\underbrace{\int_0^1 \frac{x\ln(1+x^2)}{(1+x^2)^2}dx}_{B}-2\underbrace{\int_0^1 \frac{x\arctan(x)}{(1+x^2)^2}dx}_{C}=\frac{\pi}{8}\ln(2)-A-B-2C$$

$$a)\ A=\underbrace{\int_0^1 \frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx}_{x=\tan(t)}=\int_0^{\frac{\pi}{4}} t\ln\left(\frac{1}{\cos^2(t)}\right)dt=-2\int_0^{\frac{\pi}{4}} t\ln(\cos(t))dt=$$

$$2\int_0^{\frac{\pi}{4}} t(\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(2nt))\,dt$$

$$=\ln(2)\,t^2\Big|_0^{\frac{\pi}{4}}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} t\cos(2nt)dt \overset{I.B.P}{\triangleq} \frac{\pi^2}{16}\ln(2)+$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{2nt\sin(2nt)+\cos(2nt)}{4n^2}\Bigg|_0^{\frac{\pi}{4}}$$

$$=\frac{\pi^2}{16}\ln(2)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}(n\frac{\pi}{2}\sin\left(n\frac{\pi}{2}\right)+\cos\left(n\frac{\pi}{2}\right)-1)=$$

$$=\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(n\frac{\pi}{2}\right)+$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\left(n\frac{\pi}{2}\right)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{7}{16}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

$$=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)$$

$$B=\int_0^1 \frac{x\ln(1+x^2)}{(1+x^2)^2}dx$$

$$=-\frac{1}{2}\int_0^1 \ln(1+x^2)\frac{-2x}{(1+x^2)^2}dx$$

$$=-\frac{1}{2}\int_0^1 \ln(1+x^2)\,d\frac{1}{1+x^2}=-\frac{1}{2}\frac{\ln(1+x^2)}{1+x^2}\Bigg|_0^1$$

$$+\frac{1}{2}\int_0^1 \frac{d(1+x^2)}{(1+x^2)^2}=-\frac{\ln(2)}{4}-\frac{1}{2}\frac{1}{1+x^2}\Bigg|_0^1=-\frac{\ln(2)}{4}+\frac{1}{4}$$

$$C=\underbrace{\int_0^1\frac{\text{xarctan(x)}}{(1+x^2)^2}dx}_{x=\tan(t)}=\int_0^{\frac{\pi}{4}}\frac{\tan(t)\,t\left(1+tan^2(t)\right)}{\left(1+tan^2(t)\right)^2}dt=\frac{1}{2}\int_0^{\frac{\pi}{4}}tsin(2t)dt\overset{I.B.P}{\cong}\frac{1}{2}\frac{\sin(2t)-2tcos(2t)}{4}\Bigg|_0^{\frac{\pi}{4}}$$

$$=\frac{1}{8}$$

$$Then:-2\Omega=\frac{\pi}{8}\ln(2)-A-B-2C$$

$$=\frac{\pi}{8}\ln(2)-\left\{\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)\right\}-\left\{-\frac{\ln(2)}{4}+\frac{1}{4}\right\}-2.\frac{1}{8}=$$

$$\frac{\pi}{4}G-\frac{21}{64}\zeta(3)-\frac{\pi^2}{16}\ln(2)+\frac{\pi}{8}\ln(2)+\frac{\ln(2)}{4}-\frac{1}{2}$$

$$Therefore:\ \ \Omega=\frac{21}{128}\zeta(3)+\frac{\pi^2}{32}\ln(2)-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}G-\frac{\ln(2)}{8}+\frac{1}{4}$$

Solution 2 by Exodo Halcalias-Angola

$$\Omega=\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

$$I=\int_0^1\frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx-\int_0^1\frac{\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

$$N=\int_0^1\frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx$$

$$=-2\int_0^{\frac{\pi}{4}}xln(cosx)dx=-2\int_0^{\frac{\pi}{4}}x\left(\ln\left(\frac{1}{2}\right)-\sum_{k\in N}\frac{(-1)^k}{k}\cos(2kx)\right)dx=$$

$$\frac{\pi^2}{16}\ln(2)+2\sum_{k\in N}\frac{(-1)^k}{k}\left(\frac{\pi}{8k}\sin\left(\frac{\pi k}{2}\right)+\frac{1}{4k^2}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{4k^2}\right)dx=\frac{\pi^2}{16}\ln(2)$$

$$+\frac{\pi}{4}\sum_{k\in N}\frac{(-1)^k}{(2k-1)^2}+$$

$$\frac{1}{2}\sum_{k\in N}\frac{(-1)^k}{(2k)^3}-\frac{1}{2}\sum_{k\in N}\frac{(-1)^k}{k^3}=\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)$$

$$M=\int_0^1\frac{\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$

$$=(\arctan(x)\ln(1+x^2)\int d\left(\frac{x}{2(1+x^2)}+\frac{\arctan(x)}{2}\right))\Bigg|_0^1-$$

$$\frac{1}{2}\int_0^1\frac{\ln(1+x^2)+2xarctan(x)}{1+x^2}\left(\frac{x}{2(1+x^2)}+\arctan(x)\right)dx=\frac{\pi}{16}\ln(2)+\frac{\pi^2}{32}\ln(2)-$$

$$-\frac{1}{2}\int_0^1\frac{xln(1+x^2)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{arctan(x)\ln(1+x^2)}{1+x^2}dx-\int_0^1\frac{x^2arctan(x)}{(1+x^2)^2}dx-\int_0^1\frac{xarctan^2(x)}{1+x^2}dx=$$

$$=\frac{\pi}{16}\ln(2)\left(1+\frac{\pi}{2}\right)-\frac{1}{4}\int_0^1\frac{\ln(1+x)}{(1+x)^2}dx-\frac{N}{2}+\frac{1}{2}\int_0^{\frac{\pi}{4}}xdx+\frac{1}{2}\int_0^{\frac{\pi}{4}}xcos(2x)dx\,(x^2\int dln\left(\frac{1}{cos(x)}\right))\Big|_0^{\frac{\pi}{4}}+$$
$$\frac{1}{2}\int_0^{\frac{\pi}{4}}xlncos(x)dx\quad M$$
$$=\frac{\pi}{16}\ln(2)\left(1+\frac{\pi}{2}\right)-\frac{1}{4}\left(\frac{1}{2}-\frac{\ln(2)}{2}\right)-\frac{N}{2}-\frac{\pi^2}{64}+\frac{\pi}{16}-\frac{1}{8}-\frac{\pi^2}{32}\ln(2)+N$$
$$I=N-M$$
$$I=N-\left(-\frac{1}{4}+\frac{\ln(2)}{8}-\frac{N}{2}-\frac{\pi^2}{64}+\frac{\pi}{16}+\frac{\pi}{16}\ln(2)+N\right)$$
$$=\frac{1}{2}\left(\frac{\pi^2}{16}\ln(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)\right)+$$
$$\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)+\frac{1}{4}-\frac{ln(2)}{8}$$
$$\therefore\Omega=\frac{21}{128}\zeta(3)+\frac{\pi^2}{32}\ln(2)-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}G-\frac{\ln(2)}{8}+\frac{1}{4}$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx$$
$$We\ \ have,\int_0^1\frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx\overset{I.B.P}{\cong}$$
$$-\left(\frac{xarctan(x)}{2}\int d\left(\frac{1+\ln(1+x^2)}{1+x^2}\right)\right)\Big|_0^1+$$
$$\frac{1}{2}\int_0^1\left(\frac{1+\ln(1+x^2)}{1+x^2}\right)\left(\frac{x}{1+x^2}+\arctan(x)\right)dx$$
$$=-\frac{\pi}{16}\ln(2)-\frac{\pi}{16}$$
$$+\frac{1}{2}\int_0^1\frac{x}{(1+x^2)^2}dx+\frac{1}{2}\int_0^1\frac{xln(1+x^2)}{(1+x^2)^2}dx$$
$$+\frac{1}{2}\int_0^1\frac{arctan(x)}{1+x^2}dx+\underbrace{\int_0^1\frac{\ln(1+x^2)\arctan(x)}{1+x^2}dx}_{x\to\tan(x)}$$
$$=-\frac{\pi}{16}\ln(2)-\frac{\pi}{16}-$$
$$-\frac{1}{4}\int_0^1 d\left(\frac{1}{1+x^2}\right)-\frac{1}{4}\int_0^1 d\left(\frac{1+\ln(1+x^2)}{1+x^2}\right)+\int_0^1\frac{darctan^2(x)}{4}-\int_0^{\frac{\pi}{4}}xlncos(x)dx$$
$$\left(\therefore\sum_{n\in N}\frac{(-1)^n\cos(2nx)}{n}=\ln\left(\frac{sec|x|}{2}\right),\ \ |x|<\frac{\pi}{2}\right)$$

$$\int_0^1 \frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx = \frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)+\frac{1}{4}+\ln(2)\int_0^{\frac{\pi}{4}} xdx+\sum_{n\in N}\frac{(-1)^n}{n^2}\underbrace{\int_0^{\frac{\pi}{4}} x\cos(2nx)}_{I.B.P} =$$

$$=\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}+$$

$$+\frac{\ln(2)}{2}\int_0^{\frac{\pi}{4}} d(x^2)+\sum_{n\in N}\frac{(-1)^n}{n}\left[\left(\frac{x}{2n}\int d\sin(2nx)\right)\Big|_0^{\frac{\pi}{4}}-\frac{1}{2n}\int_0^{\frac{\pi}{4}}\sin(2nx)\,dx\right]$$

$$=\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}+\frac{\pi}{8}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^2}\sin\left(\frac{n\pi}{2}\right)}_{n\to 2n+1}+\frac{1}{4}\sum_{n\in N}\frac{(-1)^n}{n^3}\int_0^{\frac{\pi}{4}} d\cos(2nx) =$$

$$=\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}-$$

$$-\frac{\pi}{8}\sum_{n\in N_0}\frac{1}{(2n+1)^2}\sin\left(n\pi+\frac{\pi}{2}\right)+\frac{1}{4}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^3}\cos\left(\frac{n\pi}{2}\right)}_{n\to 2n}$$

$$-\frac{1}{4}\sum_{n\in N}\frac{(-1)^n}{n^3}\ \overset{\therefore\cos(n\pi)=(-1)^n,\forall n\in Z}{\underset{\therefore\sin(n\pi)=0,\ \forall n\in Z}{=>}}$$

$$\int_0^1 \frac{x^2\arctan(x)\ln(1+x^2)}{(1+x^2)^2}dx = \frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}-\frac{\pi}{8}\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^2}}_{=G(Catalan's\ constant)}+$$

$$+\frac{1}{32}\sum_{n\in N}\frac{(-1)^n}{n^3}-\frac{1}{4}\sum_{n\in N}\frac{(-1)^n}{n^3}=-\frac{\pi}{8}G+\frac{\pi^2}{32}\ln(2)+\frac{\pi^2}{64}-\frac{\pi}{16}-\frac{\pi}{16}\ln(2)-\frac{\ln(2)}{8}+\frac{1}{4}+\frac{7}{32}\eta(3)$$

$$\therefore\ \Omega=\frac{21}{128}\zeta(3)+\frac{\pi^2}{32}\ln(2)-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}G-\frac{\ln(2)}{8}+\frac{1}{4}$$

2509. Prove that:

$$\int_0^1 \frac{x.\arccos(x)}{\sqrt{2-2x^4}}dx=\frac{\pi}{8\sqrt{2}}\,{}_4F_3\left(\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},\frac{3}{2};1\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1 \frac{x.\arccos(x)}{\sqrt{2-2x^4}}dx=\int_0^1 \frac{x.\arccos(x)}{\sqrt{2(1-x^4)}}dx=\frac{1}{\sqrt{2}}\int_0^1 \frac{x.\arccos(x)}{\sqrt{(1-x^4)}}dx$$

$$u=\arccos(x)\quad;\quad \frac{du}{dx}=-\frac{1}{\sqrt{(1-x^2)}}$$

$$dv=\frac{x}{\sqrt{(1-x^4)}}\ ;\ v=\frac{1}{2}\arcsin(x^2)$$

$$I=\frac{1}{\sqrt{2}}\left[\frac{1}{2}\arccos(x)\,.\arcsin(x^2)\Big|_0^1+\frac{1}{2}\int_0^1\frac{\arcsin(x^2)}{\sqrt{(1-x^2)}}dx\right]=\frac{1}{2\sqrt{2}}\int_0^1\frac{\arcsin(x^2)}{\sqrt{(1-x^2)}}dx=$$

$$\underbrace{=}_{dx=\frac{dy}{2\sqrt{y}}}^{x^2=y,\ x=\sqrt{y}}\frac{1}{2\sqrt{2}}\int_0^1\frac{\arcsin(y)}{\sqrt{(1-y)}}\frac{dy}{2\sqrt{y}}=\frac{1}{4\sqrt{2}}\int_0^1\frac{\arcsin(y)}{\sqrt{y}\sqrt{(1-y)}}dy=$$

$$=\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\int_0^1\frac{y^{2n+1}}{\sqrt{y}\sqrt{(1-y)}}dy=$$

$$=\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\int_0^1 y^{2n+1}.(1-y)^{-\frac{1}{2}}dy=$$

$$\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\int_0^1 y^{\frac{1}{2n+\frac{3}{2}}}.(1-y)^{-\frac{1}{2}-1}dy$$

$$=\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\beta\left(2n+\frac{3}{2},\frac{1}{2}\right)=$$

$$\frac{1}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2n+\frac{3}{2}+\frac{1}{2}\right)}=\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}=$$

$$\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n!)}{(2^n n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}=\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{(n!)(2n+1)}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}=$$

$$=\frac{\sqrt{\pi}}{4\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{2\left(n+\frac{1}{2}\right)n!}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)}$$

This :

$$\frac{\sqrt{\pi}}{8\sqrt{2}}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{2\left(n+\frac{1}{2}\right)n!}.\frac{4(n+1)\left(2n+\frac{3}{2}\right)n!}{4\left(n+\frac{3}{4}\right)(2(n+1))!}$$

$$=\frac{\sqrt{\pi}}{8\sqrt{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{4}\right)(n+1)\Gamma\left(n+\frac{5}{4}\right)}{n!\,\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)(n+1)}=$$

$$=\frac{\sqrt{\pi}}{8\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n\left(\frac{5}{4}\right)_n(1)_n}{n!\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n(1)_n}=\frac{\sqrt{\pi}}{8\sqrt{2}}4F_3\left[\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4};1,\frac{3}{2},\frac{3}{2};1\right]$$

2510.

$$f,g:R\to R,\ f(1)=3,\ g(1)=2,\ xf(y)+yf(x)=2f(xy)$$
$$xg(y)+yg(x)=2g(xy),\forall\, x,y\in R$$

Find :

$$\Omega=\int_0^1\frac{f\left(\frac{\sinh(x)}{3}\right)}{f\left(\frac{\sinh(x)}{3}\right)+f\left(\frac{\cosh(x)}{2}\right)}dx$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$xf(y)+yf(x)=2f(xy)\Rightarrow\ xf(1)+f(x)=2f(x)\Rightarrow f(x)=xf(1)=3x$$
$$xg(y)+yg(x)=2g(xy)\Rightarrow xg(1)+2g(x)=2g(x)\Rightarrow g(x)=xg(1)=2x$$
$$f\left(\frac{\sinh(x)}{3}\right)=\sinh(x)\ ,\qquad f\left(\frac{\cosh(x)}{2}\right)=\cosh(x)$$

$$\int_0^1\frac{f\left(\frac{\sinh(x)}{3}\right)}{f\left(\frac{\sinh(x)}{3}\right)+f\left(\frac{\cosh(x)}{2}\right)}dx$$
$$=\int_0^1\frac{\sinh(x)}{\sinh(x)+\cosh(x)}dx=\int_0^1\frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}}dx=$$
$$\int_0^1\frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x-e^{-x}+e^x+e^{-x}}{2}}dx$$
$$=\int_0^1\frac{\frac{e^x-e^{-x}}{2}}{\frac{2e^x}{2}}dx=\int_0^1\frac{e^x-e^{-x}}{2e^x}dx=\left(\frac{1}{4e^2}+\frac{1}{2}-\frac{1}{4}\right)=\left(\frac{1}{4e^2}+\frac{1}{4}\right)$$

2511. Find a closed form:

$$\int_0^\infty\frac{e^{-x}(1-\cos(3x))}{x^2}dx$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Alireza Askari-Iran

$$Let\ f(t)=\int_0^\infty \frac{e^{-x}(1-\cos(tx))}{x^2}dx \to \frac{d}{dt}f(t)=\frac{d}{dt}\left(\int_0^\infty \frac{e^{-x}(1-\cos(tx))}{x^2}dx\right)$$

$$\frac{d}{dt}f(t)=\int_0^\infty \frac{\partial}{\partial t}\frac{e^{-x}(1-\cos(tx))}{x^2}dx=\int_0^\infty \frac{e^{-x}}{x^2}(x\sin(tx))\,dx=\int_0^\infty \frac{e^{-x}}{x}\sin(tx)\,dx$$

$$\frac{d^2}{dt^2}f(t)=\int_0^\infty e^{-x}\cos(tx)dx=\mathcal{R}e\left\{\int_0^\infty e^{-x+itx}\,dx\right\}=\mathcal{R}e\left\{\frac{e^{x(-1+it)}}{1-it}\Big]_0^\infty\right\}=\mathcal{R}e\left\{\frac{1}{1-it}\right\}$$

$$\frac{d^2}{dt^2}f(t)=\mathcal{R}e\left\{\frac{1+it}{1+t^2}\right\}=\frac{1}{1+t^2}\to\frac{d}{dt}f(t)=\tan^{-1}(t)+C\underset{t=0}{\longrightarrow}\frac{d}{dt}f(0)$$
$$=\tan^{-1}(0)+C_1$$

$$\int_0^\infty \frac{e^{-x}}{x}\sin(0\times x)\,dx=0=0+C_1\to C_1=0\to\frac{d}{dt}f(t)=\tan^{-1}(t)$$

$$f(t)=\int\tan^{-1}(t)dt\underbrace{=}_{IBP}t\tan^{-1}(t)-\frac{1}{2}\ln(t^2+1)+C_2\underset{t=0}{\longrightarrow}C_2=0$$

$$\int_0^\infty \frac{e^{-x}(1-\cos(tx))}{x^2}dx=t\tan^{-1}(t)-\frac{1}{2}\ln(t^2+1)\underset{t=3}{\longrightarrow}$$

$$\text{ANSWER}=3\tan^{-1}(3)-\frac{1}{2}\ln(10)$$

2512. Prove the below closed form

$$\Omega=\int_0^1\int_0^1\frac{\log(1-x^4y^4)}{xy}dxdy=-\frac{\zeta(3)}{16}$$

where $\zeta(3)$ is the Apery's constant.

Proposed by Ankush Kuma Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

We know that:

$$\int_0^1\int_0^1 f(xy)\,dxdy=-\int_0^1\log(x)\,f(x)\,dx$$

Then we can write:

$$\Omega=\int_0^1\int_0^1\frac{\log(1-x^4y^4)}{xy}dxdy=-\int_0^1\frac{\log(x)\log(1-x^4)}{x}dx\Bigg]_{x^4\to x}=$$

$$=-\frac{1}{16}\int_0^1\frac{\log(x)\log(1-x)}{x}dx=-\frac{1}{16}\int_0^1\frac{\log(x)}{x}\left(-\sum_{k=1}^{\infty}\frac{x^k}{k}\right)dx=$$

$$=\frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{k-1}\log(x)\,dx=-\frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^3}=-\frac{\zeta(3)}{16}$$

2513. Prove the below closed form

$$\Omega=\int_0^1\int_0^1\tan^{-1}\left(\frac{1-xy}{1+xy}\right)\cot^{-1}\left(\frac{1-xy}{1+xy}\right)\frac{dxdy}{\log(xy)}=-G+\frac{\pi}{4}\log(2)$$

where G is the Catalan's constant.

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

We know that:

$$\int_0^1\int_0^1 f(xy)\,dxdy=-\int_0^1\log(x)\,f(x)\,dx$$

Then we can write

$$\Omega=\int_0^1\int_0^1\tan^{-1}\left(\frac{1-xy}{1+xy}\right)\cot^{-1}\left(\frac{1-xy}{1+xy}\right)\frac{dxdy}{\log(xy)}=-\int_0^1\tan^{-1}\left(\frac{1-x}{1+x}\right)\cot^{-1}\left(\frac{1-x}{1+x}\right)dx=$$

$$-\int_0^1\tan^{-1}\left(\frac{1-x}{1+x}\right)\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1-x}{1+x}\right)\right)dx=-\int_0^1\left(\frac{\pi}{4}-\tan^{-1}(x)\right)\left(\frac{\pi}{4}+\tan^{-1}(x)\right)dx=$$

$$-\int_0^1\left(\frac{\pi^2}{16}-\arctan^2(x)\right)dx=-\frac{\pi^2}{16}+\int_0^1\arctan^2(x)\,dx=-\frac{\pi^2}{16}+\left(\frac{\pi^2}{16}+\frac{\pi}{4}\log(2)-G\right)=$$

$$\frac{\pi}{4}\log(2)-G.$$

$$\text{Note: }\int_0^1\arctan^2(x)\,dx=\frac{\pi^2}{16}+\frac{\pi}{4}\log(2)-G$$

2514. Find:

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos^3(x)+\cos^5(x)}{\sin^2(x)+\sin^4(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{cos^3(x)+cos^5(x)}{sin^2(x)+sin^4(x)}=\cos(x).\frac{cos^2(x)+cos^4(x)}{sin^2(x)+sin^4(x)}$$

$$=\cos(x).\frac{1-sin^2(x)+(1-sin^2(x))^2}{sin^2(x)+sin^4(x)}=$$

$$\cos(x).\frac{sin^4(x)-3sin^2(x)+2}{sin^2(x)+sin^4(x)}=\cos(x).\left(1-\frac{4sin^2(x)-2}{sin^2(x)+sin^4(x)}\right)=$$

$$\cos(x).\left(1-\frac{4sin^2(x)-2}{sin^2(x)\left(1+sin^2(x)\right)}\right)=\cos(x).(1-A)$$

$$A=\frac{4sin^2(x)-2}{sin^2(x)\left(1+sin^2(x)\right)}=-\frac{2}{sin^2(x)}+\frac{6}{1+sin^2(x)}\cos(x).(1-A)$$

$$\to\ \cos(x).\left(1+\frac{2}{sin^2(x)}-\frac{6}{1+sin^2(x)}\right)$$

$$=\cos(x)+\frac{2\cos(x)}{sin^2(x)}-\frac{6\cos(x)}{1+sin^2(x)}=$$

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{cos^3(x)+cos^5(x)}{sin^2(x)+sin^4(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\cos(x)+\frac{2\cos(x)}{sin^2(x)}-\frac{6\cos(x)}{1+sin^2(x)}\right)dx=$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos(x)\,dx+2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos(x)}{sin^2(x)}dx$$

$$-6\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos(x)}{1+sin^2(x)}dx$$

$$=\sin(x)|_{\frac{\pi}{6}}^{\frac{\pi}{2}}+2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\mathrm{d}(\sin(x))}{sin^2(x)}-6\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{d(\sin(x))}{1+sin^2(x)}=$$

$$\sin(x)|_{\frac{\pi}{6}}^{\frac{\pi}{2}}-2\left(\frac{1}{\sin(x)}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}-6(\arctan(sin(x))|_{\frac{\pi}{6}}^{\frac{\pi}{2}}=$$

$$=\left(1-\frac{1}{2}\right)-2(1-2)-6(\frac{\pi}{4}-\arctan\left(\frac{1}{2}\right)=\frac{5-3\pi}{2}+6\arctan\left(\frac{1}{2}\right)$$

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{cos^3(x)+cos^5(x)}{sin^2(x)+sin^4(x)}dx=\frac{5-3\pi}{2}+6\arctan\left(\frac{1}{2}\right)$$

2515. Find a closed form:

$$\Omega=\int_0^1\left(x^2\ln(ln(x))+\frac{xln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\therefore \int_0^1 \frac{ln^n(x)}{a+x}dx = -\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k \int_0^1 x^{k-1}ln^n(x)dx = -\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k x^k n!\sum_{i=0}^{n}\frac{(-1)^i ln^{n-i}(x)}{(n-i)!\,k^{i+1}}\Bigg|_0^1 = (-1)^{n+1}n!\,Li_{n+1}\left(-\frac{1}{a}\right)$$

$$\therefore \int_0^1 \frac{ln^n(x)}{1+x^2}dx = -\sum_{k=1}^{\infty}(-1)^k \int_0^1 x^{2k-1-1}ln^n(x)dx = -\sum_{k=1}^{\infty}(-1)^k x^{2k-1}n!\sum_{i=0}^{n}\frac{(-1)^i ln^{n-i}(x)}{(n-i)!\,(2k-1)^{i+1}}\Bigg|_0^1 = n!\,(-1)^n\beta(n+1)$$

$$\therefore I_a = \int_0^1 x^{n-1}dx = a^{-1}: \frac{\partial^p}{\partial a^p} \quad I_a = \int_0^1 x^{n-1}ln^p(x)dx = (-1)^p\Gamma(p+1)a^{-1-p}$$

$$K_{a,p} = \int_0^1 x^{a-1}ln^p(x)\ln(ln(x))\,dx = \frac{\partial}{\partial p}\left(\frac{\partial^p}{\partial a^p}I_a\right) = \frac{\partial}{\partial p}\left(\int_0^1 x^{a-1}ln^p(x)dx\right) =$$
$$(-1)^p\Gamma(p+1)a^{-1-p}\{\ln(-1)+\psi(p+1)-\ln(a)\}$$

$$\Omega = \int_0^1\left(x^n\ln(ln(x)) + \frac{xln^k\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}\right)dx = \int_0^1 x^n\ln(ln(x))\,dx + (-1)^k\int_0^1 \frac{xln^k(x)}{(1+x^2)(x+2)}dx = I + (-1)^k J$$

$$1) I = \int_0^1 x^n\ln(ln(x))\,dx = K_{n+1,0} = (-1)^0\Gamma(0+1)(n+1)^{-1-0}\{\ln(-1)+\psi(0+1)-\ln(n+1)\} =$$
$$\frac{\ln(-1)+\psi(1)-\ln(n+1)}{n+1} = \frac{\ln(e^{i\pi})-\gamma-\ln(n+1)}{n+1} = \frac{i\pi-\gamma-\ln(n+1)}{n+1}$$

$$2) J = \int_0^1 \frac{xln^k(x)}{(1+x^2)(x+2)}dx = \frac{2}{5}\int_0^1 \frac{xln^k(x)}{(1+x^2)}dx + \frac{1}{5}\int_0^1 \frac{ln^k(x)}{(1+x^2)}dx - \frac{2}{5}\int_0^1 \frac{ln^k(x)}{(2+x)}dx = \frac{1}{5.2^k}\int_0^1 \frac{ln^k(x^2)}{1+x^2}d(x^2) +$$
$$\frac{1}{5}k!\,(-1)^k\beta(k+1) - \frac{2}{5}k!\,(-1)^{k+1}Li_{k+1}\left(-\frac{1}{2}\right) = \frac{1}{5.2^k}k!\,(-1)^{k+1}Li_{k+1}(-1) + \frac{1}{5}k!\,(-1)^k\beta(k+1) +$$
$$\frac{2}{5}k!\,(-1)^k Li_{k+1}\left(-\frac{1}{2}\right) = \frac{(-1)^k k!}{5}\left\{\beta(k+1) + 2Li_{k+1}\left(-\frac{1}{2}\right) + \frac{(1-2^{-k})\zeta(k+1)}{2^{2k}}\right\}$$

$$Then:\ \Omega = \frac{i\pi-\gamma-\ln(n+1)}{n+1} + \frac{(-1)^k k!}{5}\left\{\beta(k+1) + 2Li_{k+1}\left(-\frac{1}{2}\right) + \frac{(1-2^{-k})\zeta(k+1)}{2^{2k}}\right\} =$$
$$\frac{i\pi-\gamma-\ln(n+1)}{n+1} + \frac{k!}{5}\left\{\beta(k+1) + 2Li_{k+1}\left(-\frac{1}{2}\right) + \frac{(2^k-1)\zeta(k+1)}{2^{2k}}\right\}$$
$$iff:\ n=k=2;\ \Omega = \frac{i\pi-\gamma-\ln(3)}{3} + \frac{2}{5}\left\{\beta(3) + 2Li_3\left(-\frac{1}{2}\right) + \frac{3\zeta(3)}{16}\right\}$$
$$\Omega = \frac{i\pi-\gamma-\ln(3)}{3} + \frac{1}{80}\left\{\pi^3 + 64Li_3\left(-\frac{1}{2}\right) + 6\zeta(3)\right\}$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$I = \int_0^1\left(x^2\ln(ln(x)) + \frac{xln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}\right)dx = \int_0^1 (\underbrace{x^2\ln(ln(x))}_{A} + \underbrace{\frac{xln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}}_{B})dx$$

$*$ ***working on A***

$$A = \int_0^1 x^2\ln(ln(x))\,dx \underset{\substack{dx=e^p\\ [-\infty,0]}}{\overset{\substack{\ln(x)=p\\ x=e^p}}{=}} \int_{-\infty}^0 e^{2p}\ln(p).e^p dp = \int_{-\infty}^0 e^{3p}\ln(p)\,dp$$

$$= \frac{d}{dn}\int_{-\infty}^0 \lim_{n\to 0} p^n.e^{3p}dp =$$

$$= \lim_{n\to 0}\frac{d}{dn}\int_{-\infty}^{0} p^n.e^{3p}dp = \lim_{n\to 0}\frac{d}{dn}[(-1)^n 3^{-n-1}\Gamma(n+1)] =$$

$$= \lim_{n\to 0}\left[(-1)^n 3^{-n-1}\Gamma(n+1)[\psi(n+1)+i\pi-\ln(3)]\right] = \frac{1}{3}[-\gamma+i\pi-\ln(3)] = \frac{i\pi-\gamma-\ln(3)}{3}$$

$*$ ***working on B***

$$B = \int_0^1 \frac{x\ln^2\left(\frac{1}{x}\right)}{(1+x^2)(x+2)}dx = \int_0^1 \frac{x\ln^2(x)}{(1+x^2)(x+2)}dx$$

Decompose into partial fraction…

$$B = \int_0^1 \frac{(2x+1)\ln^2(x)}{5(x^2+1)}dx - \frac{2}{5}\int_0^1 \frac{\ln^2(x)}{x+2}dx = \frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln^2(x)dx + \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln^2(x)dx -$$

$$\frac{2}{5}\sum_{n=0}^{\infty}(-1)^n 2^{-1-n}\int_0^1 x^n\ln^2(x)dx = \frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{4(n+1)^3}\right] + \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{(2n+1)^3}\right] - \frac{2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{1+n}}\left[\frac{2}{(n+1)^3}\right] =$$

$$\frac{1}{10}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} + \frac{2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} - \frac{4}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{1+n}(n+1)^3} = \frac{1}{10}\left[\frac{3}{4}\zeta(3)\right] + \frac{2}{5}\left[\frac{\pi^3}{32}\right] + \frac{4}{5}\left[Li_3\left(-\frac{1}{2}\right)\right] =$$

$$= \frac{3}{40}\zeta(3) + \frac{\pi^3}{80} + \frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$

But $I = A + B$

$$I = \frac{i\pi-\gamma-\ln(3)}{3} + \frac{3}{40}\zeta(3) + \frac{\pi^3}{80} + \frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$

2516. Find a closed form:

$$\Omega = \int_0^1 \frac{\ln(x)\ln^3(x+1)}{x+1}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Kartick Chandra Betal-India

$$I = \int_0^1 \frac{\ln(x)\ln^3(x+1)}{x+1}dx$$

$$= \int_1^2 \frac{\ln(x-1)\ln^3(x)}{x}dx = \int_{\frac{1}{2}}^1 \frac{-\{\ln(1-x)-\ln(x)\}\ln^3(x)}{x}dx =$$

$$\int_{\frac{1}{2}}^1 \frac{\ln^4(x)}{x}dx - \int_{\frac{1}{2}}^1 \frac{\ln(1-x)\ln^3(x)}{x}dx = \left[\frac{\ln^5(x)}{5}\right]_{\frac{1}{2}}^1 + \sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{1}{2}}^1 x^{n-1}\ln^3(x)dx =$$

$$\frac{\ln^5(2)}{5} + \sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{x^n}{n}\ln^3(x) - \frac{3x^n\ln^2(x)}{n^2} + \frac{6x^n\ln(x)}{n^3} - \frac{6x^n}{n^4}\right]_{\frac{1}{2}}^1 = \frac{\ln^5(2)}{5} +$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\left[-\frac{6}{n^4} - \left\{\frac{-\ln^3(2)}{n.2^n} - \frac{3\ln^2(2)}{n^2.2^n} - \frac{6\ln(2)}{n^3.2^n} - \frac{6}{n^4.2^n}\right\}\right] = \frac{\ln^5(2)}{5} +$$

$$\sum_{n=1}^{\infty}\left\{-\frac{6}{n^5} + \frac{\ln^3(2)}{n^2.2^n} + \frac{3\ln^2(2)}{n^3.2^n} + \frac{6\ln(2)}{n^4.2^n} + \frac{6}{n^5.2^5}\right\} = \frac{\ln^5(2)}{5} - 6\zeta(5) + \ln^3(2).Li_2\left(\frac{1}{2}\right) +$$

$$3\ln^2(2).\mathrm{Li}_3\left(\frac{1}{2}\right)+6\ln(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right)+6\mathrm{Li}_5\left(\frac{1}{2}\right)$$

$$=\frac{\ln^5(2)}{5}-6\zeta(5)+\ln^3(2)\left\{\frac{\pi^2}{12}-\frac{\ln^2(2)}{2}\right\}+$$

$$3\ln^2(2)\left\{\frac{\ln^3(2)}{6}-\frac{\pi^2}{12}\ln(2)+\frac{7}{8}\zeta(3)\right\}+6\ln(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right)+6\mathrm{Li}_5\left(\frac{1}{2}\right)=$$

$$6\ln(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right)+6\mathrm{Li}_5\left(\frac{1}{2}\right)-6\zeta(5)-\zeta(2)\ln^3(2)+\frac{21}{8}\ln^2(2)\zeta(3)+\frac{\ln^5(2)}{5}$$

Solution 2 by Bui Hong Suc-Vietnam

$$\int_0^1\frac{\ln(x)\,ln^3(x+1)}{x+1}dx$$

$$\Omega_n=\int_0^1\frac{\ln(x)\,ln^{n-1}(x+1)}{x+1}dx$$

$$=\frac{1}{n}\int_0^1\ln(x)\,d\big(ln^n(x+1)\big)=\frac{1}{n}\left(\ln(x)ln^n(x+1)\Big|_0^1-\int_0^1\frac{ln^n(x+1)}{x}dx\right)=$$

$$-\frac{1}{n}\int_0^1\frac{ln^n(x+1)}{x}dx\overset{x+1=\frac{1}{v}}{=}\frac{(-1)^{n-1}}{n}\int_{\frac{1}{2}}^1\frac{ln^n(v)}{v(1-v)}dv=\frac{(-1)^{n-1}}{n}I$$

$$I=\int_{\frac{1}{2}}^1\frac{ln^n(v)}{v(1-v)}dv=\int_{\frac{1}{2}}^1\frac{ln^n(v)}{v}dv+\int_{\frac{1}{2}}^1\frac{ln^n(v)}{1-v}dv=\frac{1}{n+1}ln^{n+1}(v)\Big|_{\frac{1}{2}}^{1}-\int_0^{\frac{1}{2}}\frac{ln^n(v)}{1-v}dv=$$

$$\frac{(-1)^n}{n+1}ln^{n+1}(2)+\sum_{k=1}^{\infty}\int_0^1 v^{k-1}ln^n(v)dv-\sum_{k=1}^{\infty}\int_0^{\frac{1}{2}}v^{k-1}ln^n(v)dv=\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n n!\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}-$$

$$\sum_{k=1}^{\infty}v^k n!\sum_{i=0}^{n}(-1)^n i!\binom{n}{i}\frac{(-1)^{-i}}{k^{i+1}}ln^{n-i}(v)\Big|_0^{\frac{1}{2}}=\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n.n!.\zeta(n+1)-$$

$$\sum_{k=1}^{\infty}n!\sum_{i=0}^{n}(-1)^n i!\binom{n}{i}ln^{n-i}(2)\frac{\left(\frac{1}{2}\right)^n}{k^{i+1}}=\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n.n!.\zeta(n+1)-$$

$$(-1)^n n!\sum_{i=0}^{n}i!\binom{n}{i}ln^{n-i}(2)Li_{i+1}\left(\frac{1}{2}\right)$$

Then :

$$\Omega_n=\frac{(-1)^{n-1}}{n}\left\{\frac{(-1)^n}{n+1}ln^{n+1}(2)+(-1)^n.n!.\zeta(n+1)-(-1)^n n!\sum_{i=0}^{n}i!\binom{n}{i}ln^{n-i}(2)Li_{i+1}\left(\frac{1}{2}\right)\right\}$$

$$iff\ \ n=4\quad\Omega_4$$

$$=\int_0^1\frac{\ln(x)\,ln^3(x+1)}{x+1}dx=\frac{120\sum_{i=0}^{\infty}i!\binom{4}{i}ln^{4-i}(2)Li_{i+1}\left(\frac{1}{2}\right)-120\zeta(5)-ln^5\left(\frac{1}{2}\right)}{20}$$

$$\int_0^1 \frac{\ln(x)\, ln^3(x+1)}{x+1}dx = 6\ln(2)\, Li_4\left(\frac{1}{2}\right) + 6Li_5\left(\frac{1}{2}\right) - 6\zeta(5) - \zeta(2)ln^3(2) + \frac{21}{8}ln^2(2)\zeta(3) + \frac{ln^5(2)}{5}$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$\int_0^1 \frac{\ln(x)\, ln^3(x+1)}{x+1}dx$$

$$u = \ln(x)ln^3(x+1)\,,\quad \frac{du}{dx} = \frac{3ln^2(1+x)\ln(x)}{1+x} + \frac{ln^3(x+1)}{x}$$

$$v = \ln(1+x)\,,\qquad dv = \frac{1}{1+x}$$

$$I = \ln(x)\, ln^4(x+1)\Big|_0^1 - 3\underbrace{\int_0^1 \frac{\ln(x)\, ln^3(x+1)}{x+1}dx}_{I} - \int_0^1 \frac{ln^4(x+1)}{x}dx$$

$$I = -3I - \int_0^1 \frac{ln^4(x+1)}{x}dx$$

$$I = -\frac{1}{4}\int_0^1 \frac{ln^4(x+1)}{x}dx$$

Now the main task, let's calculate this integral …

$$\int_0^1 \frac{ln^4(x+1)}{x}dx$$

Recall that:

$$\int_0^1 \frac{ln^a(x+1)}{x}dx = \frac{ln^{a+1}(2)}{a+1} + a!\,\zeta(a+1) - (-1)^a\int_0^{\frac{1}{2}} \frac{ln^a(x)}{1-x}dx$$

$$\int_0^1 \frac{ln^4(x+1)}{x}dx = \frac{ln^5(2)}{5} + 24\zeta(5) - \int_0^{\frac{1}{2}} \frac{ln^4(x)}{1-x}dx$$

$*$ Working on A

$$A = \int_0^{\frac{1}{2}} \frac{ln^4(x)}{1-x}dx \quad x = \frac{p}{2},\qquad dx = \frac{dp}{2}\quad [0,1]$$

$$A = \int_0^1 \frac{ln^4\left(\frac{p}{2}\right)}{1-\frac{p}{2}}\frac{dp}{2} = \frac{1}{2}\int_0^1 \frac{ln^4\left(\frac{p}{2}\right)}{1-\frac{p}{2}}dp$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 \left(\frac{p}{2}\right)^n ln^4\left(\frac{p}{2}\right)dp = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^4\left(\frac{p}{2}\right)dp =$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n[ln^4(p) - 4ln^3(p)\ln(2) + 6ln^2(p)ln^2(2) - 4\ln(p)\, ln^3(2) + ln^4(2)]dp =$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^4(p)dp - 2\ln(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^3(p)dp$$

$$-3ln^2(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln^2(p)dp -$$

$$2ln^3(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n ln(p)dp + \frac{ln^4(2)}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 p^n dp = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d^4}{dn^4}\left(\frac{p^{n+1}}{n+1}\bigg|\begin{matrix}1\\0\end{matrix}\right) -$$

$$2\ln(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d^3}{dn^3}\left(\frac{p^{n+1}}{n+1}\bigg|\begin{matrix}1\\0\end{matrix}\right) - 3ln^2(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d^2}{dn^2}\left(\frac{p^{n+1}}{n+1}\bigg|\begin{matrix}1\\0\end{matrix}\right)$$

$$-2ln^3(2)\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{d}{dn}\left(\frac{p^{n+1}}{n+1}\bigg|\begin{matrix}1\\0\end{matrix}\right) +$$

$$\frac{ln^4(2)}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\left(\frac{1}{n+1}\right) = 12\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^5} + 12\ln(2)\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^4} - 6ln^2(2)\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^3} +$$

$$2ln^3(2)\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^2} + \frac{ln^4(2)}{2}\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)} = 24Li_5\left(\frac{1}{2}\right) + 24\ln(2)\,Li_4\left(\frac{1}{2}\right) - 6ln^2(2)[\frac{7}{4}\zeta(3) +$$

$$\frac{1}{3}ln^3(2) - \zeta(2)\ln(2)] + 2ln^3(2)[\zeta(2) - ln^2(2)] + \frac{ln^4(2)}{2}.2\ln(2)$$

$$A = 24Li_5\left(\frac{1}{2}\right) + 24\ln(2)\,Li_4\left(\frac{1}{2}\right) + \frac{21}{2}\zeta(3)ln^2(2) + 2ln^5(2) - 6\zeta(2)ln^3(2) + 2\zeta(2)ln^3(2) - ln^5(2)$$

$$\int_0^1\frac{ln^4(x+1)}{x}dx = \frac{ln^5(2)}{5} + 24\zeta(5) - A$$

$$\int_0^1\frac{ln^4(x+1)}{x}dx = \frac{ln^5(2)}{5} + 24\zeta(5) - 24Li_5\left(\frac{1}{2}\right) - 24\ln(2)\,Li_4\left(\frac{1}{2}\right) - \frac{21}{2}\zeta(3)ln^2(2)$$

$$-$$

$$-ln^5(2) + 6\zeta(2)ln^3(2) - 2\zeta(2)ln^3(2)$$

$$I = -\frac{1}{4}\int_0^1\frac{ln^4(x+1)}{x}dx =$$

$$= -\frac{1}{4}[24\zeta(5) - 24\ln(2)\,Li_4\left(\frac{1}{2}\right) - 24Li_5\left(\frac{1}{2}\right) - \frac{21}{2}\zeta(3)ln^2(2) - \frac{4}{5}ln^5(2) + 4\zeta(2)ln^3(2)]$$

$$\int_0^1\frac{\ln(x)\,ln^3(x+1)}{x+1}dx = 6\ln(2)\,Li_4\left(\frac{1}{2}\right) + 6Li_5\left(\frac{1}{2}\right) - 6\zeta(5) - \zeta(2)ln^3(2) + \frac{21}{8}ln^2(2)\zeta(3) + \frac{ln^5(2)}{5}$$

Solution 4 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^1\frac{\log(x)\log^3(1+x)}{1+x}dx = \int_0^1\frac{\left(\log(1+x) + \log\left(\frac{x}{1+x}\right)\right)\log^3(1+x)}{1+x}dx =$$

$$= \int_0^1\frac{\log^4(1+x)}{1+x}dx + \int_0^1\frac{\log\left(\frac{x}{1+x}\right)}{1+x}\log^3(1+x)\,dx = \frac{1}{5}\log^2(5) +$$

$$+\int_0^1\left(\mathrm{Li}_2\left(\frac{1}{1+x}\right)\right)'\log^3(1+x)\,dx=\frac{1}{5}\log^2(5)+\mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2)-$$

$$-3\int_0^1\frac{\mathrm{Li}_2\left(\frac{1}{1+x}\right)}{1+x}\log^2(1+x)\,dx=\frac{1}{5}\log^2(5)+\mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2)+$$

$$+3\int_0^1\left(\mathrm{Li}_3\left(\frac{1}{1+x}\right)\right)'\log^2(1+x)\,dx=\frac{1}{5}\log^2(5)+\mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2)+$$

$$+3\left\{\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2)-2\int_0^1\frac{\mathrm{Li}_3\left(\frac{1}{1+x}\right)}{1+x}\log(1+x)\,dx\right\}=\frac{1}{5}\log^2(5)+$$

$$+\mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2)+3\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2)+6\int_0^1\left(\mathrm{Li}_4\left(\frac{1}{1+x}\right)\right)'\log(1+x)\,dx=$$

$$=\frac{1}{5}\log^2(5)+\mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2)+3\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2)$$

$$+6\left\{\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)-\int_0^1\frac{\mathrm{Li}_4\left(\frac{1}{1+x}\right)}{1+x}dx\right\}=$$

$$=\frac{1}{5}\log^2(5)+\mathrm{Li}_2\left(\frac{1}{2}\right)\log^3(2)+3\mathrm{Li}_3\left(\frac{1}{2}\right)\log^2(2)+6\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)+6\mathrm{Li}_5\left(\frac{1}{2}\right)-6\zeta(5)$$

2517. Find:

$$\int_0^1\left(Li_3(-x)+x^2\arctan^2(x)\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$$I=\int_0^1\left(Li_3(-x)+x^2\arctan^2(x)\right)dx$$

$$*\ S=\int_0^1 Li_3(-x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^n dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n+1)}=$$

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^3}-\frac{1}{n^2}-\frac{1}{n+1}+\frac{1}{n}\right)=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1$$

$$*\ Y=\int_0^1 x^2\arctan^2(x)dx\overset{I.B.P}{=}\frac{1}{3}x^3\arctan^2(x)\Big|_0^1-\frac{2}{3}\int_0^1\frac{x^3\arctan^2(x)|}{1+x^2}dx\overset{I.B.P}{=}$$

$$\frac{\pi^2}{48}-\frac{2}{3}(\arctan(x)\left(\frac{x^2}{2}-\frac{1}{2}\ln(1+x^2)\right)\Big|_0^1$$

$$-\frac{1}{2}\int_0^1\frac{x^2-\ln(1+x^2)}{1+x^2}dx)=\frac{\pi^2}{48}-\frac{2}{3}(\frac{\pi}{8}-\frac{\pi}{8}\ln(2)-$$

$$\frac{1}{2}\left(1-\frac{\pi}{4}\right)+\underbrace{\frac{1}{2}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx)}_{\frac{1}{x}\to x}$$

$$=\frac{\pi^2}{48}-\frac{2}{3}(-\frac{1}{2}+\frac{\pi}{4}-\frac{\pi}{8}\ln(2)+\frac{1}{2}\int_1^\infty\frac{\ln\left(\frac{1+x^2}{x^2}\right)}{1+x^2}dx)=$$

$$\frac{\pi^2}{48}-\frac{2}{3}(-\frac{1}{2}+\frac{\pi}{4}-\frac{\pi}{8}\ln(2)+\frac{1}{4}\int_1^\infty\frac{\ln(1+x^2)}{1+x^2}dx-\frac{1}{4}\int_1^\infty\frac{\ln(x^2)}{1+x^2}dx)=$$

$$=\frac{\pi^2}{48}-\frac{2}{3}(-\frac{1}{2}+\frac{\pi}{4}-\frac{\pi}{8}\ln(2)-$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(cos(x))\,dx+\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx)=\frac{\pi^2}{48}-\frac{2}{3}\left(-\frac{1}{2}+\frac{\pi}{4}-\frac{\pi}{8}\ln(2)+\frac{\pi}{4}\ln(2)-\frac{G}{2}\right)$$

$$=$$

$$=\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$I=S+Y=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1+\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$I=\frac{G}{3}+\frac{5}{8}\zeta(2)-\frac{3}{4}\zeta(3)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}-2\ln(2)+\frac{4}{3}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1(Li_3(-x)+x^2arctan^2(x))dx$$

$$E=\int_0^1 Li_3(-x)dx=\left(Li_3(-x)\int dx\right)\Big|_0^1-\int_0^1 Li_2(-x)\,dx$$

$$=Li_3(-1)-\left(Li_2(-x)\int dx\right)\Big|_0^1-$$

$$\int_0^1\ln(1+x)\,dx=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1$$

$$E=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1$$

$$H=\int_0^1 x^2arctan^2(x)dx=\left(arctan^2(x)\int d\left(\frac{x^3}{3}\right)\right)\Big|_0^1-\frac{2}{3}\int_0^1\frac{x^3\arctan(x)}{1+x^2}dx=$$

$$\frac{arctan^2(1)}{3}-\frac{2}{3}\int_0^{\frac{\pi}{4}} xtan^3(x)dx=\frac{1}{3}\left(\frac{\pi}{4}\right)^2-\frac{2}{3}\left(x\int d\left(\frac{1}{2cos^2(x)}+\ln(cos(x))\right)\right)\Big|_0^{\frac{\pi}{4}}+$$

$$\frac{1}{3}\int_0^{\frac{\pi}{4}}\frac{dx}{cos^2(x)}+\frac{2}{3}\int_0^{\frac{\pi}{4}}\ln(cos(x))\,dx$$

$$\left\{\therefore\int_0^{\theta}\ln(cos(z))\,dz=\frac{1}{2}Cl_2(\pi-2\theta)-\theta\ln(2)\,;\ Cl_2\left(\frac{\pi}{2}\right)=\beta(2)=G\right\}$$

$$=\frac{1}{8}\zeta(2)-\frac{2}{3}\left(\frac{\pi}{4}-\frac{\pi}{8}\ln(2)\right)+\frac{1}{3}\int_0^{\frac{\pi}{4}}dtan(x)+\frac{G}{3}-\frac{\pi}{6}\ln(2)$$

$$H=\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$E+H=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1+\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$\int_0^1(Li_3(-x)+x^2arctan^2(x))dx=\frac{G}{3}+\frac{5}{8}\zeta(2)-\frac{3}{4}\zeta(3)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}-2\ln(2)+\frac{4}{3}$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1(Li_3(-x)+x^2arctan^2(x))dx=\int_0^1 Li_3(-x)dx+\int_0^1 x^2arctan^2(x)dx=A+B$$

$*$ ***working on A***

$$A=\int_0^1 Li_3(-x)dx\quad\left[u=Li_3(-x)\,,\frac{du}{dx}=\frac{Li_2(-x)}{x}\ ;\ v=x\,,dv=dx\right]$$

$$A=xLi_3(-x)|_0^1-\int_0^1 Li_2(-x)=-\frac{3}{4}\zeta(3)-[xLi_3(-x)|_0^1+\int_0^1\frac{xln(1+x)}{x}dx]$$

$$A=-\frac{3}{4}\zeta(3)-[-\frac{1}{2}\zeta(2)+\underbrace{\int_0^1\ln(1+x)\,dx}_{x+1\to y}=-\frac{3}{4}\zeta(3)-[-\frac{1}{2}\zeta(2)+\int_1^2\ln(y)\,dy]$$

$$A=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1$$

$*$ ***working on B***

$$B=\int_0^1 x^2arctan^2(x)dx\quad[x=\tan(y)\ ;dx=sec^2(y)]\ \left[0,\frac{\pi}{4}\right]$$

$$B=\int_0^{\frac{\pi}{4}}tan^2(y)arctan^2(\tan(y)).sec^2(y)dy=\int_0^{\frac{\pi}{4}}y^2tan^2(y)\,sec^2(y)dy$$

$$B=\frac{y^2tan^3(y)}{3}\Big|_0^{\frac{\pi}{4}}-\frac{2}{3}\int_0^{\frac{\pi}{4}}ytan^3(y)dy=\frac{1}{3}\left(\frac{\pi}{4}\right)^2-\frac{2}{3}\int_0^{\frac{\pi}{4}}ytan^3(y)dy$$

$$B=\frac{\pi^2}{48}-\frac{2}{3}\int_0^{\frac{\pi}{4}}ytan^3(y)dy=$$

$$=\frac{\pi^2}{48}-\frac{2}{3}[\left(\frac{sec^2(y)}{2}+\ln(cos(y))\right)y|_0^{\frac{\pi}{4}}-\frac{1}{2}\int_0^{\frac{\pi}{4}}sec^2(y)dy-\int_0^{\frac{\pi}{4}}\ln(cos(y))\,dy]$$

$$B=\frac{\pi^2}{48}-\frac{2}{3}[\frac{\pi}{4}\left(1-\frac{\ln(2)}{2}\right)-\frac{1}{2}\tan\left(\frac{\pi}{4}\right)+\ln(2)\int_0^{\frac{\pi}{4}}dy+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2ny)\,dy]$$

$$B=\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$A+B=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1+\frac{G}{3}+\frac{1}{8}\zeta(2)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}+\frac{1}{3}$$

$$I=\int_0^1(Li_3(-x)+x^2arctan^2(x))dx=\frac{G}{3}+\frac{5}{8}\zeta(2)-\frac{3}{4}\zeta(3)-\frac{\pi}{12}\ln(2)-\frac{\pi}{6}-2\ln(2)+\frac{4}{3}$$

2518. Find:

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

$$We\ have,\quad \int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx\overset{x\to\frac{1}{x}}{\Leftrightarrow}\int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)^2}dx=\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x}dx-$$

$$\underbrace{\frac{1}{4}\int_0^1\frac{xln^2(x)}{1+x^2}dx}_{x^2\to x}+\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx-\underbrace{\frac{1}{2}\int_0^1\frac{xln^2(x)}{(1+x^2)^2}dx}_{I.B.P}+\underbrace{\frac{1}{2}\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx}_{I.B.P}=\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx+$$

$$\frac{7}{32}\int_0^1\frac{ln^2(x)}{1+x^2}dx-\underbrace{\left(\frac{ln^2(x)}{4}\int d\left(\frac{x^2}{1+x^2}\right)\right)_0^1}_{=0}+\underbrace{\frac{1}{2}\int_0^1\frac{xln(x)}{1+x^2}dx}_{x^2\to x}+\underbrace{\left(\frac{ln^2(x)}{4}\int d\left(\frac{x^2}{1+x^2}+\arctan(x)\right)\right)_0^1}_{=0}$$

$$-\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx-\underbrace{\frac{1}{2}\int_0^1\frac{\ln(x)\arctan(x)}{x}dx}_{I.B.P}=\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx+\frac{7}{32}\int_0^1\frac{ln^2(x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx+$$

$$\frac{1}{8}\int_0^1\frac{\ln(x)}{1+x}dx-\underbrace{\left(\frac{\arctan(x)}{4}\int d\left(ln^2(x)\right)\right)_0^1}_{=0}+\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx\overset{|x|<1\ x\in(0,1)}{\Leftrightarrow}\frac{1}{2}\sum_{n\in N_0}(-1)^n\int_0^1x^{2n}ln^2(x)dx+$$

$$\frac{7}{32}\sum_{n\in N_0}(-1)^{n-1}\int_0^1x^{n-1}ln^2(x)dx-\frac{1}{2}\sum_{n\in N_0}(-1)^n\int_0^1x^{2n}ln(x)dx+\frac{1}{8}\sum_{n\in N_0}(-1)^{n-1}\int_0^1x^{n-1}\ln(x)dx$$

$$\left(\therefore\int_0^1t^mln^n(t)\,dt=\frac{(-1)^n.n!}{(m+1)^{n+1}}\quad n>-1\bigwedge m\neq-1\right)$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx=\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^3}}_{=\beta(3)}+\frac{7}{16}\underbrace{\sum_{n\in N_0}\frac{(-1)^{n-1}}{(n)^3}}_{=\eta(3)}+\frac{1}{2}\underbrace{\sum_{n\in N_0}\frac{(-1)^n}{(2n+1)^2}}_{=G\ (Catalan's\ constant)}$$

$$-\frac{1}{8}\underbrace{\sum_{n\in N}\frac{(-1)^{n-1}}{(n)^2}}_{=\zeta(2)}$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx=\frac{21}{64}\zeta(3)+\frac{G}{2}+\beta(3)-\frac{\zeta(2)}{16}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)^2}dx=$$

$$\int_0^1 ln^2(x)\left(\frac{1-x}{4(x^2+1)}+\frac{1-x}{2(1+x^2)^2}+\frac{1}{4(x+1)}\right)dx=\frac{1}{4}\int_0^1\frac{ln^2(x)}{1+x^2}dx-\frac{1}{4}\int_0^1\frac{xln^2(x)}{1+x^2}dx+$$

$$\int_0^1\frac{ln^2(x)}{2(1+x^2)^2}dx-\frac{1}{2}\int_0^1\frac{xln^2(x)}{(1+x^2)^2}dx+\frac{1}{4}\int_0^1\frac{ln^2(x)}{x+1}dx=\frac{1}{4}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{2k-2}\,ln^2(x)dx$$

$$+$$

$$\frac{7}{32}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{k-1}ln^2(x)dx+\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{2k-2}\,ln^2(x)dx-$$

$$\frac{1}{16}\sum_{k\in N}(-1)^{k-1}k\int_0^1 x^{k-1}ln^2(x)dx=\frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^3}+\frac{7}{16}\sum_{k\in N}\frac{(-1)^{k-1}}{(k)^3}+$$

$$\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\left(\frac{1}{(2k-1)^2}+\frac{1}{(2k-1)^3}\right)-\frac{1}{8}\sum_{k\in N}\frac{(-1)^{k-1}}{(k)^2}$$

$$\left\{\therefore\sum_{n\in N}\frac{(-1)^n}{(2k+1)^z}=\beta(z);\ \ \beta(2)=G;\ \ \beta(3)=\frac{3}{16}\pi\zeta(2)\right\}$$

$$=\frac{1}{2}\cdot\frac{3}{16}\pi\zeta(2)+\frac{7}{16}\cdot\frac{\zeta(3)}{4}+\frac{1}{2}\left(G+\frac{3}{16}\pi\zeta(2)\right)-\frac{1}{8}\cdot\frac{\zeta(2)}{2}$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx=\frac{21}{64}\zeta(3)+\frac{G}{2}+\beta(3)-\frac{\zeta(2)}{16}$$

Solution 3 by Pham Duc Nam-Vietnam

$$*\ A=\int_0^1\frac{ln^2(x)}{1-x^2}dx=\sum_{n=0}^{\infty}\int_0^1 x^{2n}ln^2(x)dx=2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}=\frac{7}{4}\zeta(3)$$

$$B=\int_0^1\frac{ln^2(x)}{(1+x)^2}dx=-\sum_{n=1}^{\infty}(-1)^n\,n\int_0^1 x^{n-1}ln^2(x)dx=-2\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}=\zeta(2)$$

$$a\geq 1: C=\int_0^1\frac{ln^2(x)}{a+x^2}dx=\frac{1}{a}\int_0^1\frac{ln^2(x)}{1+\left(\sqrt{\frac{1}{a}}x\right)^2}dx=\frac{1}{a}\sum_{n=0}^{\infty}\left(-\frac{1}{a}\right)^n\int_0^1 x^{2n}ln^2(x)dx=$$

$$=\frac{2}{a}\sum_{n=0}^{\infty}\left(-\frac{1}{a}\right)^n\frac{1}{(2n+1)^3}=\frac{1}{4a}\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)$$

$$a=1: C_1=\int_0^1\frac{ln^2(x)}{1+x^2}dx=\frac{1}{4a}\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)\ \therefore\ \Phi\left(-1,s,\frac{1}{2}\right)=2^s\beta(s)$$

$$\to\int_0^1\frac{ln^2(x)}{1+x^2}dx=2\beta(3)$$

$$\frac{d}{da}\Big|_{a=1}\int_0^1\frac{ln^2(x)}{a+x^2}dx=C_2=-\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=\frac{d}{da}\Big|_{a=1}\frac{1}{4a}\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)=$$

$$=-\frac{2\Phi\left(-\frac{1}{a},2,\frac{1}{2}\right)+\Phi\left(-\frac{1}{a},3,\frac{1}{2}\right)}{8a^2}\Bigg|_{a=1}=-G-\beta(3)\to C_2=G+\beta(3)$$

$$I=\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_0^1\frac{ln^2(x)}{(1+x)(1+x^2)^2}dx$$

$$=\int_0^1\frac{ln^2(x)}{(1-x^2)(1+x^2)^2}dx-\int_0^1\frac{xln^2(x)}{(1-x^2)(1+x^2)^2}dx$$

$$*E=\int_0^1\frac{ln^2(x)}{(1-x^2)(1+x^2)^2}dx\overset{x\to\frac{1}{x}}{\cong}\frac{1}{8}\int_0^1\frac{ln^2(x)}{(1-x)(1+x)^2}dx$$

$$=\frac{1}{16}\int_0^1 ln^2(x)\left(\frac{1}{1-x^2}+\frac{1}{(1+x)^2}\right)dx=$$

$$=\frac{1}{16}(A+B)=\frac{7}{64}\zeta(3)+\frac{1}{16}\zeta(2)$$

$$*H=\int_0^1\frac{ln^2(x)}{(1-x^2)(1+x^2)^2}dx=\frac{1}{4}\underbrace{\int_0^1\frac{ln^2(x)}{1-x^2}dx}_{A}+\frac{1}{4}\underbrace{\int_0^1\frac{ln^2(x)}{1+x^2}dx}_{C_1}+\frac{1}{2}\underbrace{\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx}_{C_2}=$$

$$=\beta(3)+\frac{7}{16}\zeta(3)+\frac{G}{2}$$

$$I=H-E=\beta(3)+\frac{7}{16}\zeta(3)+\frac{G}{2}-\frac{7}{64}\zeta(3)-\frac{1}{16}\zeta(2)=\frac{21}{64}\zeta(3)+\frac{G}{2}+\beta(3)-\frac{\zeta(2)}{16}$$

$$\int_1^{\infty}\frac{x^3ln^2(x)}{(1+x)(1+x^2)^2}dx=\frac{21}{64}\zeta(3)+\frac{G}{2}+\beta(3)-\frac{\zeta(2)}{16}$$

2519. Find:

$$\int_0^1\frac{x^2(1+\sqrt{x})cos^2(x)}{\sqrt{1-x^2}}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Exodo Halcalias-Angola

$$I=\int_0^1\frac{x^2(1+\sqrt{x})cos^2(x)}{\sqrt{1-x^2}}dx=\int_0^1\frac{x^2cos^2(x)}{\sqrt{1-x^2}}dx+\int_0^1\frac{x^2\sqrt{x}cos^2(x)}{\sqrt{1-x^2}}dx=E+H$$

$$E=\int_0^1\frac{x^2cos^2(x)}{\sqrt{1-x^2}}dx=\int_0^1\frac{x^2}{\sqrt{1-x^2}}(1+\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}x^{2k}}{(2k)!})dx=$$

$$=\int_0^1\frac{x^2}{\sqrt{1-x^2}}dx+\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}}{(2k)!}\int_0^1\frac{x^{2k+2}}{\sqrt{1-x^2}}dx=\frac{1}{2}\int_0^1\frac{x^{\frac{3}{2}-1}}{\sqrt{1-x}}dx+$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}}{(2k)!}\int_0^1\frac{x^{k+\frac{3}{2}-1}}{\sqrt{1-x}}dx=\frac{1}{2}(\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right))$$

$$+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}}{(2k)!}\left(\Gamma\left(\frac{1}{2}\right)\frac{\Gamma\left(k+\frac{3}{2}\right)}{\Gamma(k+1)}\right)=$$

$$=\frac{1}{4}\Gamma^2\left(\frac{1}{2}\right)+\frac{\sqrt{\pi}}{2}(-\frac{\sqrt{\pi}}{4}+\frac{\sqrt{\pi}}{4}\sum_{k=0}^{\infty}\frac{(-1)^{k-1}2^{2k-1}(2(1)_k-(2)_k)}{k!\,(1)_k(2)_k})=$$

$$=\frac{\pi}{4}+\frac{\sqrt{\pi}}{2}\left(-\frac{\sqrt{\pi}}{4}+\frac{\sqrt{\pi}}{4}J_{(0)}(2)-\frac{\sqrt{\pi}}{2}J_{(1)}(2)\right)$$

$$E=\left(\frac{\pi}{8}+\frac{\pi}{8}J_{(0)}(2)-\frac{\pi}{4}J_{(1)}(2)\right)$$

$$H=\int_0^1\frac{x^2\sqrt{x}cos^2(x)}{\sqrt{1-x^2}}dx=\int_0^1\frac{x^2\sqrt{x}}{\sqrt{1-x^2}}dx(1+\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}x^{2k}}{(2k)!})dx=$$

$$\int_0^1\frac{x^2\sqrt{x}}{\sqrt{1-x^2}}dx+\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}}{(2k)!}\int_0^1\frac{x^{2k+\frac{5}{2}}}{\sqrt{1-x^2}}dx=\frac{1}{2}\int_0^1\frac{x^{\frac{7}{4}-1}}{\sqrt{1-x}}dx+$$

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}}{(2k)!}\int_0^1\frac{x^{k+\frac{7}{4}-1}}{\sqrt{1-x}}dx=\frac{1}{2}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{9}{4}\right)}\right)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}}{(2k)!}\left(\Gamma\left(\frac{1}{2}\right)\frac{\Gamma\left(k+\frac{7}{4}\right)}{\Gamma\left(k+\frac{9}{4}\right)}\right)=$$

$$=\frac{\sqrt{\pi}}{2}\left(\frac{12\pi\sqrt{2}}{5\Gamma^2\left(\frac{1}{4}\right)}\right)+\frac{\sqrt{\pi}}{2}\left(\frac{6\Gamma\left(\frac{3}{4}\right)\ {}_2^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right)}{5\Gamma\left(\frac{1}{4}\right)}-\frac{6\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(\frac{1}{4}\right)}\right)=$$

$$=\frac{3\sqrt{\pi}}{5}\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2}}\right)^{-1}+\frac{3\sqrt{\pi}}{5}\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2}}\right)^{-1}{}_2^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right)-\frac{3\sqrt{\pi}}{10}\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2}}\right)^{-1}$$

$$H=\frac{3\sqrt{\pi}}{10\varpi}\left(\left({}_2^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right)+1\right)\right)$$

$$I=E+H=\left(\frac{\pi}{8}+\frac{\pi}{8}J_{(0)}(2)-\frac{\pi}{4}J_{(1)}(2)\right)+\frac{3\sqrt{\pi}}{10\varpi}\left(\left({}_1^1F\left(\frac{7}{4};\frac{1}{2};\frac{9}{4};-1\right)+1\right)\right)$$

$J_{(a)}(x)\to$ *Bassel function of first the kind*

$\varpi\to$ *Lemniscate constant*

2520. Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}}-\ln(n)\right),\qquad \omega_n=\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k(x)}{\cos^2(x)}$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\omega_n=\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k(x)}{\cos^2(x)}=\lim_{x\to\frac{\pi}{2}}n!\frac{1-\sin^n(x)}{\cos^{2n}(x)}=n!\lim_{x\to\frac{\pi}{2}}\frac{1-\sin^n(x)}{\cos^{2n}(x)}\overbrace{\underbrace{=}_{k->0}}^{x=k+\frac{\pi}{2}}$$

$$=n!\lim_{k\to 0}\frac{-1-\cos^n(k)}{\sin^{2n}(k)}=n!\lim_{k\to 0}\frac{1-\cos^n(k)}{k^{2n}.\left(\frac{\sin^{2n}(k)}{k^{2n}}\right)}\overset{\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)}{\cong}n!\lim_{k\to 0}\frac{1-\cos^n(k)}{k^{2n}.\left(\frac{\sin(k)}{k}\right)^{2n}}=$$

$$=n!\lim_{k\to 0}\frac{\left(1-2\sin^2\left(\frac{k}{2}\right)\right)^{2n}-1}{k^{2n}}\overset{\lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)}{\cong}n!\lim_{k\to 0}\frac{2^n-(-1)^n(k^2-2)^n}{2^n.k^{2n}}=\frac{n!}{2^n}$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}}-\ln(n)\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\frac{k!}{2^k}}{2\,\frac{(k+1)!}{2^{k+1}}}-\ln(n)\right)=$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{2^{k+1}.k!}{2^{k+1}.(k+1)!}-\ln(n)\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{k!}{(k+1)}-\ln(n)\right)=\gamma-1$$

2521. Find:

$$\Omega=\lim_{n\to\infty}\left(n\int_0^1 x^n\exp(x)\,dx\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Dimitris Kastriotis – Greece

$$n\cdot\int_0^1 x^n e^x\,dx=n\cdot\int_0^1 x^n\sum_{k=0}^{\infty}\frac{x^k}{k!}dx=n\cdot\int_0^1\sum_{k=0}^{\infty}\frac{x^{k+n}}{k!}dx=$$

$$= n\cdot\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1 x^{n+k}\,dx = n\cdot\sum_{k=0}^{\infty}\left(\frac{1}{k!}\cdot\frac{1}{n+k+1}\right) =$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{k!}\cdot\frac{1}{1+\frac{k+1}{n}}\right)\overset{n\to\infty}{\to}\sum_{k=0}^{\infty}\frac{1}{k!} = e$$

2522. Find:

$$\Omega = \lim_{n\to\infty}\left(n\int_0^1\frac{x^n\exp(x^2)}{1+x^{2n}}dx\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hikmat Mammadov-Azerbaijan

$$\Omega = \lim_{n\to\infty}\left(n\int_0^1\frac{x^n\exp(x^2)}{1+x^{2n}}dx\right)$$

$$\int_0^1\frac{x^n\exp(x^2)}{1+x^{2n}}dx = \int_0^1 x^n\exp(x^2)\sum_k(-1)^k x^{2nk}\,dx = \sum_{k=0}^{\infty}(-1)^k\int_0^1\exp(x^2)\,x^{n(2k+1)}dx$$

$$\int_0^1\exp(x^2)\,x^{n(2n+1)}\,dx =$$

$$= \frac{\exp(x^2)\,x^{n(2k+1)+1}}{n(2k+1)+1}\Bigg|_0^1 - \frac{2}{n(2k+1)+1}\int_0^1\exp(x^2)\,x^{n(2k+1)+2}\,dx$$

$$= \frac{\exp(1)}{n(2k+1)+1} - \frac{2}{n(2k+1)+1}\int_0^1\exp(x^2)\,x^{n(2k+1)+2}\,dx$$

$$\lim_{n\to\infty} n\sum_{k=0}^{\infty}(-1)^k\frac{\exp(1)}{n(2k+1)+1} = \lim_{n\to\infty}\sum_{k=0}^{\infty}(-1)^k\frac{\exp(1)\cdot n}{n(2k+1)+1} =$$

$$= \sum_{k=0}^{\infty}(-1)^k\lim_{n\to\infty}\frac{n\cdot\exp(1)}{n(2k+1)+1} =$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{\exp(1)}{2k+1} = \exp(1)\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \exp(1)\frac{\pi}{4} = e\frac{\pi}{4}$$

For another case: $\lim\limits_{n\to\infty} n\sum_{k=0}^{\infty}(-1)^k\frac{2}{n(2k+1)+1}\int_0^{12}\exp(x^2)\,x^{n(2k+1)+2}\,dx$

$$=\sum_{k=0}^{\infty}(-1)^k \lim_{n\to\infty}\frac{2n}{n(2k+1)+1}\cdot\lim_{n\to\infty}\exp(x^2)\,x^{m(2k\times1)+2}=0$$

So we get:

$$\int_0^1 \frac{x^n \exp(x^2)}{1+x^{2n}}dx = e\frac{\pi}{4}$$

Solution 2 by Exodo Halcalias-Angola

$$\lim_{n\to\infty}\left(n\int_0^1 \frac{x^n\exp(x^2)}{1+x^{2n}}dx\right)=\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\frac{y^{\frac{2k}{n}+\frac{1}{n}}}{1+y^2}dy\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k\in\mathbb{N}}\frac{1}{k!}\left(\sum_{j\in\mathbb{N}^*}(-1)^{j-1}\int_0^1 y^{2j-2+\frac{2k}{n}+\frac{1}{n}}dy\right)\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k\in\mathbb{N}}\frac{1}{k!}\left(\sum_{j\in\mathbb{N}^*}\frac{(-1)^{j-1}}{2j-1+\frac{2k}{n}+\frac{1}{n}}\right)\right)=\sum_{k\in\mathbb{N}}\frac{1}{k!}\left(\sum_{j\in\mathbb{N}^*}\frac{(-1)^{j-1}}{2j-1}\right)=$$

$$=e\int_0^1\frac{1}{1+y^2}dy=e\int_0^1 d\tan^{-1}(y)=\frac{\pi e}{4}$$

$$\therefore \lim_{n\to\infty}\left(n\int_0^1\frac{x^n\exp(x^2)}{1+x^{2n}}dx\right)=\frac{\pi e}{4}$$

2523. Find:

$$\Omega=\lim_{n\to\infty}\left(2^{-n}\cdot\prod_{k=1}^{n}\sqrt[k+1]{k}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Adrian Popa – Romania

$$\prod_{k=1}^{n}\sqrt[k+1]{k}=\sqrt{1}\cdot\sqrt[3]{2}\cdot\sqrt[4]{3}\cdot\ldots\cdot\sqrt[n+1]{n}=$$

$$=\sqrt{1\cdot1}\cdot\sqrt[3]{1\cdot1\cdot2}\cdot\sqrt[4]{1\cdot1\cdot1\cdot3}\cdot\ldots\cdot\sqrt[n+1]{\underbrace{1\cdot1\cdot\ldots\cdot1}_{n\ times}\cdot n}\overset{MG\le MA}{<}$$

$$< \frac{2}{2}\cdot\frac{2\cdot 2}{3}\cdot\frac{2\cdot 3}{4}\cdot\ldots\cdot\frac{2\cdot n}{n+1}\Rightarrow$$

$$\Rightarrow 2^{-n}\prod_{k=1}^{n}\sqrt[k+1]{k} < \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{n}{n+1}=\frac{1}{n+1}\overset{n\to\infty}{\rightarrow} 0$$

$$\Rightarrow \lim_{n\to\infty} 2^{-n}\prod \sqrt[k+1]{k} = 0$$

2534. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{n(n+1)}{2}\right)!}\cdot\prod_{k=1}^{n} k!\cdot\binom{\frac{k(k+1)}{2}}{\frac{k(k-1)}{2}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Bui Hong Suc-Vietnam

$$\text{Let: } S_n = \prod_{k=1}^{n} k!\binom{\frac{k(k+1)}{2}}{\frac{k(k-1)}{2}} = \prod_{k=1}^{n} k!\frac{\left(\frac{k(k+1)}{2}\right)!}{\left(\frac{k(k-1)}{2}\right)!k!}$$

$$= \prod_{k=1}^{n}\frac{\left(\frac{k(k+1)}{2}\right)!}{\left(\frac{k(k-1)}{2}\right)!} = \frac{\left(\frac{1\cdot 2}{2}\right)!\left(\frac{2\cdot 3}{2}\right)!\left(\frac{3\cdot 4}{2}\right)!\cdot\ldots\cdot\left(\frac{(n-1)n}{2}\right)!\left(\frac{n(n+1)}{2}\right)!}{\left(\frac{1\cdot 0}{2}\right)!\left(\frac{2\cdot 1}{2}\right)!\left(\frac{3\cdot 2}{2}\right)!\left(\frac{4\cdot 3}{2}\right)!\cdot\ldots\cdot\left(\frac{n(n-1)}{2}\right)!}$$

$$= \left(\frac{n(n+1)}{2}\right)!$$

Then:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{n(n+1)}{2}\right)!}S_n\right) = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{n(n+1)}{2}\right)!}\cdot\left(\frac{n(n+1)}{2}\right)!\right) = \lim_{n\to\infty}(1) = 1$$

Therefore: $\Omega = 1$

2525. Find:

$$\int_1^{\infty}\int_0^1 \frac{ln^2(\sqrt{x})ln^2(xy)}{(1+y^2)(2+x)^2}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Quadri Faruk Temitope-Nigeria

$$I=\int_1^\infty\int_0^1\frac{ln^2(\sqrt{x})ln^2(xy)}{(1+y^2)(2+x)^2}dxdy$$

$*$ ***But*** :

$$ln^2(AB)=ln^2(A)+2\ln(A)\ln(B)+ln^2(B)$$

$$I=\int_1^\infty\int_0^1\frac{ln^2(\sqrt{x})ln^2(x)}{(1+y^2)(2+x)^2}dxdy+2\int_1^\infty\int_0^1\frac{ln^2(\sqrt{x})ln(x)\ln(y)}{(1+y^2)(2+x)^2}dxdy+$$

$$\int_1^\infty\int_0^1\frac{ln^2(\sqrt{x})ln^2(y)}{(1+y^2)(2+x)^2}dxdy=\frac{1}{4}\int_1^\infty\int_0^1\frac{ln^4(x)}{(1+y^2)(2+x)^2}dxdy+\frac{2}{4}\int_1^\infty\int_0^1\frac{ln^3(x)ln(y)}{(1+y^2)(2+x)^2}dxdy+$$

$$\frac{1}{4}\int_1^\infty\int_0^1\frac{ln^2(x)ln^2(y)}{(1+y^2)(2+x)^2}dxdy=\frac{1}{4}\int_1^\infty\frac{dy}{1+y^2}\int_0^1\frac{ln^4(x)}{(2+x)^2}dx+\frac{1}{2}\int_1^\infty\frac{\ln(y)}{1+y^2}dy\int_0^1\frac{ln^3(x)}{(2+x)^2}dx+$$

$$+\frac{1}{4}\int_1^\infty\frac{ln^2(y)}{1+y^2}dy\int_0^1\frac{ln^2(x)}{(2+x)^2}dx$$

By working at the integrals

$$1)\int_1^\infty\frac{dy}{1+y^2}=\tan(y)|_1^\infty=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

$$2)\int_1^\infty\frac{ln(y)}{1+y^2}dy\overset{y\to\frac{1}{y}}{=}\int_1^0\frac{\ln\left(\frac{1}{y}\right)}{\frac{1+y^2}{y^2}}\frac{-dy}{y^2}=-\int_0^1\frac{\ln(y)}{1+y^2}dy=-\sum_{n=0}^\infty(-1)^n\int_0^1y^{2n}\ln(y)\,dy$$

$$=\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^2}=G$$

$$3)\int_1^\infty\frac{ln^2(y)}{1+y^2}dy\overset{y\to\frac{1}{y}}{=}\int_1^0\frac{ln^2\left(\frac{1}{y}\right)}{\frac{1+y^2}{y^2}}\frac{-dy}{y^2}=\int_0^1\frac{ln^2(y)}{1+y^2}dy=\sum_{n=0}^\infty(-1)^n\int_0^1y^{2n}ln^2(y)\,dy$$

$$=2\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{16}$$

$$4)\ A=\int_0^1\frac{ln^4(x)}{(2+x)^2}dx=\sum_{n=0}^\infty\binom{-2}{n}2^{-2-n}\int_0^1x^n\ln^4(x)dx=\sum_{n=0}^\infty\binom{-2}{n}2^{-2-n}\frac{d}{dn}\int_0^1x^n\,dx=$$

$$\sum_{n=0}^\infty\binom{-2}{n}\frac{1}{2^{2+n}}\cdot\frac{24}{(n+1)^5}=24\sum_{n=0}^\infty\frac{\binom{-2}{n}}{2^{2+n}.(n+1)^5}=24\left[-\frac{1}{2}Li_4\left(-\frac{1}{2}\right)\right]=-12\,Li_4\left(-\frac{1}{2}\right)$$

$$5)\ B=\int_0^1\frac{ln^3(x)}{(2+x)^2}dx=\sum_{n=0}^\infty\binom{-2}{n}2^{-2-n}\int_0^1x^n\ln^3(x)dx=\sum_{n=0}^\infty\binom{-2}{n}\frac{1}{2^{2+n}}\cdot\int_0^1x^n\ln^3(x)dx=$$

$$\sum_{n=0}^\infty\binom{-2}{n}\frac{1}{2^{2+n}}\cdot\left[-\frac{6}{(n+1)^4}\right]=-6\sum_{n=0}^\infty\frac{\binom{-2}{n}}{2^{2+n}.(n+1)^4}=-6\left[-\frac{1}{2}Li_3\left(-\frac{1}{2}\right)\right]=3Li_3\left(-\frac{1}{2}\right)$$

$$6)\ C=\int_0^1\frac{ln^2(x)}{(2+x)^2}dx=\sum_{n=0}^{\infty}\binom{-2}{n}2^{-2-n}\int_0^1 x^n\,ln^2(x)dx=\sum_{n=0}^{\infty}\binom{-2}{n}\frac{1}{2^{2+n}}.\int_0^1 x^n\,ln^2(x)dx=$$

$$\sum_{n=0}^{\infty}\binom{-2}{n}\frac{1}{2^{2+n}}.\left[\frac{2}{(n+1)^3}\right]=2\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{2^{2+n}.(n+1)^3}=2\left[-\frac{1}{2}Li_2\left(-\frac{1}{2}\right)\right]=-Li_2\left(-\frac{1}{2}\right)$$

This :

$$I=\frac{1}{4}\int_1^{\infty}\frac{dy}{1+y^2}\int_0^1\frac{ln^4(x)}{(2+x)^2}dx+\frac{1}{2}\int_1^{\infty}\frac{\ln(y)}{1+y^2}dy\int_0^1\frac{ln^3(x)}{(2+x)^2}dx+\frac{1}{4}\int_1^{\infty}\frac{ln^2(y)}{1+y^2}dy\int_0^1\frac{ln^2(x)}{(2+x)^2}dx=$$

$$\frac{1}{4}\left(\frac{\pi}{4}\right)\left(-12Li_4\left(-\frac{1}{2}\right)\right)+\frac{1}{2}G\left(3Li_3\left(-\frac{1}{2}\right)\right)+\frac{1}{4}\left(\frac{\pi^3}{16}\right)\left(-Li_2\left(-\frac{1}{2}\right)\right)$$

$$I=-\frac{3}{4}Li_4\left(-\frac{1}{2}\right)+\frac{3}{2}GLi_3\left(-\frac{1}{2}\right)-\frac{\pi^3}{64}Li_2\left(-\frac{1}{2}\right)$$

$$\int_1^{\infty}\int_0^1\frac{ln^2(\sqrt{x})ln^2(xy)}{(1+y^2)(2+x)^2}dxdy=\frac{3}{2}GLi_3\left(-\frac{1}{2}\right)-\frac{3}{4}Li_4\left(-\frac{1}{2}\right)-\frac{\pi^3}{64}Li_2\left(-\frac{1}{2}\right)$$

2526. Find:

$$\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Quadri Faruk Temitope-Nigeria

$$I=\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy=\underbrace{\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy}_{A}\underbrace{\int_0^1\frac{xarctan^3(x)}{x^2+1}dx}_{B}$$

$*$ ***Working on A***

$$A=\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy=\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{\frac{n}{2}+1}\ln(y)\,dy=-4\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+4)^2}=-4\left[\frac{31}{36}-\frac{\pi^2}{12}\right]=$$

$$-\frac{31}{9}+\frac{\pi^2}{3}=-\frac{31}{9}+2\zeta(2)$$

$*$ ***Working on B***

$$B=\int_0^1\frac{xarctan^3(x)}{x^2+1}dx\underbrace{\overset{x=\tan(y)}{=}}_{dx=sec^2(y)dy}\int_0^{\frac{\pi}{4}}\frac{\tan(y)arctan^3(\tan(y))}{1+tan^2(y)}.sec^2(y)dy=$$

$$\int_0^{\frac{\pi}{4}} \frac{y^3\tan(y)}{sec^2(y)}.sec^2(y)dy = \int_0^{\frac{\pi}{4}} y^3\tan(y) \overset{I.B.P}{\triangleq} -y^3\ln(cos(y))|_0^{\frac{\pi}{4}}$$

$$+3\int_0^{\frac{\pi}{4}} y^2\ln(cos(y))\,dy =$$

$$\frac{\pi^3}{128}\ln(2)+3\underbrace{\int_0^{\frac{\pi}{4}} y^2\ln(cos(y))\,dy}_{B_1}$$

$$B_1 = \int_0^{\frac{\pi}{4}} y^2\ln(cos(y))\,dy \quad \left\{\ln(cos(y)) = -\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2ny)}{n}\right\}$$

$$B_1 = -\ln(2)\int_0^{\frac{\pi}{4}} y^2\,dy - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} y^2\cos(2ny)\,dy = -\frac{1}{3}\left(\frac{\pi}{4}\right)^3\ln(2) -$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi^2}{32n}\sin\left(\frac{\pi n}{2}\right)-\frac{\sin\left(\frac{\pi n}{2}\right)}{4n^3}+\frac{\pi}{8n^2}\cos\left(\frac{\pi n}{2}\right)\right] = -\frac{\pi^3}{192}\ln(2)+\frac{\pi^2}{32}G+$$

$$\frac{1}{8}iLi_4(-i)-\frac{1}{8}iLi_4(i)+\frac{3\pi}{256}\zeta(3)$$

$$=\frac{\pi^2}{32}G-\frac{\pi}{8}Li_3(-i)-\frac{1}{4}iLi_4(-i)-\frac{187}{46080}i\pi^4-\frac{\pi^3}{192}\ln(2)$$

$$*\ But\ \ Li_4(x)+Li_4(-x)=\frac{1}{8}\ Li_4(x^2)$$

$$B = \frac{\pi^3}{128}\ln(2)+3B_1$$

$$=\frac{\pi^3}{128}\ln(2)+\frac{3\pi^2}{32}G-\frac{3}{8}\pi Li_3(-i)-\frac{3}{4}iLi_4(-i)-\frac{187}{15360}i\pi^4-\frac{\pi^3}{64}\ln(2)$$

$$B = -\frac{\pi^3}{128}\ln(2)+\frac{3\pi^2}{32}G-\frac{3}{8}\pi Li_3(-i)-\frac{3}{4}iLi_4(-i)-\frac{187}{15360}i\pi^4$$

$$I = A.B$$

$$\iint_0^1 \frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

$$=\left[2\zeta(2)-\frac{31}{9}\right]\left[-\frac{\pi^3}{128}\ln(2)+\frac{3\pi^2}{32}G-\frac{3}{8}\pi Li_3(-i)-\frac{3}{4}iLi_4(-i)-\frac{187}{15360}i\pi^4\right]$$

Solution 2 by Ankush Kumar Parcha-India

We have ,

$$\Omega = \iint_0^1 \frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy = \underbrace{\int_0^1 \frac{yln(y)}{\sqrt{y}+1}dy}_{M}\underbrace{\int_0^1 \frac{xarctan^3(x)}{x^2+1}dx}_{N} \quad (1)$$

$$M=\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy\overset{y\to x^2}{\Rightarrow}\ 4\int_0^1\frac{x^3\ln(x)}{1+x}dx=4\sum_{n\in N}(-1)^{n-1}\underbrace{\int_0^1 x^{n+2}\ln(x)\,dx}_{I.B.P}=$$

$$-4\sum_{n\in N}(-1)^{n-1}[\underbrace{(\frac{\ln(x)}{n+3}\int d(x^{n+3}))\Big|_0^1}_{0}$$

$$-\frac{1}{n+3}\int_0^1 x^{n+2}dx]=-4\sum_{n\in N}\frac{(-1)^{n-1}}{(n+3)^2}\int d(x^{n+3})\overset{n\to n-3}{\cong}$$

$$-4\left(1-\frac{1}{4}+\frac{1}{9}\right)+4\underbrace{\sum_{n\in N}\frac{(-1)^{n-1}}{n^2}}_{=\eta(2)}=\left(\frac{\pi^2}{3}-\frac{31}{9}\right)$$

$$N=\int_0^1\frac{xarctan^3(x)}{x^2+1}dx\overset{x\to\tan(x)}{\cong}\int_0^{\frac{\pi}{4}}x^3\tan(x)\overset{I.B.P}{\cong}-(x^3\int dlncos(x))\Big|_0^{\frac{\pi}{4}}$$

$$+3\int_0^{\frac{\pi}{4}}x^2lncos(x)dx$$

$$(\therefore\sum_{k\in N}\frac{(-1)^k\cos(2kx)}{k}=\ln\left(\frac{sec(x)}{2}\right)\quad |x|<\frac{\pi}{2})$$

$$N=\int_0^1\frac{xarctan^3(x)}{x^2+1}dx$$

$$=\frac{\pi^3}{128}\ln(2)+3\ln(2)\int_0^{\frac{\pi}{4}}x^2dx-3\sum_{n\in N}\frac{(-1)^n}{n}\underbrace{\int_0^{\frac{\pi}{4}}x^2\cos(2nx)\,dx}_{I.B.P}=$$

$$\frac{\pi^3}{128}-ln(2)\int_0^{\frac{\pi}{4}}d(x^3)-3\sum_{n\in N}\frac{(-1)^n}{n}\left[\frac{x^2}{2n}\int dsin(2nx))\Big|_0^{\frac{\pi}{4}}-\frac{1}{n}\underbrace{\int_0^{\frac{\pi}{4}}xsin(2nx)}_{I.B.P}\right]$$

$$=\frac{\pi^3}{128}ln(2)-$$

$$\frac{\pi^3}{64}\ln(2)-\frac{3\pi^2}{32}\sum_{n\in N}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)+3\sum_{n\in N}\frac{(-1)^n}{n^2}\left[-\frac{x}{2n}\int dcos(2nx))\Big|_0^{\frac{\pi}{4}}\right.$$

$$\left.+\frac{1}{2n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx\right]=$$

$$-\frac{\pi^3}{128}ln(2)-\frac{3\pi^2}{32}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)}_{n\to 2n+1}-\frac{3\pi}{8}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^3}\cos\left(\frac{\pi n}{2}\right)}_{n\to 2n}$$

$$+\frac{3}{4}\underbrace{\sum_{n\in N}\frac{(-1)^n}{n^4}\int dsin(2nx)}_{n\to 2n+1}$$

$$N=\frac{3\pi^2}{32}G-\frac{\pi^3}{128}\ln(2)-\frac{3}{4}\beta(4)+\frac{9}{256}\pi\zeta(3)$$

$$\Omega=M.N$$

$$\Omega=\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

$$=\left[2\zeta(2)-\frac{31}{9}\right]\left[\frac{3\pi^2}{32}G-\frac{\pi^3}{128}\ln(2)-\frac{3}{4}\beta(4)+\frac{9}{256}\pi\zeta(3)\right]$$

Solution 3 by Exodo Halcalias-Angola

$$\iint_0^1\frac{xyln(y)arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

$$I=\left(\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy\right)\left(\int_0^1\frac{xarctan^3(x)}{x^2+1}dx\right)$$

$$E=\int_0^1\frac{yln(y)}{\sqrt{y}+1}dy=4\int_0^1\frac{y^3\ln(y)}{y+1}dy=4\int_0^1 y^2\ln(y)\,dy-4\int_0^1 yln(y)dy+$$

$$4\int_0^1\ln(y)\,dy-4\int_0^1\frac{\ln(y)}{y+1}dy=-4\left(\frac{1}{3^2}-\frac{1}{2^2}+1\right)-4\left(\ln(y)\int dln(y+1)\right)\Big|_0^1+$$

$$4\int_0^1\frac{\ln(y+1)}{y}dy=-\frac{31}{9}-4\int_0^1 dLi_2(-x)=-\frac{31}{9}+4.\frac{\pi^2}{12}=2\zeta(2)-\frac{31}{9}$$

$$H=\int_0^1\frac{xarctan^3(x)}{x^2+1}dx$$

$$=\int_0^{\frac{\pi}{4}}x^3\tan(x)\,dx=\left(x^3\int dln(\sec(x))\right)\Big|_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}x^2\ln(cos(x))\,dx=$$

$$\left(\frac{\pi}{4}\right)^3\ln\left(sec\left(\frac{\pi}{4}\right)\right)+3\int_0^{\frac{\pi}{4}}x^2\left(\ln\left(\frac{1}{2}\right)+\sum_{k\in N}\frac{(-1)^{k-1}\cos(2kx)}{k}dx\right)$$

$$=\frac{\pi^3}{128}\ln(2)-\left(\frac{\pi}{4}\right)^3\ln(2)+$$

$$3\sum_{k\in N}\frac{(-1)^{k-1}}{k}\int_0^{\frac{\pi}{4}}x^2\cos(2kx)\,dx$$

$$=\frac{\pi^3}{128}\ln\left(\frac{1}{2}\right)+3\sum_{k\in N}\frac{(-1)^{k-1}}{k}\left(\frac{\pi^2}{32k}-\frac{\sin\left(\frac{\pi k}{2}\right)}{4k^3}+\frac{\pi\cos\left(\frac{\pi k}{2}\right)}{4k^2}\right)=$$

$$\frac{\pi^3}{128}\ln\left(\frac{1}{2}\right)+\frac{3\pi^2}{32}\sum_{k\in N}\frac{(-1)^{k-1}}{k^2}-\frac{3}{4}\sum_{k\in N}\frac{(-1)^{k-1}\sin\left(\frac{\pi k}{2}\right)}{k^4}+\frac{3\pi}{8}\sum_{k\in N}\frac{(-1)^{k-1}\cos}{k^3}$$

$$=\frac{\pi^3}{128}\ln\left(\frac{1}{2}\right)+$$

$$\frac{3\pi^2}{32}.\frac{1}{2}\sum_{k\in N}\frac{1}{k^2}-\frac{3}{4}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^4}+\frac{3\pi}{8}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k)^3}=$$

$$\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{3\pi^2}{64}.\frac{\pi^2}{6}-\frac{3}{4}\beta(4)+\frac{9}{256}\pi\zeta(2)$$

$$\left(\therefore\beta(u)=\frac{(-1)^u 2^{1-2u}}{(u-1)!}\psi^{(u-1)}\left(\frac{1}{4}\right)+\left(\frac{1}{2^u}-1\right)\zeta(u),\qquad u\in Z^+\right)$$

$$\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{\zeta(4)}{8}-\frac{3}{4}\left(\frac{1}{768}\psi^{(3)}\left(\frac{1}{4}\right)-\frac{15\zeta(4)}{16}\right)+\frac{9}{128}\pi\zeta(3)$$

$$H=\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{53\zeta(4)}{8}-\frac{1}{1024}\psi^{(3)}\left(\frac{1}{4}\right)+\frac{9}{128}\pi\zeta(3)$$

$$I=\iint_0^1\frac{xy\ln(y)\arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy=E.H$$

$$I=\left(2\zeta(2)-\frac{31}{9}\right)\left(\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{53\zeta(4)}{8}-\frac{1}{1024}\psi^{(3)}\left(\frac{1}{4}\right)+\frac{9}{128}\pi\zeta(3)\right)$$

Solution 4 by Cosghun Memmedov-Azerbaijan

$$\Omega=\iint_0^1\frac{xy\ln(y)\arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy$$

$$\iint_0^1\frac{xy\ln(y)\arctan^3(x)}{(\sqrt{y}+1)(x^2+1)}dxdy=\left(\int_0^1\frac{y\ln(y)}{\sqrt{y}+1}dy\right).\left(\int_0^1\frac{x\arctan^3(x)}{x^2+1}dx\right)=\Psi.\Phi$$

$$\Psi=\int_0^1\frac{y\ln(y)}{\sqrt{y}+1}dy\overset{\sqrt{y}\to y}{\triangleq}4\int_0^1\frac{y^3\ln(y)}{y+1}dy=4\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{n+3}\ln(y)\,dy=$$

$$4\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+4)^2}=-\frac{31}{9}+4.\frac{\pi^2}{12}=2\zeta(2)-\frac{31}{9}$$

$$\Phi=\int_0^1\frac{x\arctan^3(x)}{x^2+1}dx\overset{\arctan(x)\to x}{\triangleq}\int_0^{\frac{\pi}{4}}x^3\tan(x)\,dx\overset{I.B.P}{\triangleq}-\frac{\pi^3}{64}\ln\left(\frac{1}{\sqrt{2}}\right)+3\underbrace{\int_0^{\frac{\pi}{4}}x^2\ln(\cos(x))\,dx}_{\xi}$$

$$=\frac{\pi^3}{128}\ln(2)+3\xi$$

$$\xi=\int_0^{\frac{\pi}{4}}x^2\ln(\cos(x))\,dx=-\ln(2)\int_0^{\frac{\pi}{4}}x^2dx-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}x^2\cos(2nx)\,dx=-\frac{\pi^3}{192}\ln(2)-$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}\int_0^{\frac{\pi}{4}}x^2d(\sin(2nx))\overset{I.B.P}{\cong}-\frac{\pi^3}{192}\ln(2)-\frac{\pi^2}{32}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2}+$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^{\frac{\pi}{4}}x\sin(2nx)dx=-\frac{\pi^3}{192}\ln(2)+\frac{\pi^2}{32}G-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}\int_0^{\frac{\pi}{4}}xd(\cos(2nx))=$$

$$-\frac{\pi^3}{192}\ln(2)+\frac{\pi^2}{32}G-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx=$$

$$-\frac{\pi^3}{192}\ln(2)+\frac{\pi^2}{32}G+$$

$$\frac{1}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}+\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{4n^4}=-\frac{\pi^3}{192}\ln(2)+\frac{\pi^2}{32}G+\frac{3}{256}\zeta(3)-\frac{1}{4}\beta(4)$$

$$\xi=-\frac{\pi^3}{192}\ln(2)+\frac{\pi^2}{32}G+\frac{3}{256}\zeta(3)-\frac{1}{4}\left(\frac{\psi^{(3)}\left(\frac{1}{4}\right)}{768}-\frac{\pi^4}{96}\right)$$

$$\Phi=\frac{\pi^3}{128}\ln(2)+3\xi=\frac{\pi^3}{128}\ln(2)+\frac{9}{256}\pi\zeta(3)+\frac{\pi^4}{128}+\frac{3\pi^2}{32}G-\frac{\psi^{(3)}\left(\frac{1}{4}\right)}{1024}$$

$$\Omega=\Psi.\Phi=\left(2\zeta(2)-\frac{31}{9}\right)\left(\frac{3}{64}\pi\zeta(2)\ln\left(\frac{1}{2}\right)+\frac{53\zeta(4)}{8}-\frac{1}{1024}\psi^{(3)}\left(\frac{1}{4}\right)+\frac{9}{128}\pi\zeta(3)\right)$$

2527. Find:

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\left(1+\sqrt{x}\right)^2\sin^2(x)}{x}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\left(1+\sqrt{x}\right)^2\sin^2(x)}{x}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{(1+2\sqrt{x}+x)(1-\cos(2x))}{x}dx=$$

$$\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\frac{1-\cos(2x)}{x}dx+2\int_0^{\frac{\pi}{2}}\frac{1-\cos(2x)}{\sqrt{x}}dx+\int_0^{\frac{\pi}{2}}(1-\cos(2x))dx\right)=$$

$$\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\frac{1-\cos(2x)}{2x}d(2x)+2\int_0^{\frac{\pi}{2}}\frac{1-\cos(2x)}{\sqrt{x}}dx+\int_0^{\frac{\pi}{2}}(1-\cos(2x))dx\right)=$$

$$-\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\frac{\cos(x)-1}{x}dx+4\sqrt{x}\Big|_0^{\frac{\pi}{2}}-2\sqrt{2}\underbrace{\int_0^{\frac{\pi}{2}}\cos(2x)\,d(\sqrt{2x})}_{\sqrt{2x}\to x}+\left(x-\frac{\sin(2x)}{2}\right)\Big|_0^{\frac{\pi}{2}}\right)=$$

$$\frac{1}{2}\left(\gamma+\ln(\pi)-Ci(\pi)+2\sqrt{2\pi}-2\sqrt{2}\int_0^{\sqrt{\pi}}\cos(x^2)\,dx+\frac{\pi}{2}\right)=$$

$$=\frac{1}{2}\left(\gamma+\ln(\pi)-Ci(\pi)+2\sqrt{2\pi}-2\sqrt{2}C(\sqrt{\pi})+\left(\frac{\pi}{2}\right)\right)$$

Solution 2 by Shobhit Jain-India

$$I=\int_0^{\frac{\pi}{2}}\frac{(1+\sqrt{x})^2 sin^2(x)}{x}dx$$

$$I=\int_0^{\frac{\pi}{2}}\frac{(1+\sqrt{x})^2 sin^2(x)}{x}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+x+2\sqrt{x})\left(\frac{1-\cos(2x)}{2x}\right)dx\overset{x\to\frac{x}{2}}{\cong}$$

$$\int_0^{\pi}\left(1+\frac{x}{2}+\sqrt{2x}\right)\left(\frac{1-\cos(x)}{x}\right)dx=\frac{1}{2}\int_0^{\pi}\left(\frac{1-\cos(x)}{x}\right)dx+$$

$$\frac{1}{4}\int_0^{\pi}(-\cos(x))dx+\frac{1}{\sqrt{2}}\int_0^{\pi}\left(\frac{1-\cos(x)}{\sqrt{x}}\right)dx$$

$$I=\frac{1}{2}\big(\gamma+\ln(\pi)-Ci(\pi)\big)+\frac{\pi}{4}+\frac{1}{\sqrt{2}}\int_0^{\pi}\left(\frac{1-\cos(x)}{\sqrt{x}}\right)dx$$

$$I=\frac{1}{2}\big(\gamma+\ln(\pi)-Ci(\pi)\big)+\frac{\pi}{4}+\sqrt{2\pi}-\frac{1}{\sqrt{2}}\int_0^{\pi}\left(\frac{\cos(x)}{\sqrt{x}}\right)dx$$

$$I=\frac{1}{2}\big(\gamma+\ln(\pi)-Ci(\pi)\big)+\frac{\pi}{4}+\sqrt{2\pi}-\frac{\sqrt{\pi}}{2}\int_0^{\pi}J_{-\frac{1}{2}}(x)dx$$

$$I=\frac{1}{2}\left(\gamma+\ln(\pi)-Ci(\pi)+\frac{\pi}{2}\right)+\sqrt{2\pi}-\frac{\sqrt{\pi}}{2}C$$

Note section:

$$\gamma\to\ \textit{Euler Mascheroni's constant}$$

$$C=\int_0^{\pi}J_{-\frac{1}{2}}(x)dx\,,\ J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos(x)\to\textit{Numerical constant}$$

$$Ci(x)=-\int_x^{\infty}\frac{\cos(t)}{t}dt=\gamma+\ln(x)\,n-\int_0^{x}\frac{1-\cos(t)}{t}dt$$

2528. Find:

$$\Omega=\int_0^1\frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Shohbit Jain-India

$$\int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx =$$

$$=\frac{1}{5}\int_0^1 \ln\left(\frac{1}{x}\right)\left(\frac{x+2}{1+x^2}-\frac{2}{1+2x}\right)dx = \frac{1}{5}\int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{x^2+1}dx+\frac{2}{5}\int_0^1 \frac{\ln\left(\frac{1}{x}\right)}{x^2+1}dx -$$

$$-\frac{1}{5}\int_0^2 \frac{\ln\left(\frac{2}{u}\right)}{1+u}du = \frac{1}{5}\int_0^1 \ln\left(\frac{1}{x}\right)\{x-x^3+x^5-x^7+\cdots\}dx +$$

$$+\frac{2}{5}\int_0^1 \ln\left(\frac{1}{x}\right)\{1-x^2+x^4-x^6+\cdots\}dx - \frac{1}{5}\int_0^2 \frac{\ln(2)}{1+u}du+\frac{1}{5}\int_0^2 \frac{\ln(u)}{1+u}du$$

$$=\frac{1}{5}\Gamma 2\left\{\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}-\frac{1}{8^2}+\cdots\right\}+\frac{2}{5}\Gamma 2\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right\}-$$

$$\frac{\ln(2)\ln(3)}{5}+\frac{1}{5}\int_0^1 \frac{\ln(u)}{1+u}du+\frac{1}{5}\int_0^2 \frac{\ln(u)}{1+u}du$$

$$=\frac{1}{5}\cdot\frac{\eta(2)}{4}+\frac{2G}{5}-\frac{\ln(2)\ln(3)}{5}+\frac{1}{5}\cdot\left(-\frac{\pi^2}{12}\right)-$$

$$\frac{1}{5}\int_{\frac{1}{2}}^1 \frac{\ln(t)}{t(1+t)}dt = \frac{\pi^2}{240}+\frac{2G}{5}-\frac{\ln(2)\ln(3)}{5}-\frac{\pi^2}{60}-\frac{1}{5}\int_{\frac{1}{2}}^1 \frac{\ln(t)}{t}dt+\frac{1}{5}\int_{\frac{1}{2}}^1 \frac{\ln(t)}{1+t}dt =$$

$$=\frac{2G}{5}-\frac{\ln(2)\ln(3)}{5}-\frac{\pi^2}{80}+\frac{\ln^2(2)}{10}+\frac{P}{5}$$

$$Now\quad P=\int_{\frac{1}{2}}^1 \frac{\ln(t)}{1+t}dt = \sum_{n=1}^{\infty}(-1)^{n+1}\int_{\frac{1}{2}}^1 t^{n-1}\ln(t)\,dt$$

$$Now\quad \int_{\frac{1}{2}}^1 t^{n-1}\ln(t)\,dt = \frac{t^n}{n}\ln(t)\Big|_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^1 \frac{t^{n-1}}{n}dt = \frac{\ln(2)}{n2^n}+\frac{1}{n^2 2^n}-\frac{1}{n^2}$$

$$Therefore\ :\ P=\sum_{n=1}^{\infty}(-1)^{n+1}\int_{\frac{1}{2}}^1 t^{n-1}\ln(t)\,dt$$

$$=\sum_{n=1}^{\infty}\left\{\frac{\ln(2)(-1)^{n+1}}{n2^n}-\frac{1}{n^2}\left(-\frac{1}{2}\right)^n+\frac{(-1)^n}{n^2}\right\}$$

$$P=\ln(2)\ln\left(1+\frac{1}{2}\right)-Li_2\left(-\frac{1}{2}\right)-\eta(2)=\ln(2)\ln(3)-\ln^2(2)-Li_2\left(-\frac{1}{2}\right)-\frac{\pi^2}{12}$$

$$\int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx = \frac{2G}{5}-\frac{\ln(2)\ln(3)}{5}-\frac{\pi^2}{80}+\frac{\ln^2(2)}{10}+\frac{\ln(2)\ln(3)}{5}-\frac{\ln^2(2)}{5}-$$

$$\frac{Li_2\left(-\frac{1}{2}\right)}{5}-\frac{\pi^2}{60}=\frac{2G}{5}-\frac{\ln^2(2)}{10}-\frac{7}{40}\zeta(2)-\frac{1}{5}Li_2\left(-\frac{1}{2}\right)$$

$$\int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx = \frac{2G}{5} - \frac{\ln^2(2)}{10} - \frac{7}{40}\zeta(2) - \frac{1}{5}Li_2\left(-\frac{1}{2}\right)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega = \int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx$$

$$\therefore \int_0^1 \frac{\ln^n(x)}{a+x}dx = -\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k \int_0^1 x^{k-1}\ln^n(x)dx = -\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k x^k n! \sum_{i=0}^{n} \frac{(-1)^i \ln^{n-i}(x)}{(n-i)!\,k^{i+1}}\Bigg|_0^1 =$$

$$= (-1)^{n+1} n!\, Li_{n+1}\left(-\frac{1}{a}\right)$$

$$\therefore \int_0^1 \frac{\ln^n(x)}{a+x^2}dx = -\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k \int_0^1 x^{2k-1-1}\ln^n(x)dx = -\sum_{k=1}^{\infty}\left(-\frac{1}{a}\right)^k x^{2k-1} n! \sum_{i=0}^{n} \frac{(-1)^i \ln^{n-i}(x)}{(n-i)!\,(2k-1)^{i+1}}\Bigg|_0^1 =$$

$$= (-1)^{n+1} n! \sum_{k=1}^{\infty} \frac{\left(-\frac{1}{a}\right)^k}{(2k-1)^{n+1}}$$

$$1)\psi_n = \int_0^1 \frac{x\ln^n\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx = \frac{(-1)^n}{5}\underbrace{\int_0^1 \frac{x\ln^n(x)}{1+x^2}dx}_{x^2\to x} + \frac{2(-1)^n}{5}\int_0^1 \frac{\ln^n(x)}{1+x^2}dx$$

$$-\frac{2(-1)^n}{5}\int_0^1 \frac{\ln^n(x)}{1+2x}dx =$$

$$\frac{(-1)^n}{5.2^{n+1}}\int_0^1 \frac{\ln^n(x)}{1+x}dx + \frac{2(-1)^n}{5}\int_0^1 \frac{\ln^n(x)}{1+x^2}dx - \frac{(-1)^n}{5}\int_0^1 \frac{\ln^n(x)}{x+\frac{1}{2}}dx = \frac{(-1)^n}{5.2^{n+1}}(-1)^{n+1}n!\,Li_{n+1}(-1) +$$

$$\frac{2(-1)^n}{5}.n!.(-1)^{n+1}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^{n+1}} - \frac{(-1)^n}{5}.n!.(-1)^{n+1}.Li_{n+1}(-2) =$$

$$\frac{n!}{5}\left\{\frac{1}{2^{n+1}}(1-2^{-n})\zeta(n+1) + 2\beta(n+1) + Li_{n+1}(-2)\right\}$$

$$iff \quad n=1: \ \psi = \int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx = \frac{1}{5}\left\{\frac{1}{2^2}(1-2^{-1})\zeta(2) + 2\beta(2) + Li_2(-2)\right\} =$$

$$\frac{1}{5}\left\{\frac{1}{8}\zeta(2) + 2G - Li_2\left(-\frac{1}{2}\right) - \frac{\ln^2(2)}{2} + 2Li_2(-1)\right\}$$

$$= \frac{1}{5}\left\{\frac{1}{8}\zeta(2) + 2G - Li_2\left(-\frac{1}{2}\right) - \frac{\ln^2(2)}{2} - \zeta(2)\right\}$$

$$\int_0^1 \frac{x\ln\left(\frac{1}{x}\right)}{(x^2+1)(2x+1)}dx = \frac{2G}{5} - \frac{\ln^2(2)}{10} - \frac{7}{40}\zeta(2) - \frac{1}{5}Li_2\left(-\frac{1}{2}\right)$$

2529. Find:

$$\Omega=\int_1^{\infty}\frac{x\ln^2(x)}{(1+x)^2(3+x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Shobhit Jain-India

$$\Omega=\int_1^{\infty}\frac{x\ln^2(x)}{(1+x)^2(3+x)}dx=\int_0^1\frac{\ln^2(x)}{(1+x)^2(3x+1)}dx\ \ put\ x=\frac{1}{x}$$

$$\int_0^1\ln^2(x)\left(\frac{\frac{9}{4}}{1+3x}-\frac{\frac{3}{4}}{1+x}-\frac{\frac{1}{2}}{(1+x)^2}\right)dx$$

$$=\frac{9}{4}\int_0^1\frac{\ln^2(x)}{1+3x}dx-\frac{3}{4}\int_0^1\frac{\ln^2(x)}{1+x}dx-\frac{1}{2}\int_0^1\frac{\ln^2(x)}{(1+x)^2}=$$

$$\frac{9}{4}A-\frac{3}{4}B-\frac{1}{2}C$$

$$B=\int_0^1\ln^2(x)(1-x+x^2-x^3+\cdots)dx=\Gamma 3\left(\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\cdots\right)=2\eta(3)$$

$$=\frac{3}{2}\zeta(3)$$

$$C=\int_0^1\ln^2(x)(1-2x+3x^2-4x^3+\cdots)dx=\Gamma 3\left(\frac{1}{1^3}-\frac{2}{2^3}+\frac{3}{3^3}-\frac{4}{4^3}+\cdots\right)=2\eta(2)$$

$$=\zeta(2)$$

$$A=\int_0^1\frac{\ln^2(x)}{1+3x}dx\overset{x=\frac{x}{3}}{=}\frac{1}{3}\int_0^3\frac{(\ln(x)-\ln(3))^2}{1+x}dx=\frac{1}{3}\int_0^3\frac{\ln^2(x)}{1+x}dx$$

$$-\frac{2}{3}\ln(3)\int_0^3\frac{\ln(x)}{1+x}dx+$$

$$\frac{\ln^2(3)}{3}\int_0^3\frac{dx}{1+x}=\frac{1}{3}\int_0^3\frac{\ln^2(x)}{1+x}dx-\frac{2}{3}\ln(3)\int_0^3\frac{\ln(x)}{1+x}dx+\frac{2}{3}\ln^2(3)\ln(2)$$

$$Therefore:\ \Omega$$

$$=\frac{3}{4}\int_0^3\frac{\ln^2(x)}{1+x}dx$$

$$-\frac{3}{2}\ln(3)\int_0^3\frac{\ln(x)}{1+x}dx+\frac{3}{2}\ln^2(3)\ln(2)-\frac{9}{8}\zeta(3)-\frac{1}{2}\zeta(2)$$

$$Now:$$

$$\int_0^3\frac{\ln(x)}{1+x}dx=\ln(x)\ln(1+x)\Big|_0^3-\int_0^3\frac{\ln(1+x)}{x}dx=2\ln(2)\ln(3)+Li_2(-3)$$

$$\int_0^3 \frac{ln^2(x)}{1+x}dx = ln^2(x)\ln(1+x)\Big|_0^3 - 2\int_0^3 \frac{\ln(x)\ln(1+x)}{x}dx = 2\ln(2)ln^2(3) +$$
$$2\int_0^3 \ln(x)\, d\big(Li_2(-x)\big) = 2\ln(2)ln^2(3) + 2ln(x)Li_2(-x)\Big|_0^3 - \int_0^3 \frac{Li_2(-x)}{x}dx =$$
$$2\ln(2)ln^2(3) + 2\ln(3)\, Li_2(-3) - 2Li_3(-3)$$
$$Hence\ :\ \Omega = \frac{3}{4}\{2\ln(2)ln^2(3) + 2\ln(3)\, Li_2(-3) - 2Li_3(-3)\} -$$
$$\frac{2}{3}\ln(3)\{2\ln(2)\ln(3) + Li_2(-3)\} + \frac{3}{2}ln^2(3)\ln(2) - \frac{9}{8}\zeta(3) - \frac{1}{2}\zeta(2)$$
$$\int_1^\infty \frac{xln^2(x)}{(1+x)^2(3+x)}dx = -\frac{9}{8}\zeta(3) - \frac{1}{2}\zeta(2) - \frac{3}{2}Li_3(-3)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\psi = \int_1^\infty \frac{xln^2(x)}{(1+x)^2(3+x)}dx$$
$$\psi_n = \int_1^\infty \frac{xln^n(x)}{(1+x)^2(3+x)}dx \overset{x=\frac{1}{x}}{=} \int_0^1 \frac{\frac{1}{x}ln^n(\frac{1}{x})}{\left(\frac{1}{x}+1\right)^2\left(\frac{1}{x}+3\right)}\frac{dx}{x^2}$$
$$= (-1)^n\int_0^1 \frac{ln^n(x)}{(1+x)^2(3x+1)}dx =$$
$$\frac{(-1)^n}{4}\left\{3.3\underbrace{\int_0^1 \frac{ln^n(x)}{3x+1}dx}_{A} - 3\underbrace{\int_0^1 \frac{ln^n(x)}{1+x}dx}_{B} - 2\underbrace{\int_0^1 \frac{ln^n(x)}{(1+x)^2}dx}_{C}\right\}$$
$$= \frac{(-1)^n}{4}(3A - 3B - 2C)$$
$$A = 3\int_0^1 \frac{ln^n(x)}{3x+1}dx = \int_0^1 \frac{ln^n(x)}{x+\frac{1}{3}}dx = (-1)^{n+1}n!\, Li_{n+1}(-3)$$
$$B = \int_0^1 \frac{ln^n(x)}{1+x}dx = (-1)^{n+1}n!\, Li_{n+1}(-1) = (-1)^n n!\,(1-2^{-n})\zeta(n+1)$$
$$C = \int_0^1 \frac{ln^n(x)}{(1+x)^2}dx =$$
$$= \int_0^1 ln^n(x)d\left(\frac{x}{x+1}\right) = \frac{xln^n(x)}{x+1}\Big|_0^1 - n\int_0^1 \frac{ln^{n-1}(x)}{x+1}dx = (-1)^{n+1}n!\, Li_n(-1)$$
$$\psi_n = \frac{(-1)^n}{4}(3A - 3B - 2C) =$$
$$(-1)^n\frac{3(-1)^{n+1}n!\, Li_{n+1}(-3) - 3(-1)^n n!\,(1-2^{-n})\zeta(n+1) - 2(-1)^{n+1}n!\, Li_n(-1)}{4} =$$
$$-\frac{n!}{4}\big(3Li_{n+1}(-3) + 3(1-2^{-n})\zeta(n+1) + 2\eta(n)\big)$$

$$If\ :\ n=2$$

$$\psi=\int_1^{\infty}\frac{x\ln^2(x)}{(1+x)^2(3+x)}dx=-\frac{3}{4}.4\left(3Li_3(-3)+3(1-2^{-2})\zeta(2+1)+2\eta(2)\right)$$

$$\int_1^{\infty}\frac{x\ln^2(x)}{(1+x)^2(3+x)}dx=-\frac{9}{8}\zeta(3)-\frac{1}{2}\zeta(2)-\frac{3}{2}Li_3(-3)$$

Solution 3 by Obiajunwa Januarius-Nigeria

$$\int_1^{\infty}\frac{x\ln^2(x)}{(1+x)^2(3+x)}dx$$

$$\int_1^{\infty}\frac{x\ln^2(x)}{(1+x)^2(3+x)}dx\overset{x\to\frac{1}{x}}{=}\int_0^1\frac{\ln^2(x)}{(1+x)^2(3x+1)}dx=\frac{9}{4}\int_0^1\frac{\ln^2(x)}{1+3x}dx-$$

$$\frac{3}{4}\int_0^1\frac{\ln^2(x)}{1+x}dx-\frac{1}{2}\int_0^1\frac{\ln^2(x)}{(1+x)^2}dx=\frac{9}{4}\sum_{n=0}^{\infty}(-1)^n3^n\int_0^1x^n\ln^2(x)dx-$$

$$\frac{3}{4}\sum_{n=0}^{\infty}(-1)^n\int_0^1x^n\ln^2(x)dx-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}n\int_0^1x^{n-1}\ln^2(x)dx=$$

$$\frac{9}{4}\sum_{n=0}^{\infty}(-1)^n3^n\frac{\partial^2}{\partial n^2}\left[\int_0^1x^ndx\right]-\frac{3}{4}\sum_{n=0}^{\infty}(-1)^n\frac{\partial^2}{\partial n^2}\left[\int_0^1x^ndx\right]-$$

$$\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}n\frac{\partial^2}{\partial n^2}\left[\int_0^1x^{n-1}dx\right]=\frac{9}{4}\sum_{n=0}^{\infty}(-1)^n3^n\frac{\partial^2}{\partial n^2}\left[\frac{1}{n+1}\right]-$$

$$\frac{3}{4}\sum_{n=0}^{\infty}(-1)^n\frac{\partial^2}{\partial n^2}\left[\frac{1}{n+1}\right]-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}n\frac{\partial^2}{\partial n^2}\left[\frac{1}{n}\right]=\frac{9}{4}(-1)^2 2!\sum_{n=0}^{\infty}\frac{(-1)^n3^n}{(n+1)^3}-$$

$$\frac{3}{4}(-1)^2 2!\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}-\frac{1}{2}(-1)^2 2!\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\overset{n+1=m}{=}\frac{9}{2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}3^{m-1}}{m^3}-$$

$$\frac{3}{2}\eta(3)-\eta(2)=-\frac{3}{2}Li_3(-3)-\frac{3}{2}.\frac{3}{4}\zeta(3)-\frac{1}{2}\zeta(2)$$

$$\int_1^{\infty}\frac{x\ln^2(x)}{(1+x)^2(3+x)}dx=-\frac{9}{8}\zeta(3)-\frac{1}{2}\zeta(2)-\frac{3}{2}Li_3(-3)$$

2530. Find:

$$\int_0^1\left(\arctan^3(x)+x^2Li_3(-x^3)\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

General problem and full solution :

$$\therefore \ln(cos(y)) = -\ln(2) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\cos(2ny),\ \ |y| < \frac{\pi}{2}$$

$$\therefore \int y^m \cos(ay)\,dy = \sum_{j=0}^{\infty} j!\binom{m}{j}\frac{y^{m-j}}{a^{j+1}}\sin(ay + j\frac{\pi}{2})$$

For $m, n, k \in Z^+ : S = \int_0^1 \left(arctan^m(x) + x^{k-1}Li_n(ax^k)\right)dx = \int_0^1 arctan^m(x)dx +$

$$\int_0^1 x^{k-1}Li_n(ax^k)dx = I + J$$

$$1) I = \int_0^1 arctan^{m+2}(x)dx \overset{x=\tan(y)}{\cong} \int_0^{\frac{\pi}{4}} y^{m+2}d(\tan(y)) = y^{m+2}\tan(y)\Big|_0^{\frac{\pi}{4}} -$$

$$(m+2)\int_0^{\frac{\pi}{4}} y^{m+1}\tan(y)\,dy = \left(\frac{\pi}{4}\right)^{m+2} - (m+2)y^{m+1}\ln(cos(y))\Big|_0^{\frac{\pi}{4}} +$$

$$(m+2)(m+1)\underbrace{\int_0^{\frac{\pi}{4}} y^m \ln(cos(y))\,dy}_{A}$$

$$= \left(\frac{\pi}{4}\right)^{m+2} + (m+2)\left(\frac{\pi}{4}\right)^{m+1}\ln(\sqrt{2}) + (m+2)(m+1)A$$

$$a)\ A = \int_0^{\frac{\pi}{4}} y^m \ln(cos(y))\,dy = -\int_0^{\frac{\pi}{4}} y^m(\ln(2) + \sum_{a=1}^{\infty}\frac{(-1)^a}{a}\cos(2ay))dy = -\ln(2)\int_0^{\frac{\pi}{4}} y^m dy +$$

$$\sum_{a=1}^{\infty}\frac{(-1)^a}{a}\int_0^{\frac{\pi}{4}} y^m\cos(2ay)\,dy = -\left(\frac{\pi}{4}\right)^m\ln(2) + \sum_{a=1}^{\infty}\frac{(-1)^a}{a}\sum_{j=0}^{\infty} j!\binom{m}{j}\frac{y^{m-j}}{(2a)^{j+1}}\sin\left(ay + j\frac{\pi}{2}\right)\Bigg|_0^{\frac{\pi}{4}}$$

$$=$$

$$-\left(\frac{\pi}{4}\right)^m\ln(2) + \sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\sin(\frac{a\pi}{2} + j\frac{\pi}{2}) - \frac{m!}{2^{m+1}}\sin(\frac{m\pi}{2})\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{m+2}} =$$

$$-\left(\frac{\pi}{4}\right)^m\ln(2) + \sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\underbrace{\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\begin{pmatrix}\sin\left(\frac{a\pi}{2}\right)\cos\left(\frac{j\pi}{2}\right)\\ +cos\left(\frac{a\pi}{2}\right)sin\left(\frac{j\pi}{2}\right)\end{pmatrix}}_{D} +$$

$$\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)(1 - 2^{-1-m})\zeta(m+2)$$

$$= \frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)(1 - 2^{-1-m})\zeta(m+2) - \left(\frac{\pi}{4}\right)^m\ln(2) +$$

$$\sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j} D$$

$$D=\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\left(\sin\left(\frac{a\pi}{2}\right)\cos\left(\frac{j\pi}{2}\right)+\cos\left(\frac{a\pi}{2}\right)\sin\left(\frac{j\pi}{2}\right)\right)=$$

$$\cos\left(\frac{j\pi}{2}\right)\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\sin\left(\frac{a\pi}{2}\right)+\sin\left(\frac{j\pi}{2}\right)\sum_{a=1}^{\infty}\frac{(-1)^a}{a^{j+2}}\cos\left(\frac{a\pi}{2}\right)$$

$$=-\cos\left(\frac{j\pi}{2}\right)\sum_{a=0}^{\infty}\frac{(-1)^a}{(2a+1)^{j+2}}+$$

$$\sin\left(\frac{j\pi}{2}\right)\sum_{a=0}^{\infty}\frac{1}{(2a)^{j+2}}=-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\sin\left(\frac{j\pi}{2}\right)\zeta(j+2)$$

$$A=\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)(1-2^{-1-m})\zeta(m+2)-\left(\frac{\pi}{4}\right)^m\ln(2)+$$

$$\sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\left(\frac{1}{2^{j+2}}\sin\left(\frac{j\pi}{2}\right)\zeta(j+2)-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\right)$$

$$\text{Then }: I=\left(\frac{\pi}{4}\right)^{m+2}+(m+2)\left(\frac{\pi}{4}\right)^{m+1}\ln(\sqrt{2})+$$

$$(m+2)(m+1)\left\{\begin{matrix}\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)(1-2^{-1-m})\zeta(m+2)-\left(\frac{\pi}{4}\right)^m\ln(2)+\\ \sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\left(\frac{1}{2^{j+2}}\sin\left(\frac{j\pi}{2}\right)\zeta(j+2)-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\right)\end{matrix}\right\}$$

$$2)J=\int_0^1 x^{k-1}Li_n(ax^k)dx\overset{ax^k\to x}{\cong}\frac{1}{ka}\int_0^a Li_n(x)d)=\frac{1}{ka}K_n$$

$$K_n=\int_0^a Li_n(x)dx$$

$$=xLi_n(x)\Big|_0^a-\int_0^a x\frac{\partial Li_n(x)}{\partial x}dx=aLi_n(a)-\int_0^a Li_{n-1}(x)\,dx=aLi_n(a)-K_{n-1}$$

$$=(-1)^0aLi_n(a)+(-1)^1aLi_{n-1}(a)+(-1)^2aLi_{n-2}(a)+(-1)^3K_{n-3}=$$

$$a\sum_{k=0}^{n-1}(-1)^kLi_{n-k}(a)+(-1)^{n-1}K_{n-1}=a\sum_{k=0}^{n-1}(-1)^kLi_{n-k}(a)+(-1)^{n-1}\int_0^a Li_1(x)dx=$$

$$a\sum_{k=0}^{n-1}(-1)^kLi_{n-k}(a)+(-1)^{n-1}\int_0^a\ln(1-x)d(1-x)=a\sum_{k=0}^{n-1}(-1)^kLi_{n-k}(a)+$$

$$(-1)^{n-1}\{(1-x)\ln(1-x)+x\}\Big|_0^a+a\sum_{k=0}^{n-1}(-1)^kLi_{n-k}(a)$$

$$+(-1)^{n-1}\{(1-a)\ln(1-a)+a\}$$

$$\text{Then }:$$

$$J=\frac{\sum_{k=0}^{n-1}(-1)^k Li_{n-k}(a)}{k}+(-1)^{n-1}\frac{\{(1-a)\ln(1-a)+a\}}{ka}$$

$$Therefore\ :\ S=I+J$$

$$S=\left(\frac{\pi}{4}\right)^{m+2}+(m+2)\left(\frac{\pi}{4}\right)^{m+1}\ln(\sqrt{2})+$$

$$(m+2)(m+1)\left\{\begin{array}{c}\frac{m!}{2^{m+1}}\sin\left(\frac{m\pi}{2}\right)(1-2^{-1-m})\zeta(m+2)-\left(\frac{\pi}{4}\right)^m\ln(2)+\\ \sum_{j=0}^{0}\frac{j!}{2^{j+1}}\binom{m}{j}\left(\frac{\pi}{4}\right)^{m-j}\left(\frac{1}{2^{j+2}}sin\left(\frac{j\pi}{2}\right)\zeta(j+2)-\cos\left(\frac{j\pi}{2}\right)\beta(j+2)+\frac{1}{2^{j+2}}\right)\end{array}\right\}+$$

$$\frac{\sum_{k=0}^{n-1}(-1)^k Li_{n-k}(a)}{k}+(-1)^{n-1}\frac{\{(1-a)\ln(1-a)+a\}}{ka}$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$\Omega=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx=\int_0^1 arctan^3(x)dx+\int_0^1 x^2Li_3(-x^3)dx$$

$$=I_1+I_2$$

$$I_1=\int_0^1 arctan^3(x)dx\quad \left\{x=\tan(y)\ ,\qquad dx=sec^2(y)\ \left[0,\frac{\pi}{4}\right]\right\}$$

$$I_1=\int_0^{\frac{\pi}{4}} arctan^3(\tan(y)).sec^2(y)dy=\int_0^{\frac{\pi}{4}} y^3\,sec^2(y)$$

$$=y^3\tan(y)|_0^{\frac{\pi}{4}}-3\int_0^{\frac{\pi}{4}} y^2\tan(y)\,dy=$$

$$\frac{\pi^3}{64}-3\underbrace{\int_0^{\frac{\pi}{4}} y^2\tan(y)\,dy}_{A}$$

$$A=\int_0^{\frac{\pi}{4}} y^2\tan(y)\,dy=-y^2\ln(cos(y))|_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}} yln(cos(y))\,dy$$

$$=\frac{\pi^2}{32}\ln(2)-2\ln(2)\int_0^{\frac{\pi}{4}} ydy-$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} ycos(2ny)dy$$

$$=\frac{\pi^2}{32}\ln(2)-2\ln(2)\left[\frac{1}{2}\cdot\frac{\pi^2}{16}\right]-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathcal{R}\int_0^{\frac{\pi}{4}} ye^{-2iny}dy=$$

$$\frac{\pi^2}{32}\ln(2)-\frac{\pi^2}{16}\ln(2)-2\mathcal{R}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{-1+e^{-\frac{i\pi n}{2}}+\frac{i\pi n}{2}e^{-\frac{i\pi n}{2}}}{4n^2}\right]=$$

$$-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi\sin\left(\frac{\pi n}{2}\right)}{8n}+\frac{\cos\left(\frac{\pi n}{2}\right)}{4n^2}-\frac{1}{4n^2}\right]=-\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2}-$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-\frac{\pi^2}{32}\ln(2)$$

$$A=\frac{\pi}{4}G+\frac{3}{64}\zeta(3)-\frac{3}{8}\zeta(3)-\frac{\pi^2}{32}\ln(2)$$

$$I_1=\frac{\pi^3}{64}-3A=\frac{\pi^3}{64}-\frac{3\pi}{4}G+\frac{63}{64}\zeta(3)+\frac{3\pi^2}{32}\ln(2)$$

$$I_2=\int_0^1 x^2Li_3(-x^3)dx\ \left\{u=x^3,\frac{du}{dx}=3x^2,dx=\frac{du}{3x^2}\right\}$$

$$I_2=\frac{1}{3}\int_0^1 Li_3(-u)du\overset{u=x}{=}\frac{1}{3}\int_0^1 Li_3(-x)dx\ \left\{u=Li_3(-x),\right.$$

$$\left.\frac{du}{dx}=\frac{Li_2(-x)}{x},du=dx\ \ v=x\right\}$$

$$I_2=\frac{1}{3}\left[xLi_3(-x)]\,\frac{1}{0}-\int_0^1 Li_2(-x)dx\right]=\frac{1}{3}\left(-\frac{3}{4}\zeta(3)-\left(-\frac{\zeta(2)}{2}+2\ln(2)-1\right)\right)=$$

$$I_2=-\frac{1}{4}\zeta(3)+\frac{\zeta(2)}{6}-\frac{2}{3}\ln(2)+\frac{1}{3}$$

$$\Omega=\int_0^1\left(\arctan^3(x)+x^2Li_3(-x^3)\right)dx$$

$$=\frac{\pi^3}{64}-\frac{3\pi}{4}G+\frac{47}{64}\zeta(3)+\frac{3\pi^2}{32}\ln(2)+\frac{\zeta(2)}{6}-\frac{2}{3}\ln(2)+\frac{1}{3}$$

Solution 3 by Shobhit Jain-India

$$I=\int_0^1\left(\arctan^3(x)+x^2Li_3(-x^3)\right)dx=M+K$$

$$M=\int_0^1\arctan^3(x)dx$$

$$=\int_0^{\frac{\pi}{4}}\theta^3 d(\tan(\theta))=\theta^3\tan(\theta)\Big|\begin{matrix}\frac{\pi}{4}\\0\end{matrix}-3\int_0^{\frac{\pi}{4}}\theta^2\tan(\theta)\,d(\theta)=\frac{\pi^3}{64}+$$

$$3\int_0^{\frac{\pi}{4}}\theta^2 d(\ln(2\cos(\theta))=\frac{\pi^3}{64}+3.\frac{\pi^2}{16}.\ln\left(\frac{2}{\sqrt{2}}\right)-3\int_0^{\frac{\pi}{4}}\ln(2\cos(\theta))=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)-$$

$$\frac{3}{2}\int_0^{\frac{\pi}{2}}\theta\ln\left(2\cos\left(\frac{\theta}{2}\right)\right)d\theta \;\; \textit{by replacing } \theta \textit{ with } \theta/2$$

$$Now\;:\;\ln\left(2\cos\left(\frac{\theta}{2}\right)\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(n\theta)}{n}$$

$$\int_0^{\frac{\pi}{2}}\theta\ln\left(2\cos\left(\frac{\theta}{2}\right)\right)d\theta=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}}\theta\cos(n\theta)\,d\theta$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left\{\frac{\theta\sin(n\theta)}{n}+\frac{\cos(n\theta)}{n^2}\right\}\Bigg|_0^{\frac{\pi}{2}}=$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left\{\frac{\frac{\pi}{2}\sin\left(\frac{n\pi}{2}\right)}{n}+\frac{\cos\left(\frac{n\pi}{2}\right)}{n^2}-\frac{1}{n^2}\right\}$$

$$=\frac{\pi}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}-$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{\pi}{2}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\right\}+\left\{\frac{1}{2^3}-\frac{1}{4^3}+\frac{1}{6^3}-\cdots\right\}-\left\{\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots\right\}=$$

$$\frac{\pi}{2}G+\frac{\eta(3)}{8}-\eta(3)=\frac{\pi}{2}G-\frac{21}{32}\zeta(3)$$

$$M=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)-\frac{3}{2}\left\{\frac{\pi}{2}G-\frac{21}{32}\zeta(3)\right\}=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)+\frac{63}{64}\zeta(3)-\frac{3\pi}{4}G$$

Now

$$K=\int_0^1 x^2Li_3(-x^3)dx=\frac{1}{3}\int_0^1 Li_3(-t)dt \;\; \textit{By substituting } \; t=x^3$$

$$Li_3(-t)=\sum_{n=1}^{\infty}(-1)^n\frac{t^n}{n^3}$$

$$Therefore\;\;K=\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n+1)}=\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{n^3}-\frac{1}{n^2}+\frac{1}{n}-\frac{1}{n+1}\right\}=$$

$$\frac{1}{3}\{-\eta(3)+\eta(2)-\ln(2)+1-\ln(2)\}=\frac{1}{3}\left\{-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)+1-2\ln(2)\right\}=$$

$$\frac{\pi^2}{36}-\frac{1}{4}\zeta(3)-\frac{2}{3}\ln(2)+\frac{1}{3}$$

$$I=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx=M+K$$

$$=\frac{\pi^3}{64}+\frac{3\pi^2}{32}\ln(2)+\frac{63}{64}\zeta(3)-\frac{3\pi}{4}G+$$

$$\frac{\pi^2}{36}-\frac{1}{4}\zeta(3)-\frac{2}{3}\ln(2)+\frac{1}{3}$$

$$I=\int_0^1\left(arctan^3(x)+x^2Li_3(-x^3)\right)dx$$
$$=\frac{\pi^3}{64}-\frac{3\pi}{4}G+\frac{47}{64}\zeta(3)+\frac{3\pi^2}{32}\ln(2)+\frac{\zeta(2)}{6}-\frac{2}{3}\ln(2)+\frac{1}{3}$$

2531. Find a closed form:

$$I=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)^{n+\frac{3}{2}}}{n\left(n+\frac{3}{2}\right)\ ^{2n}C_n}$$

Proposed by Akerele Olofin Segun-Nigeria

Solution by Shobhit Jain-India

$$I=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)^{n+\frac{3}{2}}}{n\left(n+\frac{3}{2}\right)\ ^{2n}C_n}=\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\frac{1}{3^{n+1}\ n(2n+3)\ ^{2n}C_n}=$$

$$=\frac{2}{\sqrt{3}}\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}\ n!\,n!}{3^{n+1}\ n\ (2n)!}dx=\frac{2}{\sqrt{3}}\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}\ \Gamma(\mathrm{n})\Gamma(\mathrm{n}+1)}{3^{n+1}\ \Gamma(2n+1)}dx=$$

$$=\frac{2}{\sqrt{3}}\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}}{3^{n+1}}B(n,n+1)dx=\frac{2}{\sqrt{3}}\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n+2}}{3^{n+1}}(1-t)^{n-1}t^n dtdx=$$

$$=\frac{2}{\sqrt{3}}\int_0^1\int_0^1\frac{\frac{x^4t}{3^2}}{1-\frac{x^2}{3}t(1-t)}dtdx=\frac{2}{3\sqrt{3}}\int_0^1\int_0^1\frac{x^4t}{3-x^2t(1-t)}dtdx=$$

$$\int_0^1\frac{t}{3-x^2t(1-t)}dt\underset{t\to 1-t}{=}\int_0^1\frac{1-t}{3-x^2t(1-t)}dt=\frac{1}{2}\int_0^1\frac{1}{3-x^2t(1-t)}dt=$$

$$=\frac{1}{2}\int_0^1\frac{1}{3-\frac{x^2}{4}+\left(xt-\frac{x}{2}\right)^2}dt=\frac{1}{2x\sqrt{3-\frac{x^2}{4}}}\left[tan^{-1}\left(\frac{xt-\frac{x}{2}}{\sqrt{3-\frac{x^2}{4}}}\right)\right]_{t=0}^{t=1}=$$

$$=\frac{2}{x\sqrt{12-x^2}}tan^{-1}\left(\frac{x}{\sqrt{12-x^2}}\right)=\frac{2}{x\sqrt{12-x^2}}sin^{-1}\left(\frac{x}{\sqrt{12}}\right)$$

Therefore,

$$I=\frac{2}{3\sqrt{3}}\int_0^1\frac{2x^4}{x\sqrt{12-x^2}}sin^{-1}\left(\frac{x}{\sqrt{12}}\right)dx=\frac{4}{3\sqrt{3}}\int_0^1\frac{x^3}{\sqrt{12-x^2}}sin^{-1}\left(\frac{x}{\sqrt{12}}\right)dx$$

$$=32\int_0^{\phi}\theta sin^3\theta d\theta$$

$$by\ substituting\ \ \theta = sin^{-1}\left(\frac{x}{\sqrt{12}}\right)\ or\ x = \sqrt{12}sin\theta$$

$$= 8\int_0^{\phi} \theta(3sin\theta - sin3\theta)d\theta\ ,\qquad by\ using\ \ sin^3\theta = \frac{3sin\theta - sin3\theta}{4}$$

$$here, : \phi = sin^{-1}\left(\frac{1}{\sqrt{12}}\right) = cos^{-1}\left(\frac{\sqrt{11}}{\sqrt{12}}\right) = tan^{-1}\left(\frac{1}{\sqrt{11}}\right)$$

$$\Rightarrow I = 8\left[\theta\left(\frac{cos3\theta}{3} - 3cos\theta\right)\right]_0^{\phi} - 8\int_0^{\phi}\left(\frac{cos3\theta}{3} - 3cos\theta\right)d\theta$$

$$= 8\phi\left(\frac{cos3\phi}{3} - 3cos\phi\right) - 8\left(\frac{sin3\phi}{9} - 3sin\phi\right) =$$

$$Now\ use\ the\ identity\ \ cos3\phi = 4cos^3\phi - 3cos\phi\ \ and\ sin3\phi = 3sin\phi - 4sin^3\phi$$

$$\Rightarrow I = \frac{32}{9}(6sin\phi + sin^3\phi) - \frac{32\phi}{3}(3cos\phi - cos^3\phi)$$

$$= \frac{32}{9}\left(\frac{6}{\sqrt{12}} + \frac{1}{12\sqrt{12}}\right) - \frac{32\phi}{3}\left(\frac{3\sqrt{11}}{\sqrt{12}} - \frac{11\sqrt{11}}{12\sqrt{12}}\right)$$

$$= \frac{32 * 73}{9 * 12\sqrt{12}} - \phi\frac{32 * 25\sqrt{11}}{3 * 12\sqrt{12}}$$

$$\Rightarrow I = \frac{4 * 73}{27\sqrt{3}} - \phi\frac{100\sqrt{11}}{9\sqrt{3}}\quad \Rightarrow\quad I = \frac{4}{27\sqrt{3}}(73 - 75\phi\sqrt{11}) =$$

$$= \frac{4}{27\sqrt{3}}\left(73 - 75\sqrt{11}tan^{-1}\left(\frac{1}{\sqrt{11}}\right)\right)$$

2532.

$$I = \sum_{n=1}^{\infty}\frac{\sin(n)\sin(n+1)\sin(n-1)}{n}$$

Proposed by Ankush Kumar Parcha-India

Solution by Alireza Askari-Iran

$$I = \sum_{n=1}^{\infty}\frac{\sin(n)\sin(n+1)\sin(n-1)}{n} =$$

$$= \frac{\cos(2)}{2}\sum_{n=1}^{\infty}\frac{\sin(n)}{n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(n)\cos(2n)}{n} = \left(\frac{\cos(2)}{2} + \frac{1}{4}\right)\sum_{n=1}^{\infty}\frac{\sin(n)}{n} - \sum_{n=1}^{\infty}\frac{\sin(3n)}{4n}$$

$$A = \sum_{n=1}^{\infty}\frac{\sin(n)}{n} = Im\left(\sum_{n=1}^{\infty}\frac{e^{in}}{n}\right) = Im(-\ln(1 - e^i)) = Im\left(\ln\left(\frac{1}{1 - e^i}\right)\right)$$

$$= Im\left(\ln\left(\frac{1}{\cos(1)+1-i\sin(1)}\right)\right)Im\left(\ln\left(\frac{\cos(1)+1+i\sin(1)}{(1+\cos(1))^2+(\sin(1))^2}\right)\right)$$

$$= Im\left(\ln\left(2\sin(\frac{1}{2})\,e^{i\tan^{-1}\left(\frac{\sin(1)}{1-\cos(1)}\right)}\right)\right) = \tan^{-1}\left(\frac{\sin(1)}{1-\cos(1)}\right)$$

$$= \tan^{-1}\left(\cot\left(\frac{1}{2}\right)\right) = \tan^{-1}\left(\tan\left(\frac{\pi-1}{2}\right)\right) = \left(\frac{\pi-1}{2}\right)$$

$$(\frac{\cos(2)}{2}+\frac{1}{4})\sum_{n=1}^{\infty}\frac{\sin(n)}{n} = (\frac{\cos(2)}{2}+\frac{1}{4})\left(\frac{\pi-1}{2}\right)$$

$$\text{note: }\begin{cases}\{\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b)-\cos(a+b))\}\\ \{\sin(a)\cos(b) = \frac{1}{2}(\sin(a-b)+\sin(a+b))\}\end{cases}$$

$$(according\ to\ above\ solution)\sum_{n=1}^{\infty}\frac{\sin(3n)}{4n} = \frac{\pi-3}{8}$$

$$\mathrm{I} = \left(\frac{\cos(2)}{2}+\frac{1}{4}\right)\left(\frac{\pi-1}{2}\right)+\frac{3-\pi}{8} = \frac{\pi}{4}\cos(2)+\frac{1-\cos(2)}{4}$$

$$\text{ANSWER} = \frac{\pi\cos(2)+(\sin(1))^2}{4}$$

2533. Prove that:

$$\sum_{n\epsilon N}\frac{n+1}{(-4)^{n+1}}\zeta(n+1) = \frac{G}{2}+\frac{\pi^2}{16}-\frac{\pi}{8}-\frac{3}{4}ln2$$

Proposed by Ankush Kumar Parcha-India

Solution by Shobhit Jain-India

$$\psi(1+x)-\psi(1) = \sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{x+k} = \sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{k}\left(1+\frac{x}{k}\right)^{-1} \qquad here\ \psi(x)$$
$$= digamma\ function$$

Let $|x| < 1$

$$\Rightarrow \psi(1+x)+\gamma = \sum_{k=1}\frac{1}{k}-\frac{1}{k}\left(1-\frac{x}{k}+\frac{x^2}{k^2}-\frac{x^3}{k^3}+\ldots\ldots\right)$$

$$= \sum_{k=1}\left(\frac{x}{k^2}-\frac{x^2}{k^3}+\frac{x^3}{k^4}+\ldots\ldots\right)$$

$$= x\zeta(2) - x^2\zeta(3) + x^3\zeta(4) - \ldots\ldots$$

$$\Rightarrow \psi(1+x) = -\gamma + x\zeta(2) - x^2\zeta(3) + x^3\zeta(4) - \dots\dots..$$

$$\Rightarrow -x\psi(1-x) = x\gamma + x^2\zeta(2) + x^3\zeta(3) + x^4\zeta(4) + \dots\dots..$$

$$= x\gamma + \sum_{n=1}^{\infty} x^{n+1}\zeta(n+1)$$

Differentiate w.r.t x

$$\Rightarrow -\psi(1-x) + x\psi'(1-x) = \gamma + \sum_{n=1}^{\infty}(n+1)x^n\zeta(n+1)$$

put $x = (-1/4)$

$$\Rightarrow -\psi(\frac{5}{4}) - \frac{1}{4}\psi'(\frac{5}{4}) = \gamma + \sum_{n=1}^{\infty}(n+1)(\frac{-1}{4})^n\zeta(n+1)$$

$$\Rightarrow \sum_{n\epsilon N}\frac{n+1}{(-4)^{n+1}}\zeta(n+1) = \frac{1}{4}\psi(\frac{5}{4}) + \frac{\gamma}{4} + \frac{1}{16}\psi'(\frac{5}{4})$$

Now, $\psi(\frac{5}{4}) = 4 + \psi(\frac{1}{4})$

$$= 4 - \gamma - \frac{\pi}{2} - 3ln2 \quad (By\ Gauss's\ Digamma\ Theorem)$$

Therefore, $\frac{1}{4}\psi(\frac{5}{4}) + \frac{\gamma}{4} = 1 - \frac{\pi}{8} - \frac{3}{4}ln2$ ----------------(A)

We know, $\psi'(x) = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \dots\dots\dots..$

$$\psi'(\frac{5}{4}) = 16\left\{\frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \frac{1}{17^2} + \dots\dots\dots\right\}$$

$$\Rightarrow \frac{1}{16}\psi'(\frac{5}{4}) = \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \frac{1}{17^2} + \dots\dots\dots..$$

Now,

$$\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \dots\dots\dots = G$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \dots\dots\dots = \left\{1 - \frac{1}{2^2}\right\}\zeta(2) = \frac{3}{4} \times \frac{\pi^2}{6} = \frac{\pi^2}{8}$$

By adding both series,

$$\Rightarrow 2\left\{\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\dots\dots\dots\dots\dots\right\}=G+\frac{\pi^2}{8}$$

$$\Rightarrow \frac{1}{16}\psi'\left(\frac{5}{4}\right)=\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}+\dots\dots\dots\dots=\frac{G}{2}+\frac{\pi^2}{16}-1 \qquad \text{----------------}(B)$$

$$Add\ both\ (A)\ and\ (B)$$

$$\frac{1}{4}\psi\left(\frac{5}{4}\right)+\frac{\gamma}{4}+\frac{1}{16}\psi'\left(\frac{5}{4}\right)=\frac{1}{4}\left(4-\gamma-\frac{\pi}{2}-3ln2\right)+\frac{\gamma}{4}+\frac{G}{2}+\frac{\pi^2}{16}-1$$

$$=1-\frac{\pi}{8}-\frac{3}{4}ln2+\frac{G}{2}+\frac{\pi^2}{16}-1$$

$$=\frac{G}{2}+\frac{\pi^2}{16}-\frac{\pi}{8}-\frac{3}{4}ln2$$

$$\Rightarrow \sum_{n\epsilon N}\frac{n+1}{(-4)^{n+1}}\zeta(n+1)=\frac{G}{2}+\frac{\pi^2}{16}-\frac{\pi}{8}-\frac{3}{4}ln2$$

$$here\ \ G=Catalan's\ constant\ \ and\ \ \gamma=Euler-Mascheroni's\ \ constant$$

2534. Find a closed form:

$$\sum_{n\in N}\frac{\sin(n-1)\sin(n)\sin(n+1)}{n}$$

Proposed by Ankush Kumar Parcha-India

Solution by Pham Duc Nam-Vietnam

$$S=\sum_{n\in N}\frac{\sin(n-1)\sin(n)\sin(n+1)}{n}$$

$$This\ \ summ:S(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}=\frac{\pi-x}{2},\qquad where\ \in(0,2\pi)\ is\ well-known$$

$$S(1)=\frac{\pi-1}{2},S(3)=\frac{\pi-3}{2}$$

$$Since\ :\ \sin(n-1)\sin(n+1)=\frac{1}{2}(\cos(2)-1+1-\cos(2n))$$

$$S=\frac{1}{2}(\cos(2)-1)\sum_{n=1}^{\infty}\frac{\sin(n)}{n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(n)(1-\cos(2n))}{n}$$

$$=-sin^2(1)\left(\frac{\pi-1}{2}\right)+\sum_{n=1}^{\infty}\frac{sin^3(n)}{n}=$$

$$= -sin^2(1)\left(\frac{\pi-1}{2}\right)$$
$$+\sum_{n=1}^{\infty}\frac{\frac{1}{4}(3\sin(n)-\sin(3n))}{n} = -sin^2(1)\left(\frac{\pi-1}{2}\right)+\frac{3}{4}S(1)-\frac{1}{4}S(3) =$$
$$= -sin^2(1)\left(\frac{\pi-1}{2}\right)+\frac{\pi}{4} = \frac{\pi cos(2)+2sin^2(1)}{4}$$

2535. Find:

$$\Omega = \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Shobhit Jain-India

$$\Omega = \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2} \to$$
$$\Omega + 1 = \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2}$$
$$= \sum_{k=0}^{\infty}(-1)^k\left\{\frac{12}{k+1}+\frac{4}{(k+1)^2}+\frac{1}{(k+1)^3}-\frac{24}{2k+1}+\frac{8}{(2k+1)^2}\right\} \to$$
$$\Omega+1 = 12\left\{\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots\right\}+4\left\{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right\}+\left\{\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots\right\}-$$
$$24\left\{\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\cdots\right\}+8\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\right\} = 12\ln(2)+4\eta(2)+\eta(3)-$$
$$-24\beta(1)+8\beta(2) = 12\ln(2)+2\zeta(2)+\frac{3}{4}\zeta(3)-24.\frac{\pi}{4}+8G$$
$$\Omega = \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2} = 8G+12\ln(2)+2\zeta(2)+\frac{3}{4}\zeta(3)-6\pi-1$$

Solution 2 by Kartick Chandra Betal-India

$$\Omega = \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^3(2k+1)^2} = \sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k^3(2k-1)^2} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3(2k-1)^2}-1 =$$
$$= 8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{(2k)^2(2k-1)^2}-\frac{1}{(2k)^3(2k-1)}\right\}-1 =$$
$$= 8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{2k(2k-1)^2}-\frac{1}{(2k)^2(2k-1)}-\frac{1}{(2k)^2(2k-1)}+\frac{1}{(2k)^3}\right\}-1 =$$

$$= 8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{(2k-1)^2}-\frac{3}{2k(2k-1)}+\frac{2}{(2k)^2}+\frac{1}{(2k)^3}\right\}-1=$$

$$= 8\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\frac{1}{(2k-1)^2}-\frac{3}{(2k-1)}+\frac{3}{2k}+\frac{1}{2k^2}+\frac{1}{(2k)^3}\right\}-1=$$

$$= 8\left\{\beta(2)-3.\frac{\pi}{4}+\frac{3}{2}\ln(2)+\frac{1}{2}.\frac{\pi^2}{12}+\frac{1}{8}.\frac{3}{4}\zeta(3)\right\}-1=$$

$$= 8\left\{G-\frac{3\pi}{4}+\frac{3}{2}ln(2)+\frac{\pi^2}{24}+\frac{3}{32}\zeta(3)\right\}-1$$

$$= 8G+12\ln(2)+2\zeta(2)+\frac{3}{4}\zeta(3)-6\pi-1$$

2536. Find:

$$\Omega=\sum_{k=1}^{\infty}\frac{k^2}{(2k+1)^3\binom{2k}{2k+1}}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Shobhit Jain-India

$$\Omega=\sum_{k=1}^{\infty}\frac{k^2}{(2k+1)^3\,({}^{2k}C_{k+1})}=\sum_{k=0}^{\infty}\frac{k^2\,(k-1)!\,(k+1)!}{(2k+1)^3\,(2k)!}=\sum_{k=0}^{\infty}\frac{k(k+1)\,k!\,k!}{(2k+1)^2\,(2k+1)!}=$$

$$=\sum_{k=0}^{\infty}\frac{k(k+1)\,\Gamma(k+1)\Gamma(k+1)}{(2k+1)^2\,\Gamma(2k+2)}=\sum_{k=0}^{\infty}\frac{k(k+1)}{(2k+1)^2}B(k+1,k+1)$$

Now, $\dfrac{k(k+1)}{(2k+1)^2}=\dfrac{1}{4}-\dfrac{1}{4(2k+1)^2}$ *and* $B(k+1,k+1)=\displaystyle\int_0^1 t^k(1-t)^k\,dt$

Therefore, $\Omega=\displaystyle\int_0^1\sum_{k=0}^{\infty}\left(\frac{1}{4}-\frac{1}{4(2k+1)^2}\right)t^k(1-t)^k dt$

$$=\frac{1}{4}\int_0^1\sum_{k=0}^{\infty}t^k(1-t)^k dt$$

$$-\frac{1}{4}\int_0^1\sum_{k=0}^{\infty}\frac{t^k(1-t)^k}{(2k+1)^2}dt \qquad \textit{Now use,}\ \frac{1}{(2k+1)^2}$$

$$=-\int_0^1(lnx)\,x^{2k}dx$$

$$\Rightarrow\ \Omega=\frac{1}{4}\int_0^1\frac{dt}{1-t(1-t)}+\frac{1}{4}\int_0^1\int_0^1\sum_{k=0}^{\infty}(lnx)\,x^{2k}t^k(1-t)^k dxdt$$

$$=\frac{1}{4}\int_0^1\frac{dt}{\frac{3}{4}+\left(t-\frac{1}{2}\right)^2}+\frac{1}{4}\int_0^1\int_0^1\frac{lnx}{1-x^2t(1-t)}dtdx$$

$$use\ \int_0^1\frac{dt}{\frac{3}{4}+\left(t-\frac{1}{2}\right)^2}=\left[\frac{2}{\sqrt{3}}tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\right]_0^1=\frac{2\pi}{3\sqrt{3}}$$

$$\int_0^1\frac{dt}{1-x^2t(1-t)}=\frac{1}{x^2}\int_0^1\frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{x^2}-\frac{1}{4}\right)}=\frac{1}{x^2}\frac{1}{\sqrt{\frac{1}{x^2}-\frac{1}{4}}}\left[tan^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{\frac{1}{x^2}-\frac{1}{4}}}\right)\right]_0^1$$

$$=\frac{2}{x^2\sqrt{\frac{1}{x^2}-\frac{1}{4}}}tan^{-1}\left(\frac{\frac{1}{2}}{\sqrt{\frac{1}{x^2}-\frac{1}{4}}}\right)=\frac{4}{x\sqrt{4-x^2}}tan^{-1}\left(\frac{x}{\sqrt{4-x^2}}\right)$$

$$=\frac{4}{x\sqrt{4-x^2}}sin^{-1}\left(\frac{x}{2}\right)$$

Therefore, $\Omega=\frac{\pi}{6\sqrt{3}}+\int_0^1\frac{lnx}{x\sqrt{4-x^2}}sin^{-1}\left(\frac{x}{2}\right)dx$ substitute $\theta=sin^{-1}\left(\frac{x}{2}\right)$ or $x=2sin\theta$

$$=\frac{\pi}{6\sqrt{3}}+\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{\theta\ln(2sin\theta)}{\sin\theta}d\theta=\frac{\pi}{6\sqrt{3}}+\frac{ln2}{2}\int_0^{\frac{\pi}{6}}\frac{\theta}{\sin\theta}d\theta+\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{\theta\ln(sin\theta)}{\sin\theta}d\theta=$$

$$=\frac{\pi}{6\sqrt{3}}+\frac{ln2}{2}P+\frac{1}{2}C$$

$$Now,\quad P=\int_0^{\frac{\pi}{6}}\frac{\theta}{\sin\theta}d\theta$$

$$=\int_0^{\frac{\pi}{6}}\theta d\left[\ln\left(tan\frac{\theta}{2}\right)\right]\underbrace{=}_{IBP}\left[\theta\ln\left(tan\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{6}}-\int_0^{\frac{\pi}{6}}\ln\left(tan\frac{\theta}{2}\right)d\theta$$

$$=\frac{\pi}{6}\ln\left(tan\frac{\pi}{12}\right)+\int_0^{\frac{\pi}{6}}\ln\left(cot\frac{\theta}{2}\right)d\theta$$

$$\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}\quad and \quad \ln\left(cot\frac{\theta}{2}\right)=2\sum_{n=odd>0}\frac{\cos(n\theta)}{n}$$

$$P=\frac{\pi}{6}\ln(2-\sqrt{3})+2\sum_{n=odd>0}\int_0^{\frac{\pi}{6}}\frac{\cos(n\theta)}{n}d\theta=\frac{\pi}{6}\ln(2-\sqrt{3})+\sum_{n=odd>0}\frac{2sin\left(\frac{n\pi}{6}\right)}{n^2}$$

$$Now, \sum_{n=odd>0}\frac{2sin\left(\frac{n\pi}{6}\right)}{n^2}=\left\{\frac{1}{1^2}+\frac{2}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{2}{9^2}-\frac{1}{11^2}+\dots\dots\dots\dots\dots\right\}$$

$$=\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\dots\dots\dots\dots\dots\right\}+\left\{\frac{3}{3^2}-\frac{3}{9^2}+\dots\dots\dots\dots\dots\right\}$$

$$=G+\frac{3}{3^2}G=\frac{4G}{3}$$

$$Hence, P=\frac{\pi}{6}\ln(2-\sqrt{3})+\frac{4G}{3}. \text{ Therefore, } \Omega=\frac{\pi}{6\sqrt{3}}+\frac{ln2}{2}\left\{\frac{\pi}{6}\ln(2-\sqrt{3})+\frac{4G}{3}\right\}+\frac{1}{2}C$$

$$\Omega=\frac{\pi}{6\sqrt{3}}+\frac{\pi\ln(2)\ln(2-\sqrt{3})}{12}+\frac{2G\ln2}{3}+\frac{1}{2}C$$

here $G=catalan's\ constant$

$$C=\int_0^{\frac{\pi}{6}}\frac{\theta\ln(sin\theta)}{\sin\theta}d\theta=some\ numerical\ constant$$

2537. Prove that:

$$\Omega=\sum_{k=1}^{\infty}\frac{1+k^2}{2^kk^2\binom{1+k^2}{k^2}}=Li_2\left(\frac{1}{2}\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Mohammad Rostami-Afghanistan

$$\Omega=\sum_{k=1}^{\infty}\frac{1+k^2}{2^kk^2\binom{1+k^2}{k^2}}=\sum_{k=1}^{\infty}\frac{1+k^2}{2^kk^2\frac{(1+k^2)!}{k^2!}}=\sum_{k=1}^{\infty}\frac{1+k^2}{2^kk^2(1+k^2)}=\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{k^2}=$$

$$-\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\int_0^1x^{k-1}\ln(x)\,dx=-\int_0^1\frac{\ln(x)}{x}\sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^kdx=-\int_0^1\frac{\ln(x)}{x}\frac{\frac{x}{2}}{1-\frac{x}{2}}dx=$$

$$-\int_0^1\frac{\ln(x)}{2-x}dx=-\int_0^1\frac{\ln(1-x)}{1+x}dx\ \left\{x=\frac{1-t}{1+t},\right.$$

$$\left.dx=\frac{-1-t-1+t}{(1+t)^2}dt\to dx=-\frac{2}{(1+t)^2}dt\right\}$$

$$\Omega = -\int_0^1 \frac{\ln\left(1-\frac{1-t}{1+t}\right)}{1+\frac{1-t}{1+t}} \frac{-2}{(1+t)^2} dt = \int_0^1 \frac{\ln(1+t)-\ln(2)-\ln(t)}{1+t} dt =$$

$$\int_0^1 \frac{\ln(1+t)}{1+t} dt - \ln(2)\int_0^1 \frac{1}{1+t} dt - \int_0^1 \frac{\ln(t)}{1+t} dt, \qquad \ln(1+t) = u$$

$$= \int_0^{\ln(2)} u du - \ln(2)\,[\ln(1+t)]\Big|_0^1$$

$$-\int_0^1 \sum_{n=0}^{\infty}(-t)^n \frac{\partial}{\partial a}\Big|_{a=0} t^a dt = \left[\frac{u^2}{2}\right]\Big|_0^{\ln(2)} - \ln^2(2) -$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{\partial}{\partial a}\Big|_{a=0} \int_0^1 t^{n+a} dt = \frac{\ln^2(2)}{2} - \ln^2(2) - \sum_{n=0}^{\infty}(-1)^n \frac{\partial}{\partial a}\Big|_{a=0} \frac{1}{n+a+1} =$$

$$-\frac{\ln^2(2)}{2} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} = -\frac{\ln^2(2)}{2} + \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^2} = -\frac{\ln^2(2)}{2} + \eta(2) =$$

$$-\frac{\ln^2(2)}{2} + (1-2^{-1})\zeta(2) = -\frac{\ln^2(2)}{2} + \frac{1}{2}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2) = Li_2\left(\frac{1}{2}\right)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega = \sum_{k=1}^{\infty} \frac{1+k^2}{2^k k^2 \binom{1+k^2}{k^2}}$$

$$\Omega_{a,m,n} = \sum_{k=1}^{\infty} \frac{b^2+k^n}{a^k k^m \binom{b^2+k^n}{k^n}} = \sum_{k=1}^{\infty} \frac{b^2+k^n}{a^k k^m \frac{(b^2+k^n)!}{(k^n)!(b^2)!}} =$$

$$(b^2)!\sum_{k=1}^{\infty} \frac{b^2+k^n}{a^k k^m \frac{(b^2+k^n)(k^n)!}{(k^n)!}} = (b^2)!\sum_{k=1}^{\infty}\frac{1}{a^k k^m} = (b^2)!\sum_{k=1}^{\infty}\frac{\left(\frac{1}{a}\right)^k}{k^m} = (b^2)!\,Li_m\left(\frac{1}{a}\right)$$

$$Let\ \ a=2,\ b=1,\ n=m=2:$$

$$\Omega_{2,2,2} = \sum_{k=1}^{\infty} \frac{1+k^2}{2^k k^2 \binom{1+k^2}{k^2}} = (1^2)!\,Li_2\left(\frac{1}{2}\right) = Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2)$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1+k^2}{2^k k^2 \binom{1+k^2}{k^2}} = Li_2\left(\frac{1}{2}\right)}$$

2538. Find a closed form:

$$I = \int_0^1 \frac{x\ln(x)\{Li_2(x) - Li_2(-x)\}}{1+x^2} dx$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Pratham Prasad-India

$$I=\int_0^1\frac{x\ln(x)\{Li_2(x)-Li_2(-x)\}}{1+x^2}dx=\int_0^1\frac{x\ln(x)\{2Li_2(x)-[Li_2(x)+Li_2(-x)]\}}{1+x^2}dx$$

$$=$$

$$=\int_0^1\frac{x\ln(x)\{2Li_2(x)-\frac{1}{2}Li_2(x^2)\}}{1+x^2}dx$$

$$=2\int_0^1\frac{x\ln(x)Li_2(x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{x\ln(x)Li_2(x^2)}{1+x^2}dx$$

in second integral $\{x^2=u,\ 2xdx=du\}$

$$=2\int_0^1\frac{x\ln(x)Li_2(x)}{1+x^2}dx-\frac{1}{8}\int_0^1\frac{\ln(u)Li_2(u)}{1+u}du=2A-\frac{1}{8}B$$

Where

$$A=\int_0^1\frac{x\ln(x)Li_2(x)}{1+x^2}dx,\qquad B=\int_0^1\frac{\ln(x)Li_2(x)}{1+x}dx$$

$$A=\int_0^1\frac{x\ln(x)Li_2(x)}{1+x^2}dx=-\int_0^1\int_0^1\frac{x^2\ln(x)\ln(y)}{(1+x^2)(1-xy)}dydx$$

$$=-\left(\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1-xy)}dydx\right.$$

$$\left.-\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1+x^2)(1+y^2)}dydx-\int_0^1\int_0^1\frac{xy\ln(x)\ln(y)}{(1+x^2)(1+y^2)}dydx\right)$$

$$=\frac{-1}{2}\left(\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1-xy)}dydx-\left(\int_0^1\frac{\ln(x)}{(1+x^2)}dx\right)^2-\left(\int_0^1\frac{x\ln(x)}{(1+x^2)}dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\int_0^1\int_0^1 x^{n-1}y^{n-1}\ln(x)\ln(y)dydx-\left(\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-2}\ln(x)\,dx\right)^2\right.$$

$$\left.-\left(\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-1}\ln(x)\,dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\right)^2-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2}\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\right)^2-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2}\right)^2\right)=\frac{-1}{2}\left(\zeta(4)-G^2-\left(\frac{1}{8}\zeta(2)\right)^2\right)=$$

$$=\frac{1}{2}\left(G^2-\frac{123}{128}\zeta(4)\right)$$

$$B=\int_0^1\frac{\ln(x)\,Li_2(x)}{1+x}dx=-\int_0^1\int_0^1\frac{x\ln(x)\ln(y)}{(1+x)(1-xy)}dydx=$$

$$=-\frac{1}{2}\left(\int_0^1\int_0^1\frac{x\ln(x)\ln(y)}{(1+x)(1-xy)}dydx+\int_0^1\int_0^1\frac{y\ln(x)\ln(y)}{(1+y)(1-xy)}dydx\right)$$

$$=-\frac{1}{2}\int_0^1\int_0^1\frac{(x+y+2xy)\ln(x)\ln(y)}{(1+y)(1+x)(1-xy)}dydx$$

$$=-\frac{1}{2}\int_0^1\int_0^1\frac{[(1+x)(1+y)-(1-xy)]\ln(x)\ln(y)}{(1+y)(1+x)(1-xy)}dydx$$

$$=-\frac{1}{2}\left(\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1-xy)}dydx-\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1+y)(1+x)}dydx\right)$$

$$=-\frac{1}{2}\left(\int_0^1\int_0^1\frac{\ln(x)\ln(y)}{(1-xy)}dydx-\left(\int_0^1\frac{\ln(x)}{1+x}dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\int_0^1\int_0^1 x^{n-1}y^{n-1}\ln(x)\ln(y)dydx-\left(\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 \mathrm{x}^{n-1}\ln(x)\,dx\right)^2\right)$$

$$=\frac{-1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right)^2\right)=\frac{-1}{2}\left(\zeta(4)-\left(\frac{1}{2}\zeta(2)\right)^2\right)=\frac{-1}{2}\left(\frac{\pi^4}{90}-\frac{\pi^4}{144}\right)$$

$$=\frac{-1}{2}\left(\frac{\pi^4}{90}\left(\frac{54}{144}\right)\right)=-\frac{3}{16}\zeta(4)$$

$$I=2A-\frac{1}{8}B,\quad I=2\left(\frac{1}{2}\left(G^2-\frac{123}{128}\zeta(4)\right)\right)-\frac{1}{8}\left(-\frac{3}{16}\zeta(4)\right)$$

$$I=G^2-\frac{120}{128}\zeta(4),\ \ I=G^2-\frac{120}{128}\frac{\pi^4}{90},\quad I=G^2-\frac{\pi^4}{96}$$

Or,

$$\int_0^1\frac{x\ln(x)\{Li_2(x)-Li_2(-x)\}}{1+x^2}dx=G^2-\frac{\pi^4}{96}$$

2539. Prove that:

$$\int_0^{\infty}\frac{\dfrac{\dfrac{x}{x^2+1+x^2}}{\dfrac{x^4+x^2+1+x^2+x^4}{x^2}}}{\dfrac{x}{\pi}}dx=\frac{2-\sqrt{1+\sqrt{2}}}{2\sqrt{2}}$$

Proposed by Ankush Kumar Parcha-India

Solution by Shohbit Jain-India

After simplification:

$$I=\frac{1}{\pi}\int_0^{\infty}\frac{1}{(1+2x^2)(1+2x^2+2x^4)}dx=$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{dx}{(1+2x^2)}-\frac{1}{\pi}\int_0^{\infty}\frac{(1+2x^2)}{(1+2x^2+2x^4)}dx=\frac{1}{\sqrt{2}}-\frac{1}{\pi}\int_0^{\infty}\frac{\frac{1}{x^2}+2}{2x^2+\frac{1}{x^2}+2}dx=$$

$$=\frac{1}{\sqrt{2}}-\frac{1}{\pi}\int_0^{\infty}\frac{\frac{1}{x^2}+2}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx\underset{x\to\frac{1}{\sqrt{2}x}}{=}\frac{1}{\sqrt{2}}-\frac{1}{\pi}\int_0^{\infty}\frac{\sqrt{2}\left(\frac{1}{x^2}+1\right)}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx=$$

$$I=\frac{1}{\sqrt{2}}-\frac{1}{2\pi}\int_0^{\infty}\frac{\frac{1}{x^2}+2+\sqrt{2}\left(\frac{1}{x^2}+1\right)}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx$$

$$=\frac{1}{\sqrt{2}}-\frac{1+\sqrt{2}}{2\pi}\int_0^{\infty}\frac{\sqrt{2}+\frac{1}{x^2}}{\left(\sqrt{2}x-\frac{1}{x}\right)^2+2+2\sqrt{2}}dx$$

$$\textit{Now substitute } \sqrt{2}x-\frac{1}{x}=u$$

$$I=\frac{1}{\sqrt{2}}-\frac{1+\sqrt{2}}{2\pi}\int_{-\infty}^{\infty}\frac{du}{u^2+2+2\sqrt{2}}=\frac{1}{\sqrt{2}}-\frac{1+\sqrt{2}}{2\pi}\frac{\pi}{\sqrt{2+2\sqrt{2}}}=\frac{1}{\sqrt{2}}-\frac{\sqrt{1+\sqrt{2}}}{2\sqrt{2}}$$

$$\textit{Therefore,}\quad I=\frac{2-\sqrt{1+\sqrt{2}}}{2\sqrt{2}}$$

2540. Find a closed form:

$$I=\iiint_{\{[0,1]^3\}}\frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)}dxdydz$$

Proposed by Ankush Kumar Parcha-India

Solution by Pratham Prasad-India

By, partial fractions

$$I = \iiint_{\{[0,1]^3\}} \left(-\frac{3}{8}\frac{1}{1+\frac{xyz}{4}} - \frac{1}{4}\frac{1}{1+\frac{xyz}{2}} + 1\right) dxdydz$$

$$I = \iiint_{\{[0,1]^3\}} \left(-\frac{3}{8}\sum_{r=0}^{\infty}\left(-\frac{xyz}{4}\right)^r - \frac{1}{4}\sum_{r=0}^{\infty}\left(-\frac{xyz}{2}\right)^r + 1\right) dxdydz$$

$$I = -\frac{3}{8}\sum_{r=0}^{\infty}\iiint_{\{[0,1]^3\}}\left(-\frac{xyz}{4}\right)^r - \frac{1}{4}\sum_{r=0}^{\infty}\iiint_{\{[0,1]^3\}}\left(-\frac{xyz}{2}\right)^r + 1\, dxdydz$$

$$I = \frac{3}{2}\sum_{r=0}^{\infty}\frac{\left(\frac{-1}{4}\right)^{r+1}}{(r+1)^3} + \frac{1}{2}\sum_{r=0}^{\infty}\frac{\left(\frac{-1}{2}\right)^{r+1}}{(r+1)^3} + 1, \qquad I = \frac{3}{2}Li_3\left(-\frac{1}{4}\right) + \frac{1}{2}Li_3\left(-\frac{1}{2}\right) + 1$$

$$I = \frac{1}{2}\left(3Li_3\left(-\frac{1}{4}\right) + Li_3\left(-\frac{1}{2}\right)\right) + 1$$

2541. Find a closed form:

$$\int_0^{\frac{\pi}{2}} x\, ln\left(1 + \frac{1}{csc(x)} + \frac{1}{sec(x)}\right) dx$$

Proposed by Ankush Kumar Parcha-India

Solution by Pratham Prasad-India

$$\int_0^{\frac{\pi}{2}} x\ln\left(1+\frac{1}{\csc(x)}+\frac{1}{\sec(x)}\right)dx = \int_0^{\frac{\pi}{2}} x\ln(1+\sin(x)+\cos(x))dx =$$

$$x = 2u$$

$$= 4\int_0^{\frac{\pi}{4}} u\ \ln(1+\sin(2u)+\cos(2u))\,du = 4\int_0^{\frac{\pi}{4}} u\ \ln(2cos^2(u) + 2\cos(u)\sin(u))\,du$$

$$=$$

$$= 4\int_0^{\frac{\pi}{4}} u\ \ln(2cos(u)(\cos(u)+\sin(u))\,du = 4\int_0^{\frac{\pi}{4}} u\ \ln\left(2\sqrt{2}cos(u)\cos\left(\frac{\pi}{4}-u\right)\right)du =$$

$$= 4\int_0^{\frac{\pi}{4}} u\ \ln(2\sqrt{2})\,du + 4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du + 4\int_0^{\frac{\pi}{4}} u\ \ln\left(\cos\left(\frac{\pi}{4}-u\right)\right)du =$$

$$replace\ \left(\frac{\pi}{4}-u\right)\ by\ u\ in\ the\ last\ integral\ by\ u$$

$$= 4\int_0^{\frac{\pi}{4}} u\ \ln(2\sqrt{2})\,du + 4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du + 4\int_0^{\frac{\pi}{4}} \left(\frac{\pi}{4}-u\right)\ \ln(\cos(u))\,du =$$

$$= 4\int_0^{\frac{\pi}{4}} u\ \ln(2\sqrt{2})\,du + 4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du + \pi\int_0^{\frac{\pi}{4}} \ln(\cos(u))\,du - 4\int_0^{\frac{\pi}{4}} u\ \ln(\cos(u))\,du =$$

$$=\frac{12}{2}\int_0^{\frac{\pi}{4}}u\ \ln(2)\,du+\pi\int_0^{\frac{\pi}{4}}\ln(\cos(u))\,du=\frac{3\pi^2}{16}\ln(2)+\pi(\frac{G}{2}-\frac{\pi}{4}\ln(2))$$

$$=\frac{G\pi}{2}-\frac{\pi^2}{16}\ln(2)$$

2542. Find a closed form:

$$I=\iint_0^1\tan^{-1}\left(\frac{x+y}{1-xy}\right)\ln(xy)\,dxdy$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Alireza Askari-Iran

$$I=\iint_0^1\tan^{-1}\left(\frac{x+y}{1-xy}\right)\ln(xy)\,dxdy$$

$$=\iint_0^1\tan^{-1}(x)\ln(xy)\,dxdy+\iint_0^1\tan^{-1}(y)\ln(xy)\,dxdy$$

$$A=\iint_0^1\tan^{-1}(x)\ln(xy)\,dxdy\ .\ \ B=\iint_0^1\tan^{-1}(y)\ln(xy)\,dxdy$$

$$A=\iint_0^1\tan^{-1}(x)\ln(xy)\,dxdy=\frac{\partial}{\partial a}\bigg|_{a=0}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1x^{2n+a-1}dx\int_0^1y^a dy$$

$$\frac{\partial}{\partial a}\bigg|_{a=0}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(2n+a)(a+1)}=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{4(2n-1)n^2}=B=A$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{2(2n-1)n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)n^2}$$

$$=\frac{-\pi^2}{24}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{2n-1}-\frac{2(-1)^n}{n}-\frac{(-1)^n}{n^2}$$

$$Answer=\frac{-\pi^2}{24}+4\left(\frac{-\pi}{4}\right)+2\ln(2)+\frac{\pi^2}{12}=\frac{\pi^2}{24}-\pi+\ln(4)$$

2543. Prove that:

$$\int_0^1\frac{\ln^2x}{(x+1)(x^2+x\varphi^2+\varphi)}dx=\frac{\varphi\pi^2}{6}-2\varphi^2Li_3\left(-\frac{1}{\varphi}\right)-\frac{3}{2}\varphi^2\zeta(3)$$

$$\varphi=golden\ ratio,\zeta(3)=Apery's\ Constant\ ,$$
$$Li_3(z)=trilogarithmic\ function$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shobhit Jain-India

$$use,\quad \varphi^2 = 1+\varphi \ \ and\ \ 1+\varphi^{-1} = \varphi \ \ and\ \ \frac{1}{\varphi-1} = \varphi$$

$$x^2+x\varphi^2+\varphi = x^2+x(1+\varphi)+\varphi = (x+1)(x+\varphi)$$

$$Now,\quad I = \int_0^1 \frac{ln^2x}{(x+1)(x^2+x\varphi^2+\varphi)}dx = \int_0^1 \frac{ln^2x}{(x+1)^2(x+\varphi)}dx$$

$$= \int_0^1 ln^2x\left(\frac{\frac{1}{(\varphi-1)}}{(x+1)^2} - \frac{\frac{1}{(\varphi-1)^2}}{x+1} + \frac{\frac{1}{(\varphi-1)^2}}{x+\varphi}\right)dx$$

$$= \phi\int_0^1 \frac{ln^2x}{(1+x)^2}dx - \varphi^2\int_0^1 \frac{ln^2x}{x+1}dx + \varphi^2\int_0^1 \frac{ln^2x}{x+\varphi}dx = \phi A - \varphi^2 B + \varphi^2 C$$

Now,

$$A = \int_0^1 \frac{ln^2x}{(1+x)^2}dx = \int_0^1 ln^2x\,(1-2x+3x^2-4x^3+\cdots)dx$$

$$= \Gamma 3\left(\frac{1}{1^3} - \frac{2}{2^3} + \frac{3}{3^3} - \frac{4}{4^3} + \cdots\right) = 2\eta(2) = \zeta(2)$$

$$Now, \int_0^1 \frac{ln^2x}{x-a}dx = -\frac{1}{a}\sum_{n=1}^{\infty}\int_0^1 (ln^2x)\left(\frac{x}{a}\right)^{n-1}dx = -\frac{1}{a}\sum_{n=1}^{\infty}\frac{\Gamma 3}{a^{n-1}n^3} = -2\sum_{n=1}^{\infty}\frac{\left(\frac{1}{a}\right)^n}{n^3}$$

$$= -2Li_3\left(\frac{1}{a}\right)$$

$$put\ \ a=-1 \ \Rightarrow B = \int_0^1 \frac{ln^2x}{x+1}dx = -2Li_3(-1) = 2\eta(3) = \frac{3}{2}\zeta(3)$$

$$put\ \ a=-\varphi \ \Rightarrow C = \int_0^1 \frac{ln^2x}{x+\varphi}dx = -2Li_3\left(-\frac{1}{\varphi}\right)$$

$$Therefore,\quad I = \phi\zeta(2) - \varphi^2\frac{3}{2}\zeta(3) + -2Li_3\left(-\frac{1}{\varphi}\right)\varphi^2$$

$$= \frac{\varphi\pi^2}{6} - 2\varphi^2 Li_3\left(-\frac{1}{\varphi}\right) - \frac{3}{2}\varphi^2\zeta(3)$$

2544. Prove that:

$$\int_0^\infty \frac{1}{(1+x^{2025})(1+x^2)}dx = \frac{\pi}{2}$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Tapas Das-India

$$Let\ I=\int_0^\infty \frac{1}{(1+x^{2025})(1+x^2)}dx\quad (A)$$

$$Let\ x=\frac{1}{t}\ then\ dx=-\frac{1}{t^2}dt$$

$$I=\int_\infty^0 \frac{-(t^{2025}.t^2)}{(1+t^{2025})(1+t^2)t^2}dt=\int_0^\infty \frac{(t^{2025}.)}{(1+t^{2025})(1+t^2)}dt$$

$$=\int_0^\infty \frac{(x^{2025}.)}{(1+x^{2025})(1+x^2)}dx\ (B)$$

$$adding\ (A) and\ (B)\ we\ get:$$

$$2I=\int_0^\infty \frac{(x^{2025}+1.)}{(1+x^{2025})(1+x^2)}dx=\int_0^\infty \frac{(x^{2025}+1)}{(1+x^{2025})(1+x^2)}dx=\int_0^\infty \frac{1}{(1+x^2)}dx=$$

$$=\lim_{x\to\infty}\int_0^x \frac{1}{(1+x^2)}dx=\lim_{x\to\infty}[\tan^{-1}x]_0^x=\tan^{-1}\infty=\frac{\pi}{2}$$

$$I=\frac{\pi}{4}$$

2545. Prove that:

$$\int_0^1 \frac{\ln(1+x)}{x(2x+1)(3x+1)}dx=-5Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{4}-\frac{29}{2}ln^2 2+8(ln2)(ln3)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shobhit Jain-India

$$I=\int_0^1 \frac{\ln(1+x)}{x(2x+1)(3x+1)}dx\underset{x\to x-1}{=}\int_1^2 \frac{lnx}{(x-1)(2x-1)(3x-2)}dx\underset{x\to\frac{1}{x}}{=}\int_1^{\frac{1}{2}} \frac{xlnx}{(1-x)(2-x)(3-2x)}dx$$

$$=\int_1^{\frac{1}{2}}(lnx)\left(\frac{1}{1-x}+\frac{2}{2-x}-\frac{6}{3-2x}\right)dx=\int_1^{\frac{1}{2}}(lnx)\left\{(1-x)^{-1}+\left(1-\frac{x}{2}\right)^{-1}-2\left(1-\frac{2x}{3}\right)^{-1}\right\}dx$$

$$=F(1)+F\left(\frac{1}{2}\right)-2F\left(\frac{2}{3}\right)$$

$$here,\ F(a)=\int_1^{\frac{1}{2}}(lnx)(1-ax)^{-1}dx=\sum_{n=1}^\infty a^{n-1}\int_1^{\frac{1}{2}}(lnx)x^{n-1}dx$$

$$Now,\ \int_1^{\frac{1}{2}}(lnx)x^{n-1}dx=\left[(lnx)\frac{x^n}{n}-\frac{x^n}{n^2}\right]_1^{\frac{1}{2}}=-\frac{ln2}{n2^n}+\frac{1}{n^2}-\frac{1}{n^2 2^n}$$

$$Therefore,\quad F(a)=\frac{1}{a}(ln2)\left(1-\frac{a}{2}\right)+\frac{1}{a}Li_2(a)-\frac{1}{a}Li_2\left(\frac{a}{2}\right)$$

$$Note:\ Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{ln^2 2}{2}\Rightarrow F(1)=\frac{\pi^2}{12}-\frac{ln^2 2}{2}$$

$$\Rightarrow F\left(\frac{1}{2}\right)=2(ln2)(ln3)-5ln^2 2+\frac{\pi^2}{6}-2Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow F\left(\frac{2}{3}\right)=\frac{3}{2}ln^2 2-\frac{3}{2}(ln2)(ln3)+\frac{3}{2}Li_2\left(\frac{2}{3}\right)-\frac{3}{2}Li_2\left(\frac{1}{3}\right)\Rightarrow I=F(1)+F\left(\frac{1}{2}\right)-2F\left(\frac{2}{3}\right)$$

$$=\frac{\pi^2}{12}-\frac{ln^2 2}{2}+2(ln2)(ln3)-5ln^2 2+\frac{\pi^2}{6}-2Li_2\left(\frac{1}{4}\right)-3ln^2 2+3(ln2)(ln3)-3Li_2\left(\frac{2}{3}\right)+3Li_2\left(\frac{1}{3}\right)$$

$$\Rightarrow I=\frac{\pi^2}{4}-\frac{17ln^2 2}{2}+5(ln2)(ln3)-2Li_2\left(\frac{1}{4}\right)-3\left(Li_2\left(\frac{2}{3}\right)-Li_2\left(\frac{1}{3}\right)\right)$$

Now use the following identities:

$$Li_2(z)+Li_2(1-z)=\frac{\pi^2}{6}-ln(x)ln(1-x)\underbrace{\Rightarrow}_{z=\frac{1}{3}}Li_2\left(\frac{2}{3}\right)+Li_2\left(\frac{1}{3}\right)$$

$$=\frac{\pi^2}{6}-ln^2 3+(ln2)(ln3)\quad \dots\dots\dots\dots(A)$$

$$Li_2(1-z)+Li_2(1-z^{-1})=-\frac{ln^2 z}{2}\underbrace{\Rightarrow}_{z=\frac{2}{3}}Li_2\left(\frac{1}{3}\right)+Li_2\left(-\frac{1}{2}\right)=(ln2)(ln3)-\frac{ln^2 2}{2}-\frac{ln^2 3}{2}\quad \dots\dots\dots(B)$$

$$Li_2(z)+Li_2(-z)=\frac{1}{2}Li_2(z^2)\underbrace{\Rightarrow}_{z=\frac{1}{2}}Li_2\left(\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)=\frac{1}{2}Li_2\left(\frac{1}{4}\right)\quad \dots\dots\dots(C)$$

Now use, $(A)-2(B)+2(C)$

$$\Rightarrow Li_2\left(\frac{2}{3}\right)-Li_2\left(\frac{1}{3}\right)+2Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{6}-ln^2 3+(ln2)(ln3)-2(ln2)(ln3)+ln^2 2+ln^2 3+Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow Li_2\left(\frac{2}{3}\right)-Li_2\left(\frac{1}{3}\right)+\frac{\pi^2}{6}-ln^2 2=\frac{\pi^2}{6}-(ln2)(ln3)+ln^2 2+Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow Li_2\left(\frac{2}{3}\right)-Li_2\left(\frac{1}{3}\right)=2ln^2 2-(ln2)(ln3)+Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow I=\frac{\pi^2}{4}-\frac{17ln^2 2}{2}+5(ln2)(ln3)-2Li_2\left(\frac{1}{4}\right)-3\left(2ln^2 2-(ln2)(ln3)+Li_2\left(\frac{1}{4}\right)\right)$$

$$\Rightarrow I=\frac{\pi^2}{4}-\frac{29ln^2 2}{2}+8(ln2)(ln3)-5Li_2\left(\frac{1}{4}\right)$$

2546. Prove that:

$$\iint_{[0,1]^2}tan^{-1}\left(\frac{x+y}{x+1}\right)dxdy=\frac{1}{8}(3\pi-22ln2+7ln5-2cot^{-1}2)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shobhit Jain-India

$$I=\int_{x=0}^{1}\int_{y=0}^{1}tan^{-1}\left(\frac{x+y}{x+1}\right)dydx\underbrace{=}_{\substack{x\to x-1\\ y\to 1-y}}\int_{x=1}^{2}\int_{y=0}^{1}tan^{-1}\left(\frac{x-y}{x}\right)dydx\underbrace{=}_{x\to\frac{1}{x}}$$

$$=\int_{x=\frac{1}{2}}^{1}\left(\int_{y=0}^{1}\frac{tan^{-1}(1-yx)}{x^2}dy\right)dx\underbrace{=}_{y\to\frac{y}{x}}\int_{x=\frac{1}{2}}^{1}\left(\int_{y=0}^{x}\frac{tan^{-1}(1-y)}{x^3}dy\right)dx\underbrace{=}_{y\to 1-y}$$

$$=\int_{x=\frac{1}{2}}^{1}\left(\int_{y=1-x}^{1}\frac{tan^{-1}(y)}{x^3}dy\right)dx\underbrace{=}_{x\to 1-x}\int_{x=0}^{\frac{1}{2}}\frac{\left(\int_{y=x}^{1}tan^{-1}(y)\,dy\right)}{(1-x)^3}dx=$$

$$=\int_0^{\frac{1}{2}}\frac{f(1)-f(x)}{(1-x)^3}dx \qquad Here,\ f(x)=\int_0^x tan^{-1}(y)\,dy = xtan^{-1}(x)-\frac{1}{2}\ln(1+x^2)$$

$$=\int_0^{\frac{1}{2}}(f(1)-f(x))\,d\left[\frac{1}{2(1-x)^2}\right]=\left[\frac{f(1)-f(x)}{2(1-x)^2}\right]_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}}\frac{f'(x)}{2(1-x)^2}dx=$$

$$=\frac{3}{2}f(1)-2f\left(\frac{1}{2}\right)+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{tan^{-1}(x)}{(1-x)^2}dx$$

$$Now,\quad \int_0^{\frac{1}{2}}\frac{tan^{-1}(x)}{(1-x)^2}dx=\int_0^{\frac{1}{2}}tan^{-1}(x)d\left[\frac{1}{(1-x)}\right]\underbrace{=}_{IBP}\left[\frac{tan^{-1}(x)}{1-x}\right]_0^{\frac{1}{2}}-\int_{x=0}^{\frac{1}{2}}\frac{1}{1-x}d\left[tan^{-1}(x)\right]$$

$$=2\phi-\int_0^{\phi}\frac{1}{1-tan\theta}d\theta=2\phi-\int_0^{\phi}\frac{cos\theta}{cos\theta-sin\theta}d\theta \qquad Here,\ \phi=tan^{-1}\left(\frac{1}{2}\right)=cot^{-1}(2)$$

$$=2\phi+\int_0^{\phi}\frac{cos\theta}{sin\theta-cos\theta}d\theta=2\phi+\frac{1}{2}\int_0^{\phi}\left(\frac{cos\theta+sin\theta}{sin\theta-cos\theta}-1\right)d\theta=\frac{3}{2}\phi+\frac{1}{2}ln|sin\phi-cos\phi|=$$

$$=\frac{3}{2}\phi+\frac{1}{2}ln\left|\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}\right|=\frac{3}{2}\phi-\frac{1}{4}ln5$$

$$\Rightarrow I=\frac{3}{2}f(1)-2f\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{3}{2}\phi-\frac{1}{4}ln5\right)=\frac{3}{2}f(1)-2f\left(\frac{1}{2}\right)+\frac{3}{4}\phi-\frac{ln5}{8}$$

$$Now,\quad f(x)=xtan^{-1}(x)-\frac{1}{2}\ln(1+x^2)$$

$$\Rightarrow f(1)=\frac{\pi}{4}-\frac{1}{2}ln2 \qquad and \qquad f\left(\frac{1}{2}\right)=\frac{\phi}{2}-\frac{ln5}{2}+ln2$$

$$\Rightarrow I=\frac{3}{2}\left(\frac{\pi}{4}-\frac{1}{2}ln2\right)-2\left(\frac{\phi}{2}-\frac{ln5}{2}+ln2\right)+\frac{3}{4}\phi-\frac{ln5}{8}=\frac{3\pi}{8}-\frac{11}{4}ln2-\frac{\phi}{4}+\frac{7}{8}ln5$$

$$\Rightarrow I=\frac{1}{8}(3\pi-22ln2-2cot^{-1}2+7ln5)$$

2547. Find:

$$I=\int_0^1\frac{x^2ln^2(1+x^2)}{(1+x)(1+x^2)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1\frac{x^2ln^2(1+x^2)}{(1+x)(1+x^2)}dx=$$

$$=\underbrace{\int_0^1\frac{ln^2(1+x^2)}{1+x}dx}_{A}-\frac{1}{2}\underbrace{\int_0^1\frac{ln^2(1+x^2)}{1+x^2}dx}_{B}+\frac{1}{2}\underbrace{\int_0^1\frac{xln^2(1+x^2)}{1+x^2}dx}_{C}$$

Calculate the given integral's:

$$A=\int_0^1\frac{ln^2(1+x^2)}{1+x}dx\ \left\{u=ln^2(1+x^2),\frac{du}{dx}=\frac{4xln(1+x^2)}{1+x^2},v=\ln(1+x)\ dv=\frac{1}{1+x}\right\}$$

$$A=ln^2(1+x^2)\ln(1+x)\Big|_0^1-4\underbrace{\int_0^1\frac{xln(1+x^2)\ln(1+x)}{1+x^2}dx}_{M}=$$

$$A=ln^3(2)-4\underbrace{\int_0^1\frac{xln(1+x^2)\ln(1+x)}{1+x^2}dx}_{M}$$

$$\underbrace{\int_0^1\frac{xln(1+x^2)\ln(1+x)}{1+x^2}dx}_{M}=\frac{1}{2}\int_0^1\frac{xln(1-x^2)\ln(1+x^2)}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{xln\left(\frac{1+x}{1-x}\right)\ln(1+x^2)}{1+x^2}dx$$

$$=\frac{1}{2}(A_1+A_2)$$

$$A_1=\int_0^1\frac{xln(1-x^2)\ln(1+x^2)}{1+x^2}dx\ \left[x^2=y,dy=2xdx,dx=\frac{dy}{2x}\right]$$

$$\int_0^1\frac{xln(1-x^2)\ln(1+x^2)}{1+x^2}dx$$

$$=\int_o^1\frac{xln(1-y)\ln(1+y)}{1+y}\frac{dy}{2x}=\frac{1}{2}\int_0^1\frac{\ln(1+y)\ln(1-y)}{1+y}dy$$

$$A_1=\frac{1}{16}\zeta(3)+\frac{1}{6}ln^3(2)-\frac{\ln(2)}{4}\zeta(2),\quad A_2=\frac{3}{8}\zeta(2)\ln(2)-\frac{21}{16}\zeta(3)+\frac{\pi}{2}G$$

$$M=\frac{1}{2}(A_1+A_2)=\frac{1}{2}\left(\frac{1}{16}\zeta(3)+\frac{1}{6}ln^3(2)-\frac{\ln(2)}{4}\zeta(2)\right)+$$

$$\frac{1}{2}\left(\frac{3}{8}\zeta(2)\ln(2)-\frac{21}{16}\zeta(3)+\frac{\pi}{2}G\right)=-\frac{5}{8}\zeta(3)+\frac{1}{12}ln^3(2)+\frac{1}{16}\zeta(2)\ln(2)+\frac{\pi}{4}G$$

$$A=ln^3(2)-4M=ln^3(2)-4\left(-\frac{5}{8}\zeta(3)+\frac{1}{12}ln^3(2)+\frac{1}{16}\zeta(2)\ln(2)+\frac{\pi}{4}G\right)$$

$$A=\frac{2}{3}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G$$

$$\int_0^1\frac{ln^2(1+x^2)}{1+x^2}dx=\frac{2}{3}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G$$

$$C=\int_0^1\frac{xln^2(1+x^2)}{1+x^2}dx$$

$$C=\frac{1}{2}ln^2(1+x^2)ln(1+x^2)\Big|_0^1-2\underbrace{\int_0^1\frac{xln^2(1+x^2)}{1+x^2}dx}_{C}=\frac{1}{2}ln^3(2)-2C$$

$$3C=\frac{1}{2}ln^3(2),\quad C=\frac{1}{6}ln^3(2),\quad C=\int_0^1\frac{xln^2(1+x^2)}{1+x^2}dx=\frac{1}{6}ln^3(2)$$

$$\underbrace{\int_0^1 \frac{ln^2(1+x^2)}{1+x^2}dx}_{B}$$

$$= \int_0^{\frac{\pi}{4}} \frac{ln^2\left(1+tan^2(y)\right)}{1+tan^2(y)}.sec^2(y)dy = \int_0^{\frac{\pi}{4}} \frac{ln^2\left(sec^2(y)\right)}{sec^2(y)}.sec^2(y)dy =$$

$$\int_0^{\frac{\pi}{4}} ln^2\left(sec^2(y)\right)dy = 4\int_0^{\frac{\pi}{4}} ln^2(\cos(y))dy = 4\int_0^{\frac{\pi}{4}} ln^2(\cos(x))dx = 4\int_0^{\frac{\pi}{4}} ln^2[2\cos(x)]dx -$$

$$4ln^2(2)\int_0^{\frac{\pi}{4}} dx - 2\ln(2)\int_0^{\frac{\pi}{4}} ln[\cos(x)]dx$$

Recall that :

$$ln^2[2\cos(x)] = x^2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n H_{n-1}}{n}\cos(2nx);\ ln[\cos(x)] = -\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}$$

$$B = 4\int_0^{\frac{\pi}{4}} x^2dx + 8\sum_{n=1}^{\infty}\frac{(-1)^n H_{n-1}}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx - 4ln^2(2)(\frac{\pi}{4}) + 2ln^2(2)\int_0^{\frac{\pi}{4}} dx +$$

$$2\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx = \frac{4}{3}\left(\frac{\pi}{4}\right)^3 + 8\sum_{n=1}^{\infty}\frac{(-1)^n H_{n-1}}{n}\left[\frac{\sin\left(\frac{\pi n}{2}\right)}{2n}\right] -$$

$$\pi ln^2(2) + 2ln^2(2)\left(\frac{\pi}{4}\right) + 2\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\sin\left(\frac{\pi n}{2}\right)}{2n}\right]$$

$$= \frac{\pi^3}{48} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)\left[H_n - \frac{1}{n}\right] -$$

$$\frac{1}{2}\pi ln^2(2) + \ln(2)\,(-G)$$

$$B = \frac{\pi^3}{48} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)H_n - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{\pi n}{2}\right) - \frac{1}{2}\pi ln^2(2) - G\ln(2)$$

$$B = \frac{\pi^3}{48} + 4S(n) - 4\left[-\frac{\pi^3}{32}\right] - \frac{1}{2}\pi ln^2(2) - G\ln(2)$$

$$S(n) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)H_n,\qquad S(n) = -\int_0^1 x^{n-1}\ln(1-x)\,dx$$

$$S(n) = -\int_0^1\frac{\ln(1-x)}{x}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{\pi n}{2}\right)dx,\qquad S(n) = -\int_0^1\frac{\ln(1-x)}{x}(-\arctan(x))dx$$

$$S(n) = \int_0^1\frac{\ln(1-x)\arctan(x)}{x}dx\ \ (I.B.P),$$

$$S(n) = -arctan(x)Li_2(x)\Big|_0^1 + \int_0^1\frac{Li_2(x)}{1+x^2}dx$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}Li_2(x)dx\,,S(n)$$

$$=-\frac{\pi}{4}\zeta(2)+\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2n+1}-\frac{H_{2n+1}}{(2n+1)^2}\right]$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}-\sum_{n=0}^{\infty}\frac{H_{2n+1}}{(2n+1)^2}$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\frac{\pi}{4}-\left[-\mathcal{J}Li_3(1-i)-\frac{\pi}{16}ln^2(2)-\frac{1}{2}\ln(2)\,G\right]$$

$$S(n)=-\frac{\pi}{4}\zeta(2)+\frac{\pi}{4}+\mathcal{J}Li_3(1-i)+\frac{\pi}{16}ln^2(2)+\frac{1}{2}\ln(2)\,G$$

$$B=\frac{\pi^3}{48}+4\left(-\frac{\pi}{4}\zeta(2)+\frac{\pi}{4}+\mathcal{J}Li_3(1-i)+\frac{\pi}{16}ln^2(2)+\frac{1}{2}\ln(2)\,G\right)-4\left[-\frac{\pi^3}{32}\right.$$
$$\left.-\frac{1}{2}\pi ln^2(2)-G\ln(2)\right]$$

$$B=-\frac{\pi^3}{48}+\pi+4\mathcal{J}Li_3(1-i)-\frac{\pi}{4}ln^2(2)+G\ln(2)$$

$$\int_0^1\frac{x^2ln^2(1+x^2)}{(1+x)(1+x^2)}dx=A-B+C=\left(\frac{2}{3}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G\right)-$$
$$-\frac{1}{2}\left(-\frac{\pi^3}{48}+\pi+4\mathcal{J}Li_3(1-i)-\frac{\pi}{4}ln^2(2)+G\ln(2)\right)+\frac{1}{2}\left(\frac{1}{6}ln^3(2)\right)$$

$$\int_0^1\frac{x^2ln^2(1+x^2)}{(1+x)(1+x^2)}dx$$
$$=\frac{9}{12}ln^3(2)+\frac{5}{2}\zeta(3)-\frac{1}{4}\zeta(2)\ln(2)-\pi G-\frac{\pi^3}{96}-\frac{\pi}{2}-2\mathcal{J}Li_3(1-i)+$$
$$\frac{\pi}{8}ln^2(2)-\frac{G}{2}ln(2)$$

2548. Find:

$$\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Kartick Chandra Betal-India

$$\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx=\sum_{n=1}^{\infty}(-1)^{n-1}.\frac{n(n+1)}{2}\int_0^1 x^{n+1}ln^3(x)dx=$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}.\frac{n(n+1)}{2}\left[\frac{x^{n+2}ln^3(x)}{n+2}-\frac{3x^{n+2}ln^2(x)}{(n+2)^2}+\frac{6x^{n+2}\ln(x)}{(n+2)^3}-\frac{6x^{n+2}}{(n+2)^4}\right]_0^1=$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}.\frac{n(n+1)}{2}\left[-\frac{6}{(n+2)^4}\right]=3\sum_{n=1}^{\infty}(-1)^n.\frac{n(n+1)}{(n+2)^4}=$$

$$3\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{(n+2)^2}-\frac{3}{(n+2)^3}+\frac{2}{(n+2)^4}\right\}=3\sum_{n=3}^{\infty}(-1)^{n-1}\left\{-\frac{1}{n^2}+\frac{3}{n^3}-\frac{2}{n^4}\right\}=$$

$$3\left[\sum_{n=1}^{\infty}(-1)^{n-1}\left\{-\frac{1}{n^2}+\frac{3}{n^3}-\frac{2}{n^4}\right\}\right]=3\{-\eta(2)+3\eta(3)-2\eta(4)\}=$$

$$3\left\{-\frac{\pi^2}{12}+3.\frac{3}{4}\zeta(3)-2.\frac{7}{8}.\zeta(4)\right\}=3\left(-\frac{\pi^2}{12}+\frac{9}{4}\zeta(3)-\frac{7}{4}.\frac{\pi^4}{90}\right)=$$

$$\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

$$\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx=\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

Solution 2 by Pham Duc Nam-Vietnam

$$*\ n,m\in N: I(m,n)=\int_0^1\frac{ln^n(x)}{(1+x)^m}dx=\frac{(-1)^{m-1}}{(m-1)!}.\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\int_0^1\frac{ln^n(x)}{a+x}dx=$$

$$\frac{(-1)^{m-1}}{(m-1)!}.\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\sum_{k=0}^{\infty}\frac{(-1)^k}{a^{k+1}}\int_0^1x^kln^n(x)dx=$$

$$\frac{(-1)^{m-1}(-1)^nn!}{(m-1)!}.\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\sum_{k=0}^{\infty}\frac{(-1)^k}{a^{k+1}(1+k)^{n+1}}$$

$$=\frac{(-1)^{m-1}(-1)^nn!}{(m-1)!}.\frac{\partial^{m-1}}{\partial a^{m-1}}\bigg|_{a=1}\left(-Li_{n+1}\left(-\frac{1}{a}\right)\right)$$

$$*\ I=\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx=\int_0^1\left(\frac{1}{(1+x)^3}-\frac{2}{(1+x)^2}+\frac{1}{1+x}\right)ln^3(x)dx$$

$$=I(3,3)-2I(3,2)+I(3,1)$$

$$I(3,3)=\frac{(-1)^2(-1)^33!}{(3-1)!}.\frac{\partial^2}{\partial a^2}\bigg|_{a=1}\left(-Li_4\left(-\frac{1}{a}\right)\right)=-\frac{9}{4}\zeta(3)-\frac{3}{2}\zeta(2)$$

$$I(3,2)=\frac{(-1)^1(-1)^33!}{(1)!}.\frac{\partial^1}{\partial a^1}\bigg|_{a=1}\left(-Li_3\left(-\frac{1}{a}\right)\right)=-\frac{9}{2}\zeta(3)$$

$$I(3,1)=\frac{(-1)^0(-1)^33!}{(0)!}\left(-Li_4(-1)\right)=-\frac{21}{4}\zeta(4)$$

$$*\ I=I(3,3)-2I(3,2)+I(3,1)=-\frac{9}{4}\zeta(3)-\frac{3}{2}\zeta(2)+2.\frac{9}{2}\zeta(3)-\frac{21}{4}\zeta(4)$$

$$\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx=\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

Solution 3 by Shobhit Jain-India

$$\Omega=\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx$$

$$Now\ ,\qquad (1+x)^{-3}=\sum_{n=0}{}^{n+2}_{\ \ 2}C(-1)^n x^n$$

$$\Omega=\sum_{n=0}{}^{n+2}_{\ \ 2}C(-1)^n\int_0^1 x^{n+2}ln^3(x)dx=-\Gamma 4\sum_{n=0}\frac{{}^{n+2}_{\ \ 2}C(-1)^n}{(n+3)^4}=$$

$$-6\sum_{n=2}\frac{{}^{n}_{2}C(-1)^n}{(n+1)^4}=-3\sum_{n=2}(-1)^n\frac{n(n-1)}{(n+1)^4}=-3\sum_{n=0}(-1)^n\frac{n(n-1)}{(n+1)^4}=$$

$$-3\sum_{n=1}(-1)^{n-1}\frac{(n-1)(n-2)}{n^4}=-3\sum_{n=1}(-1)^{n-1}\left\{\frac{1}{n^2}-\frac{3}{n^3}+\frac{2}{n^4}\right\}=$$

$$-3\{\eta(2)-3\eta(3)+2\eta(4)\}=-3\eta(2)+9\eta(3)-6\eta(4)=$$

$$-\frac{3}{2}\zeta(2)+\frac{27}{4}\zeta(3)-\frac{21}{4}\zeta(4)=\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

$$\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx=\frac{27}{4}\zeta(3)-\frac{\pi^2}{4}-\frac{7\pi^4}{120}$$

Solution 4 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^1\frac{x^2ln^3(x)}{(1+x)^3}dx=\int_0^1\frac{ln^3(x)}{1+x}dx-2\int_0^1\frac{ln^3(x)}{(1+x)^2}dx+\int_0^1\frac{ln^3(x)}{(1+x)^3}dx=\Omega_1-2\Omega_2+\Omega_3$$

$$\Omega_1=\int_0^1\frac{ln^3(x)}{1+x}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\,ln^3(x)dx=-6\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4}=$$

$$-6\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}=-\,6\eta(4)=-6.\frac{7\pi^4}{720}=-\frac{7\pi^4}{120}$$

$$\Omega_2=\int_0^1\frac{ln^3(x)}{(1+x)^2}dx=-\sum_{n=1}^{\infty}(-1)^n n\int_0^1 x^{n-1}ln^3(x)dx=6\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^4}=-6\eta(3)=$$

$$-6.\frac{3}{4}\zeta(3)=-\frac{9}{2}\zeta(3)$$

$$\Omega_3=\int_0^1\frac{ln^3(x)}{(1+x)^3}dx=\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)\int_0^1 x^{n-2}ln^3(x)dx=-3\sum_{n=2}^{\infty}\frac{(-1)^n n(n-1)}{(n-1)^4}=$$

$$-3\sum_{n=2}^{\infty}\frac{(-1)^n n}{(n-1)^3}=-3\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)^2}-3\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)^3}=-3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}-3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=$$

$$-3\eta(2)-3\eta(3)=-3.\frac{\pi^2}{12}-3.\frac{3}{4}\zeta(3)=-\frac{\pi^2}{4}-\frac{9}{4}\zeta(3)$$

$$\Omega=\Omega_1-2\Omega_2+\Omega_3=-\frac{7\pi^4}{120}+9\zeta(3)-\frac{\pi^2}{4}-\frac{9}{4}\zeta(3)$$

$$\int_0^1 \frac{x^2 ln^3(x)}{(1+x)^3}dx = \frac{27}{4}\zeta(3) - \frac{\pi^2}{4} - \frac{7\pi^4}{120}$$

Solution 5 by Quadri Faruk Temitope-Nigeria

$$I = \int_0^1 \frac{x^2 ln^3(x)}{(1+x)^3}dx$$

$$I = \int_0^1 \frac{ln^3(x)}{1+x}dx - 2\int_0^1 \frac{ln^3(x)}{(1+x)^2}dx + \int_0^1 \frac{ln^3(x)}{(1+x)^3}dx$$

$$I = \sum_{n=0}^{\infty}\binom{-3}{n}\int_0^1 x^n ln^3(x)dx - 2\sum_{n=0}^{\infty}\binom{-2}{n}\int_0^1 x^n\, ln^3(x)dx + \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n ln^3(x)dx$$

$$I = \sum_{n=0}^{\infty}\binom{-3}{n}\left[-\frac{6}{(n+1)^4}\right] - 2\sum_{n=0}^{\infty}\binom{-2}{n}\left[-\frac{6}{(n+1)^4}\right] + \sum_{n=0}^{\infty}(-1)^n\left[-\frac{6}{(n+1)^4}\right]$$

$$I = -6\sum_{n=0}^{\infty}\frac{\binom{-3}{n}}{(n+1)^4} + 12\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{(n+1)^4} - 6\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4}$$

$$I = -6\left[\frac{3}{8}\zeta(3) + \frac{1}{4}\zeta(2)\right] + 12\left[\frac{3}{4}\zeta(3)\right] - 6\left[\frac{7}{8}\zeta(4)\right]$$

$$\int_0^1 \frac{x^2 ln^3(x)}{(1+x)^3}dx = \frac{27}{4}\zeta(3) - \frac{\pi^2}{4} - \frac{7\pi^4}{120}$$

2549. Find:

$$\Omega = \int_0^1 \frac{ln^3(x+1)}{(1+x)(3+x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\int_0^1 \frac{ln^3(x+1)}{(1+x)(3+x)}dx = \frac{1}{2}\int_0^1 \frac{ln^3(x+1)}{(1+x)}dx - \frac{1}{2}\int_0^1 \frac{ln^3(x+1)}{(3+x)}dx$$

$$= \frac{1}{2}\int_0^1 d\left(\frac{1}{4}ln^4(1+x)\right) -$$

$$\frac{1}{2}\int_0^{\frac{1}{2}} \frac{ln^3(x)}{x}dx + \int_0^{\frac{1}{2}} \frac{ln^3(x)}{1+2x}dx = \frac{ln^4(2)}{8}$$

$$-\frac{1}{2}\int_0^{\frac{1}{2}} d\left(\frac{1}{4}ln^4(x)\right) - \int_0^1 \frac{ln^3(x)}{1+2x}dx + \int_0^{\frac{1}{2}} \frac{ln^3(x)}{1+2x}dx =$$

$$\frac{1}{2}\int_0^1 -\frac{2ln^3(x)}{1+2x}dx + \frac{1}{2}\int_0^1 \frac{ln^3\left(\frac{x}{2}\right)}{1+x}dx = -3Li_4(-2) +$$

$$\frac{1}{2}\sum_{k=0}^{\infty}\binom{3}{k}ln^{3-k}\left(\frac{1}{2}\right)\left(\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{n-1}ln^k(x)dx\right)=-3Li_4(-2)+$$

$$\frac{1}{2}\sum_{k=0}^{\infty}\binom{3}{k}ln^{3-k}\left(\frac{1}{2}\right)\left((-1)^k k!\sum_{n\in N}\frac{(-1)^{n-1}}{n^{k+1}}\right)=-3Li_4(-2)+$$

$$\frac{1}{2}\sum_{k=0}^{\infty}\binom{3}{k}ln^{3-k}\left(\frac{1}{2}\right)\left((-1)^k k!\,\eta(k+1)\right)=-3Li_4(-2)+$$

$$\frac{1}{2}\left(ln^3\left(\frac{1}{2}\right)\ln(2)-\frac{\pi^2}{4}ln^2(2)+\frac{9}{2}\zeta(3)\ln\left(\frac{1}{2}\right)-\frac{7\pi^4}{120}\right)$$

$$\int_0^1\frac{ln^3(x+1)}{(1+x)(3+x)}dx=-3Li_4(-2)-\frac{\pi^4}{240}-\frac{9}{4}\zeta(3)\ln(2)-\frac{\pi^2}{8}ln^2(2)-\frac{1}{2}ln^4(2)$$

Solution 2 by Pham Duc Nam-Vietnam

$$I=\int_0^1\frac{ln^3(x+1)}{(1+x)(3+x)}dx$$

$$Let\ :x=\frac{1}{t}-1\to\ \ I=-\int_{\frac{1}{2}}^1\frac{ln^3(t)}{2t+1}dt=-\int_{\frac{1}{2}}^1\frac{ln^3(t)}{2t+1}d\left(\frac{1}{2}\ln(2t+1)\right)=$$

$$-\frac{1}{2}ln^4(2)+\frac{3}{2}\int_{\frac{1}{2}}^1 ln^2(t)\,d\left(-Li_2(-2t)\right)=$$

$$-\frac{1}{2}ln^4(2)+\frac{3}{2}\left(-\frac{\pi^2}{12}ln^2(2)+2\int_{\frac{1}{2}}^1\ln(t)\,d(\,Li_3(-2t)\right)=$$

$$-\frac{1}{2}ln^4(2)-\frac{\pi^2}{8}ln^2(2)+3(-\frac{3}{4}\zeta(3)\ln(2)-\int_{\frac{1}{2}}^1 d\left(Li_4(-2t)\right)=$$

$$-\frac{1}{2}ln^4(2)-\frac{\pi^2}{8}ln^2(2)-\frac{9}{4}\zeta(3)\ln(2)-3\left(Li_4(-2)+\frac{7\pi^4}{120}\right)=$$

$$-3Li_4(-2)-\frac{7\pi^4}{240}-\frac{9}{4}\zeta(3)\ln(2)-\frac{\pi^2}{8}ln^2(2)-\frac{1}{2}ln^4(2)$$

Solution 3 by Shobhit Jain-India

$$\Omega=\int_0^1\frac{ln^3(1+x)}{(1+x)(3+x)}dx\underset{x\to x-1}{=}\int_1^2\frac{ln^3x}{x(2+x)}dx=\frac{1}{2}\int_1^2\frac{ln^3x}{x}dx-\frac{1}{2}\int_1^2\frac{ln^3x}{(2+x)}dx$$

$$=\frac{ln^4 2}{8}-\frac{1}{4}\int_1^2 ln^3x\left(1+\frac{x}{2}\right)^{-1}dx=\frac{ln^4 2}{8}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\int_1^2(ln^3x)x^{n-1}dx$$

$$Now,\ \int(ln^3x)x^{n-1}dx=(ln^3x)\frac{x^n}{n}-3(ln^2x)\frac{x^n}{n^2}+6(lnx)\frac{x^n}{n^3}-6\frac{x^n}{n^4}+C$$

$$\Rightarrow \int_1^2 (ln^3x)x^{n-1}dx = (ln^3 2)\frac{2^n}{n} - 3(ln^2 2)\frac{2^n}{n^2} + 6(ln2)\frac{2^n}{n^3} - 6\frac{2^n}{n^4} + \frac{6}{n^4}$$

$$\Omega = \frac{ln^4 2}{8} + \frac{1}{2}\sum_{n=1}^{\infty}\left((ln^3 2)\frac{(-1)^n}{n} - 3(ln^2 2)\frac{(-1)^n}{n^2} + 6(ln2)\frac{(-1)^n}{n^3} - 6\frac{(-1)^n}{n^4} + \frac{6(-1)^n}{n^4 2^n}\right)$$

$$= \frac{ln^4 2}{8} - \frac{ln^4 2}{2} + \frac{3}{2}(ln^2 2)\eta(2) - 3(ln2)\eta(3) + 3\eta(4) + 3Li_4\left(-\frac{1}{2}\right)$$

$$= \frac{ln^4 2}{8} - \frac{ln^4 2}{2} + \frac{3}{4}(ln^2 2)\zeta(2) - \frac{9}{4}(ln2)\zeta(3) + \frac{21}{8}\zeta(4) + 3Li_4\left(-\frac{1}{2}\right)$$

$$= -\frac{3}{8}ln^4(2) + \frac{\pi^2}{8}\left(ln^2(2)\right) - \frac{9}{4}\ln(2)\zeta(3) + \frac{7\pi^4}{240}ln(2) + 3Li_4\left(-\frac{1}{2}\right)$$

Solution 4 by Cosghun Memmedov-Azerbaijan

$$\Omega = \int_0^1 \frac{ln^3(1+x)}{(1+x)(3+x)}dx = \frac{1}{2}\underbrace{\int_0^1 \frac{ln^3(1+x)}{(1+x)}dx}_{\Omega_1} - \frac{1}{2}\underbrace{\int_0^1 \frac{ln^3(1+x)}{(3+x)}dx}_{\Omega_2}$$

$$\Omega_1 = \int_0^1 \frac{ln^3(1+x)}{(1+x)}dx \overset{1+x\to x}{\triangleq} \underbrace{\int_1^2 \frac{ln^3(x)}{x}dx}_{A} \overset{IBP}{\triangleq} ln^4(2) - 3\underbrace{\int_1^2 \frac{ln^3(x)}{x}dx}_{A}$$

$$A = ln^4(2) - 3A \rightarrow \Omega_1 = A = \frac{ln^4(2)}{4}$$

$$\Omega_2 = \int_0^1 \frac{ln^3(1+x)}{(3+x)}dx \overset{1+x\to x}{\triangleq} \int_1^2 \frac{ln^3(x)}{2+x}dx = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\int_1^2 x^n\, ln^3(x)dx \overset{IBP}{\triangleq}$$

$$ln^3(x)\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} - 3ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} + 6\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} - 6\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4} +$$

$$3\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^{n+1}}{(n+1)^4} = -\frac{\pi^2}{4}ln^2(2) - 6Li_4\left(-\frac{1}{2}\right) + \frac{9}{2}\zeta(3)\ln(2) - \frac{7\pi^4}{240} + ln^4(2)$$

$$\Omega = \frac{1}{2}(\Omega_1 - \Omega_2) = -3Li_4(-2) - \frac{7\pi^4}{240} - \frac{9}{4}\zeta(3)\ln(2) - \frac{\pi^2}{8}ln^2(2) - \frac{1}{2}ln^4(2)$$

Solution 5 by Quadri Faruk Temitope-Nigeria

$$\mathrm{I} = \int_0^1 \frac{ln^3(1+x)}{(1+x)(3+x)}dx$$

$$I=\int_0^1 \frac{ln^3(1+x)}{(1+x)(3+x)}dx=\frac{1}{2}\int_0^1 \frac{ln^3(1+x)}{(1+x)}dx-\frac{1}{2}\int_0^1 \frac{ln^3(1+x)}{(3+x)}dx$$

$$Le\ :\ 1+x=p\to\ dx=dp,\ I=\frac{1}{2}\int_1^2 \frac{ln^3(p)}{(p)}dp-\frac{1}{2}\int_1^2 \frac{ln^3(p)}{(p+2)}dp$$

$$I=\frac{1}{2}\cdot\frac{1}{4}ln^4(p)\Big|_1^2-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{1+n}}\int_1^2 p^n\, ln^3(p)dp,$$

$$I=\frac{1}{8}ln^4(2)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n}\int_1^2 p^{n-1}\, ln^3(p)dp$$

$$I=\frac{1}{8}ln^4(2)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\left[\frac{6}{n^4}-\frac{6.2^n}{n^4}+\frac{6.2^n\ln(2)}{n^3}-3ln^2(2)\frac{2^n}{n^2}+ln^3(2)\frac{2^n}{n}\right]$$

$$I=-3Li_4(-2)-\frac{7\pi^4}{240}-\frac{9}{4}\zeta(3)\ln(2)-\frac{\pi^2}{8}ln^2(2)-\frac{1}{2}ln^4(2)$$

Solution 6 by Obiajunwa Januarius-Nigeria

$$\int_0^1 \frac{ln^3(1+x)}{(1+x)(3+x)}dx=\frac{1}{2}\int_0^1 \frac{ln^3(1+x)}{(1+x)}dx-\frac{1}{2}\int_0^1 \frac{ln^3(1+x)}{(3+x)}dx=$$

$$\frac{1}{2}\lim_{n\to-1}\int_0^1 (1+x)^n\, ln^3(1+x)dx-\frac{1}{4}\int_0^1 \frac{ln^3(1+x)}{1+\frac{1+x}{2}}dx=$$

$$\frac{1}{2}\lim_{n\to-1}\frac{\partial^3}{\partial n^3}\left[\int_0^1 (1+x)^n dx\right]-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\int_0^1 (1+x)^n ln^3(1+x)dx=$$

$$\frac{1}{2}\lim_{n\to-1}\frac{\partial^3}{\partial n^3}\left[\frac{(1+x)^{n+1}}{n+1}\Big|_0^1\right]-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{\partial^3}{\partial n^3}\left[\frac{(1+x)^{n+1}}{n+1}\Big|_0^1\right]=$$

$$\frac{1}{2}\lim_{n\to-1}\frac{\partial^3}{\partial n^3}\left[\frac{2^{n+1}}{n+1}-\frac{1}{n+1}\right]-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{\partial^3}{\partial n^3}\left[\frac{2^{n+1}}{n+1}-\frac{1}{n+1}\right]=$$

$$\frac{1}{2}\lim_{n\to1}\left[\frac{6}{(n+1)^4}-\frac{12.2^n}{(n+1)^4}+\frac{12.\ln(2).2^n}{(n+1)^3}-\frac{6ln^2(2)2^n}{(n+1)^2}+\frac{2ln^3(2)2^n}{n+1}\right]-$$

$$\frac{6}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{1}{(n+1)^4}+\frac{12}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n}{(n+1)^4}-\frac{12}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n\ln(2)}{(n+1)^3}+$$

$$\frac{6}{4}ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n}{(n+1)^2}-\frac{2}{4}ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{2^n}{(n+1)}$$

$Extended\ \ L'Hopital's\ \ rule\ :$

$$\lim_{n\to p}\frac{f(x)}{g(x)}=\lim_{n\to p}\frac{f^n(x)}{g^n(x)}$$

$$\Omega = \frac{1}{2}\lim_{n\to -1}\Big[-\frac{12}{4!}2^n ln^4(2) + \frac{12}{3!}\ln(2).2^n ln^3(2) - \frac{6}{2!}ln^2(2)2^n ln^2(2) + 2ln^3(2)2^n\ln(2)\Big] +$$

$$3\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}(n+1)^4} + 3\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4} - 3\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} +$$

$$\frac{3}{2}ln^2(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} - \frac{ln^3(2)}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \frac{1}{2}[-\frac{ln^4(2)}{4} + ln^4(2) - \frac{3}{3}ln^4(2) + ln^4(2)] +$$

$$3Li_4\left(-\frac{1}{2}\right) + 3\eta(4) - 3\ln(2)\eta(3) + \frac{3}{2}ln^2(2)\eta(2) - \frac{ln^4(2)}{2}$$

$$\Omega = \frac{ln^4(2)}{8} + 3Li_4\left(-\frac{1}{2}\right) + \frac{21}{8}\zeta(4) - \frac{9}{4}\ln(2)\zeta(3) + \frac{3}{4}ln^2(2)\zeta(2) - \frac{ln^4(2)}{2}$$

$$\Omega = -3Li_4(-2) - \frac{7\pi^4}{240} - \frac{9}{4}\zeta(3)\ln(2) - \frac{\pi^2}{8}ln^2(2) - \frac{1}{2}ln^4(2)$$

2550. Find:

$$\int_0^{\infty}\frac{x|\ln(x)|}{(x^2+1)(x^2+x+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Kartick Chandra Betal-India

$$I = \int_0^{\infty}\frac{x|\ln(x)|}{(x^2+1)(x^2+x+1)}dx = -\int_0^1\frac{x\ln(x)}{(x^2+1)(x^2+x+1)}dx + \int_1^{\infty}\frac{x\ln(x)}{(x^2+1)(x^2+x+1)}dx =$$

$$-2\int_0^1\frac{x\ln(x)}{(x^2+1)(x^2+x+1)}dx = -2\int_0^1\left\{\frac{1}{1+x^2} - \frac{1}{x^2+x+1}\right\}\ln(x) =$$

$$2\int_0^1\left\{\frac{1-x}{1-x^3} - \frac{1-x^2}{1-x^4}\right\}\ln(x)$$

$$= \frac{2}{9}\int_0^1\frac{x^{\frac{1}{3}-1} - x^{\frac{2}{3}-1}}{1-x}ln(x)dx - \frac{2}{16}\int_0^1\frac{x^{\frac{1}{4}-1} - x^{\frac{2}{4}-1}}{1-x}ln(x)dx =$$

$$\frac{2}{9}\left\{\psi'\left(\frac{2}{3}\right) - \psi'\left(\frac{1}{3}\right)\right\} - \frac{1}{8}\left\{\psi'\left(\frac{3}{4}\right) - \psi'\left(\frac{1}{4}\right)\right\} =$$

$$\frac{2}{9}\left[\frac{\partial}{\partial x}(\pi ctg(\pi x))\right]_{x=\frac{1}{3}} - \frac{1}{8}\left[\frac{\partial}{\partial x}(\pi ctg(\pi x))\right]_{x=\frac{1}{4}}$$

$$= \frac{2}{9}\pi^2\left(-cosec^2\left(\frac{\pi}{3}\right)\right) - \frac{1}{8}\pi^2\left(-cosec^2\left(\frac{\pi}{4}\right)\right) =$$

$$=\frac{\pi^2}{4}-\frac{8\pi^2}{27}=-\frac{5\pi^2}{108}$$

2551.

We consider the function $f:R\to R$

$$f(x)=\begin{cases}\dfrac{\arctan(x)}{x(x^2+x+1)}, & \text{if } x\in R\setminus\{0\}\\ 1, & \text{if } x=0\end{cases}$$

$$\text{Find } \Omega=\int_{-\infty}^{\infty}f(x)dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Shobhit Jain-India

$$\Omega=\int_{-\infty}^{\infty}f(x)dx=\int_0^{\infty}f(x)\,dx+\int_0^{\infty}f(-x)\,dx=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^{\infty}f(x)\,dx=\int_0^1 f(x)\,dx+\int_1^{\infty}f(x)\,dx$$

$$use,\quad \int_1^{\infty}f(x)\,dx=\int_1^{\infty}\frac{\arctan(x)}{x(x^2+x+1)}dx\underbrace{=}_{x\to\frac{1}{x}}\int_0^1\frac{x\arctan\left(\frac{1}{x}\right)}{(x^2+x+1)}dx$$

$$=\int_0^1\frac{x\left(\frac{\pi}{2}-\arctan(x)\right)}{(x^2+x+1)}dx$$

$$Therefore,\quad \Omega_1=\frac{\pi}{2}\int_0^1\frac{x}{(x^2+x+1)}dx+\int_0^1\frac{(1-x^2)\arctan(x)}{x(x^2+x+1)}dx$$

$$Now,\quad \int_0^1\frac{x}{(x^2+x+1)}dx=\frac{1}{2}\int_0^1\frac{2x+1}{(x^2+x+1)}dx-\frac{1}{2}\int_0^1\frac{1}{(x^2+x+1)}dx$$

$$=\left[\frac{1}{2}ln(x^2+x+1)-\frac{1}{\sqrt{3}}arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right]_0^1=\frac{ln3}{2}-\frac{\pi}{6\sqrt{3}}$$

$$Therefore,\quad \Omega_1=\frac{\pi}{2}\left(\frac{ln3}{2}-\frac{\pi}{6\sqrt{3}}\right)+\int_0^1\frac{(1-x^2)\arctan(x)}{x(x^2+x+1)}dx=\frac{\pi ln3}{4}-\frac{\pi^2}{12\sqrt{3}}+J$$

$$J=\int_0^1\frac{(1-x^2)\arctan(x)}{x(x^2+x+1)}dx=\int_0^1 tan^{-1}x\left(\frac{1}{x}-\frac{2x+1}{x^2+x+1}\right)dx=$$

$$=\int_0^1 \tan^{-1}x\, d\left[\ln\left(\frac{x}{x^2+x+1}\right)\right]=-\frac{\pi\ln 3}{4}+\int_0^1 \frac{\ln\left(x+\frac{1}{x}+1\right)}{1+x^2}dx=-\frac{\pi\ln 3}{4}+P$$

Therefore, $\Omega_1=\frac{\pi\ln 3}{4}-\frac{\pi^2}{12\sqrt{3}}+\left(-\frac{\pi\ln 3}{4}+P\right)=P-\frac{\pi^2}{12\sqrt{3}}$

Now, $P=\int_0^1 \frac{\ln\left(x+\frac{1}{x}+1\right)}{1+x^2}dx \underbrace{=}_{x\to\frac{1-x}{1+x}} \int_0^1 \frac{\ln\left(\frac{3+x^2}{1-x^2}\right)}{1+x^2}dx$

$$=\underbrace{\int_0^1 \frac{\ln(3+x^2)}{1+x^2}dx}_{A}-\underbrace{\int_0^1 \frac{\ln(1-x^2)}{1+x^2}dx}_{B}$$

$$\Rightarrow\ \Omega_1=A-B-\frac{\pi^2}{12\sqrt{3}}$$

$$\Omega_2=\int_0^\infty f(-x)\,dx=\int_0^1 f(-x)\,dx+\int_1^\infty f(-x)\,dx$$

use, $\int_1^\infty f(-x)\,dx=\int_1^\infty \frac{\arctan(x)}{x(x^2-x+1)}dx \underbrace{=}_{x\to\frac{1}{x}} \int_0^1 \frac{x\arctan\left(\frac{1}{x}\right)}{(x^2-x+1)}dx$

$$=\int_0^1 \frac{x\left(\frac{\pi}{2}-\arctan(x)\right)}{(x^2-x+1)}dx$$

Therefore, $\Omega_2=\frac{\pi}{2}\int_0^1 \frac{x}{(x^2-x+1)}dx+\int_0^1 \frac{(1-x^2)\arctan(x)}{x(x^2-x+1)}dx$

Now, $\int_0^1 \frac{x}{(x^2-x+1)}dx=\frac{1}{2}\int_0^1 \frac{2x-1}{(x^2-x+1)}dx+\frac{1}{2}\int_0^1 \frac{1}{(x^2-x+1)}dx$

$$=\left[\frac{1}{2}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1=\frac{\pi}{3\sqrt{3}}$$

Therefore, $\Omega_2=\frac{\pi}{2}\left(\frac{\pi}{3\sqrt{3}}\right)+\int_0^1 \frac{(1-x^2)\arctan(x)}{x(x^2+x+1)}dx=\frac{\pi^2}{6\sqrt{3}}+K$

Now, $K=\int_0^1 \frac{(1-x^2)\arctan(x)}{x(x^2-x+1)}dx=\int_0^1 \tan^{-1}x\left(\frac{1}{x}-\frac{2x-1}{x^2-x+1}\right)dx$

$$=\int_0^1 \tan^{-1}x\, d\left[\ln\left(\frac{x}{x^2-x+1}\right)\right]=\int_0^1 \frac{\ln\left(x+\frac{1}{x}-1\right)}{1+x^2}dx$$

$$K=\int_0^1 \frac{ln\left(x+\frac{1}{x}-1\right)}{1+x^2}dx \underbrace{=}_{x\to\frac{1-x}{1+x}} \int_0^1 \frac{ln\left(\frac{1+3x^2}{1-x^2}\right)}{1+x^2}dx$$

$$=\underbrace{\int_0^1 \frac{ln(1+3x^2)}{1+x^2}dx}_{R}-\underbrace{\int_0^1 \frac{ln(1-x^2)}{1+x^2}dx}_{B}$$

$$\Rightarrow\ \Omega_2=R-B+\frac{\pi^2}{6\sqrt{3}}$$

Now, $\Omega=\ \Omega_1+\ \Omega_2=A+R-2B+\dfrac{\pi^2}{12\sqrt{3}}$

$$B=\int_0^1 \frac{ln(1-x^2)}{1+x^2}dx \underbrace{=}_{x\to tan\theta} \int_0^{\frac{\pi}{4}} ln\left(\frac{cos2\theta}{cos^2\theta}\right)d\theta=$$

$$=\int_0^{\frac{\pi}{4}} \ln(2cos2\theta)\,d\theta+\int_0^{\frac{\pi}{4}} \ln 2\,d\theta-2\int_0^{\frac{\pi}{4}} \ln(2cos\theta)\,d\theta$$

$$\int_0^{\frac{\pi}{4}} \ln(2cos2\theta)\,d\theta \underbrace{=}_{\theta\to\frac{\theta}{4}} \frac{1}{4}\int_0^{\pi} \ln(2cos\frac{\theta}{2})\,d\theta=\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\pi}\cos(n\theta)\,d\theta=0$$

$$\int_0^{\frac{\pi}{4}} \ln(2cos\theta)\,d\theta \underbrace{=}_{\theta\to\frac{\theta}{2}} \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(2cos\frac{\theta}{2})\,d\theta=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}}\cos(n\theta)\,d\theta$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}sin\left(\frac{n\pi}{2}\right)}{n^2}=\frac{G}{2}\Rightarrow B=\frac{\pi}{4}ln2-G$$

$G=Catalan's\ Constant$

$$R=\int_0^1 \frac{ln(1+3x^2)}{1+x^2}dx=2\int_0^1 \frac{ln(x)}{1+x^2}dx+\int_0^1 \frac{ln\left(\frac{1}{x^2}+3\right)}{1+x^2}dx$$

$$=-2G+\int_1^{\infty}\frac{ln(y^2+3)}{1+y^2}dy\quad\left(y=\frac{1}{x}\right)$$

$$\Rightarrow A+R=\int_0^1 \frac{ln(3+x^2)}{1+x^2}dx+\int_1^{\infty}\frac{ln(y^2+3)}{1+y^2}dy-2G=\int_0^{\infty}\frac{ln(y^2+3)}{1+y^2}dy-2G$$

$$\Rightarrow\Omega=A+R-2B+\frac{\pi^2}{12\sqrt{3}}=\int_0^{\infty}\frac{ln(y^2+3)}{1+y^2}dy-\frac{\pi}{2}ln2+\frac{\pi^2}{12\sqrt{3}}$$

$$Now\ use,\int_0^\infty \frac{ln(y^2+a^2)}{y^2+b^2}dy=\frac{\pi}{|b|}ln(|a|+|b|)$$

$$\Rightarrow \Omega=\pi ln(1+\sqrt{3})-\frac{\pi}{2}ln2+\frac{\pi^2}{12\sqrt{3}}$$

2552. Prove that:

$$\int_0^\infty \frac{xln(1+x)}{(x^2+1)(x+1)(2x+1)}dx=\frac{1}{480}(144G-19\pi^2+96ln^2 2+36\pi ln2)$$

G is the Catalan's constant

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shobhit Jain-India

$$I=\int_0^\infty \frac{xln(1+x)}{(x^2+1)(x+1)(2x+1)}dx=\int_0^\infty \frac{ln(1+x)}{10}\left(\frac{5}{1+x}-\frac{8}{1+2x}-\frac{x}{1+x^2}+\frac{3}{1+x^2}\right)dx=$$

$$=\int_0^\infty \frac{ln(1+x)}{10}\left(\frac{5}{1+x}-\frac{8}{1+2x}-\frac{x}{1+x^2}\right)dx+\frac{3}{10}\int_0^\infty \frac{ln(1+x)}{(x^2+1)}dx=$$

$$=\frac{1}{10}\int_0^\infty g(x)\,ln(1+x)\,dx+\frac{3}{10}\int_0^\infty \frac{ln(1+x)}{(x^2+1)}dx$$

$$here,\ g(x)=\frac{5}{1+x}-\frac{8}{1+2x}-\frac{x}{1+x^2}$$

$$\int_0^\infty \frac{ln(1+x)}{(x^2+1)}dx=\int_0^1 \frac{ln(1+x)}{(x^2+1)}dx+\int_1^\infty \frac{ln(1+y)}{(y^2+1)}dy\underbrace{=}_{y\to\frac{1}{x}}\int_0^1 \frac{ln(1+x)}{(x^2+1)}dx+\int_0^1 \frac{ln\left(1+\frac{1}{x}\right)}{(x^2+1)}dx$$

$$=2\int_0^1 \frac{ln(1+x)}{(x^2+1)}dx+\int_0^1 \frac{ln\left(\frac{1}{x}\right)}{(x^2+1)}dx$$

$$2\int_0^1 \frac{ln(1+x)}{(x^2+1)}dx\underbrace{=}_{x\to\frac{1-x}{1+x}}2\int_0^1 \frac{ln\left(\frac{2}{1+x}\right)}{(x^2+1)}dx=\int_0^1 \frac{ln(1+x)+ln\left(\frac{2}{1+x}\right)}{(x^2+1)}dx=\int_0^1 \frac{ln2}{(x^2+1)}dx$$

$$=\frac{\pi}{4}ln2$$

$$And,\ \int_0^1 \frac{ln\left(\frac{1}{x}\right)}{(x^2+1)}dx=\int_0^1 ln\left(\frac{1}{x}\right)(1-x^2+x^4-\ \dots\dots\dots)dx=\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\ \dots\dots\right)=G$$

$$\Rightarrow\int_0^\infty \frac{ln(1+x)}{(x^2+1)}dx=\frac{\pi}{4}ln2+G$$

$$\Rightarrow I = \frac{1}{10}\int_0^\infty g(x)\,ln(1+x)\,dx + \frac{3}{10}\left(\frac{\pi}{4}ln2 + G\right)$$

$$= \frac{1}{10}\int_0^\infty g(x)\,ln(1+x)\,dx + \frac{1}{480}(36\pi ln2 + 144G)$$

$$Now,\ \int_0^\infty g(x)\,ln(1+x)\,dx = \int_0^1 g(x)\,ln(1+x)\,dx + \int_1^\infty g(y)\,ln(1+y)\,dy$$

$$\underbrace{=}_{y\to\frac{1}{x}} \int_0^1 g(x)\,ln(1+x)\,dx + \int_0^1 \frac{1}{x^2}g\left(\frac{1}{x}\right)ln\left(1+\frac{1}{x}\right)dx$$

$$= \underbrace{\int_0^1 ln(1+x)\left(g(x)+\frac{1}{x^2}g\left(\frac{1}{x}\right)\right)dx}_{A} - \underbrace{\int_0^1 (lnx)\left(\frac{1}{x^2}g\left(\frac{1}{x}\right)\right)dx}_{B}$$

$$\Rightarrow I = \frac{1}{10}(A-B) + \frac{1}{480}(36\pi ln2 + 144G).\ \ Now,\ \ g(x) = \frac{5}{1+x} - \frac{8}{1+2x} - \frac{x}{1+x^2}$$

$$\Rightarrow \frac{1}{x^2}g\left(\frac{1}{x}\right) = \frac{5}{x(1+x)} - \frac{8}{x(x+2)} - \frac{1}{x(1+x^2)} = \frac{4}{x+2} - \frac{5}{1+x} + \frac{x}{1+x^2}$$

$$\Rightarrow g(x) + \frac{1}{x^2}g\left(\frac{1}{x}\right) = \frac{4}{x+2} - \frac{8}{1+2x} = \frac{-12}{(x+2)(1+2x)}$$

$$\Rightarrow A = -12\int_0^1 \frac{ln(1+x)}{(x+2)(1+2x)}dx \underbrace{=}_{\substack{x\to\frac{1}{t}-1 \\ t=\frac{1}{1+x}}} -12\int_1^{\frac{1}{2}} \frac{lnt}{(1+t)(2-t)}dt$$

$$= -4\int_1^{\frac{1}{2}} \frac{lnt}{(1+t)}dt - 4\int_1^{\frac{1}{2}} \frac{lnt}{(2-t)}dt = 4f(-1) - 4f(2)$$

$$\Rightarrow I = \frac{2}{5}\big(f(-1) - f(2)\big) - \frac{B}{10} + \frac{1}{480}(36\pi ln2 + 144G)$$

$$Here\ i\ define\ f(a) = \int_1^{\frac{1}{2}} \frac{lnt}{a-t}dt = \sum_{n=1}^\infty \frac{1}{a^n}\int_1^{\frac{1}{2}} (lnt)t^{n-1}dt \qquad for\ |a| \geq 1$$

$$Now,\ \int_1^{\frac{1}{2}} (lnt)t^{n-1}dt = \left[(lnt)\frac{t^n}{n} - \frac{t^n}{n^2}\right]_1^{\frac{1}{2}} = -\frac{ln2}{n2^n} - \frac{1}{n^2 2^n} + \frac{1}{n^2}$$

$$\Rightarrow f(a) = \sum_{n=1}^\infty \frac{1}{a^n}\left(-\frac{ln2}{n2^n} - \frac{1}{n^2 2^n} + \frac{1}{n^2}\right) = (ln2)ln\left(1-\frac{1}{2a}\right) - Li_2\left(\frac{1}{2a}\right) + Li_2\left(\frac{1}{a}\right)$$

$$\Rightarrow f(-1) = (ln2)ln\left(\frac{3}{2}\right) - Li_2\left(-\frac{1}{2}\right) + Li_2(-1) = (ln2)(ln3) - ln^2 2 - Li_2\left(-\frac{1}{2}\right) - \frac{\pi^2}{12}$$

$$\Rightarrow f(2)=(ln2)ln\left(\frac{3}{4}\right)-Li_2\left(\frac{1}{4}\right)+Li_2\left(\frac{1}{2}\right)=(ln2)(ln3)-2ln^2 2-Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{12}-\frac{ln^2 2}{2}$$

$$=(ln2)(ln3)-Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{12}-\frac{5}{2}ln^2 2$$

$$Now,\ B=\int_0^1 (lnx)\left(\frac{1}{x^2}g\left(\frac{1}{x}\right)\right)dx=\int_0^1 (lnx)\left(\frac{4}{x+2}-\frac{5}{1+x}+\frac{x}{1+x^2}\right)dx$$

$$=4\int_0^1 \frac{\frac{1}{2}lnx}{1+\frac{x}{2}}dx+5\int_0^1 \frac{ln\left(\frac{1}{x}\right)}{1+x}dx-\int_0^1 \frac{xln\left(\frac{1}{x}\right)}{1+x^2}dx$$

$$Now,\ Li_2(z)=-\int_0^1 \frac{zlnx}{1-zx}dx\ \underbrace{\Rightarrow}_{z=-\frac{1}{2}}\ \int_0^1 \frac{\frac{1}{2}lnx}{1+\frac{x}{2}}dx=Li_2\left(-\frac{1}{2}\right)$$

$$And\ \int_0^1 \frac{ln\left(\frac{1}{x}\right)}{1+x}dx=\int_0^1 ln\left(\frac{1}{x}\right)(1-x+x^2-\ \dots\dots\dots)dx=\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ \dots\dots\right)=\eta(2)=\frac{\pi^2}{12}$$

$$\int_0^1 \frac{xln\left(\frac{1}{x}\right)}{1+x^2}dx=\int_0^1 ln\left(\frac{1}{x}\right)(x-x^3+x^5-\ \dots\dots\dots)dx=\left(\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}-\ \dots\dots\right)=\frac{\eta(2)}{4}=\frac{\pi^2}{48}$$

$$\Rightarrow B=4Li_2\left(-\frac{1}{2}\right)+\frac{5\pi^2}{12}-\frac{\pi^2}{48}=4Li_2\left(-\frac{1}{2}\right)+\frac{19\pi^2}{48}$$

$$\Rightarrow I=\frac{2}{5}\left(f(-1)-f(2)\right)-\frac{1}{10}\left(4Li_2\left(-\frac{1}{2}\right)+\frac{19\pi^2}{48}\right)+\frac{1}{480}(36\pi ln2+144G)$$

$$\Rightarrow I=\frac{2}{5}\left(f(-1)-f(2)-Li_2\left(-\frac{1}{2}\right)\right)+\frac{1}{480}(36\pi ln2+144G-19\pi^2)$$

$$Now,\ f(-1)-f(2)-Li_2\left(-\frac{1}{2}\right)$$

$$=\left((ln2)(ln3)-ln^2 2-Li_2\left(-\frac{1}{2}\right)-\frac{\pi^2}{12}\right)-\left((ln2)(ln3)-Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{12}-\frac{5}{2}ln^2 2\right)-Li_2\left(-\frac{1}{2}\right)$$

$$=\frac{3}{2}ln^2 2-\frac{\pi^2}{6}+2\left(\frac{1}{2}Li_2\left(\frac{1}{4}\right)-Li_2\left(-\frac{1}{2}\right)\right)$$

$$Now\ use\ the\ identity,\ Li_2(z)+\ Li_2(-z)=\frac{1}{2}Li_2(z^2)\ \underbrace{\Rightarrow}_{z=\frac{1}{2}}\ Li_2\left(\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)=\frac{1}{2}Li_2\left(\frac{1}{4}\right)$$

$$\Rightarrow \frac{1}{2}Li_2\left(\frac{1}{4}\right)-Li_2\left(-\frac{1}{2}\right)=Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{ln^2 2}{2}$$

$$\Rightarrow f(-1)-f(2)-Li_2\left(-\frac{1}{2}\right)=\frac{3}{2}ln^2 2-\frac{\pi^2}{6}+2\left(\frac{\pi^2}{12}-\frac{ln^2 2}{2}\right)=\frac{1}{2}ln^2 2$$

$$\Rightarrow I=\frac{2}{5}\left(\frac{1}{2}ln^2 2\right)+\frac{1}{480}(36\pi ln2+144G-19\pi^2)$$

$$=\frac{1}{5}ln^2 2+\frac{1}{480}(36\pi ln2+144G-19\pi^2)\Rightarrow I=\frac{1}{480}(144G-19\pi^2+96ln^2 2+36\pi ln2)$$

2553. Prove that:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)\Gamma(2n)\Gamma\left(-n+\frac{1}{2}\right)\cos(n\pi)}{8^n\Gamma(n)}=\frac{\sqrt{\pi}}{6}\log_e\left(\frac{3e^{\gamma}}{2}\right)$$

$\gamma\approx 0.577$ *is Euler – Mascheroni's constant*

Proposed by Amin Hajiyev-Azerbaijan

Solution by Shobhit Jain-India

$$Let\ \ I=\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)\Gamma(2n)\Gamma\left(-n+\frac{1}{2}\right)\cos(n\pi)}{8^n\Gamma(n)}$$

$$We\ know: \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin(n\pi)}\Rightarrow\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(-n+\frac{1}{2}\right)$$

$$=\frac{\pi}{\cos(n\pi)}\ (Euler's\ reflection\ formula)\Rightarrow\Gamma\left(-n+\frac{1}{2}\right)\cos(n\pi)=\frac{\pi}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow I=\pi\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)}{8^n}\left(\frac{\Gamma(2n)}{\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)}\right)$$

$$Now, we\ know\ \prod_{k=1}^{m-1}\Gamma\left(x+\frac{k}{m}\right)=\frac{\left(\sqrt{2\pi}\right)^{m-1}}{\sqrt{m}}\frac{1}{m^{mx-1}}\frac{\Gamma(mx)}{\Gamma(x)}\quad for\ m\geq 2$$

$$\underset{m=2}{\Rightarrow}\Gamma\left(x+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2x-1}}\frac{\Gamma(2x)}{\Gamma(x)}\Rightarrow\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)\quad (legendre's\ duplication\ formula)$$

$$\Rightarrow\frac{\Gamma(2n)}{\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)}=\frac{2^{2n-1}}{\sqrt{\pi}}\Rightarrow I=\pi\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)}{8^n}\left(\frac{2^{2n-1}}{\sqrt{\pi}}\right)=\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\psi(n)}{2^n}=\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^n\psi(n)$$

$$Now\ use,\ \ \psi(n)=-\gamma+H_{n-1}\quad (property\ of\ digamma\ function)$$

$$\Rightarrow I=\sqrt{\pi}\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^{n+1}\gamma-\frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}H_{n-1}\left(\frac{-1}{2}\right)^{n-1}$$

$$We\ know,\ \ -\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}\Rightarrow I=-\frac{\ln(1-x)}{1-x}=\sum_{n=1}^{\infty}H_{n-1}x^{n-1}\quad here\ H_n=\sum_{k=1}^{n}\frac{1}{k}\ and\ H_0=0$$

$$\Rightarrow I=\sqrt{\pi}\gamma\frac{\frac{1}{4}}{1+\frac{1}{2}}+\frac{\sqrt{\pi}}{4}\frac{\ln\left(1+\frac{1}{2}\right)}{1+\frac{1}{2}}=\frac{\gamma\sqrt{\pi}}{6}+\frac{\sqrt{\pi}}{6}\ln\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{6}\left(\gamma+\ln\left(\frac{3}{2}\right)\right)\Rightarrow I=\frac{\sqrt{\pi}}{6}\ln\left(\frac{3e^{\gamma}}{2}\right)$$

2554. Find:

$$\Omega=\int_0^1\frac{1}{\sqrt{x}}\ln\left(\sum_{m=0}^{n}x^{2m}\right)dx$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Pratham Prasad-India

$$\Omega = \int_0^1 2ln\left(\sum_{m=0}^{n} x^{4m}\right)dx = 2\left(\int_0^1 ln\left(1-x^{4(n+1)}\right)dx - \int_0^1 ln(1-x^4)dx\right) =$$

$$= 2(I(4(n+1)) - I(4))$$

Thus

$$I = 2(I(4(n+1)) - I(4))$$

where

$$I(a) = \int_0^1 ln(1-x^a)dx,\quad x^a = t,\qquad dx = \frac{1}{a}t^{\frac{1}{a}-1}dt$$

$$aI(a) = \int_0^1 t^{\frac{1}{a}-1}ln(1-t)dx,\quad aI(a) = \lim_{u\to 1}\frac{d}{du}B(\frac{1}{a},u)$$

$$aI(a) = \lim_{u\to 1} B(\frac{1}{a},u)\left(\psi(u) - \psi(u+\frac{1}{a})\right),\quad aI(a) = a\left(\psi(1) - \psi\left(\frac{1}{a}\right) - a\right)$$

$$I(a) = \left(\psi(1) - \psi\left(\frac{1}{a}\right) - a\right)$$

$$I = 2\left(\psi(1) - \psi\left(\frac{1}{4(n+1)}\right) - 4(n+1) - \psi(1) + \psi\left(\frac{1}{4}\right) + 4\right)$$

$$I = 2\left(\psi\left(\frac{1}{4}\right) - \psi\left(\frac{1}{4(n+1)}\right) - 4n\right)$$

2555. Find a closed form:

$$\Omega = \int_0^1\int_0^1 \arctan(x^2+y^2-2xy)\,dxdy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pratham Prasad-India

$$\Omega = \int_0^1\int_0^1 \arctan((y-x)^2)\,dxdy = \int_0^1\int_{y-1}^{y} \arctan(x^2)\,dxdy$$

$Applying\ Integration\ by\ parts\ on\ the\ outer\ Integral\ and\ replacing\ y\ with\ x\ after\ that,$

$$= \int_0^1 \arctan(x^2)\,dx - \int_0^1 x(\arctan(x^2) - \arctan((x-1)^2))dx$$

$$= \int_0^1 \arctan(x^2)\,dx - \int_0^1 x(\arctan(x^2))dx + \int_0^1 x(\arctan((1-x)^2))dx$$

$1-x = t\ and\ replace\ t\ with\ x\ in\ the\ last\ integral, to\ get:$

$$= \int_0^1 \arctan(x^2)\,dx - \int_0^1 x(\arctan(x^2))dx + \int_0^1 (1-x)(\arctan(x^2))dx$$

$$= 2\left(\int_0^1 \arctan(x^2)\,dx - \int_0^1 x(\arctan(x^2))dx\right)$$

$$x^2 = t \textit{ and replace } t \textit{ with } x \textit{ in the last integral, to get}:$$

$$= 2\left(\int_0^1 \arctan(x^2)\,dx\right) - \int_0^1 \arctan(x)\,dx = 2(A) - B$$

$$I = 2(A) - B, \qquad A = \int_0^1 \arctan(x^2)\,dx$$

Applying Integration by parts,

$$A = \frac{\pi}{4} - \int_0^1 \frac{2x^2}{1+x^4}dx, \qquad A = \frac{\pi}{4} - \int_0^1 \frac{(x^2-1)+(x^2+1)}{1+x^4}dx$$

$$A = \frac{\pi}{4} - \int_0^1 \frac{(x^2-1)}{1+x^4}dx - \int_0^1 \frac{(x^2+1)}{1+x^4}dx$$

$$A = \frac{\pi}{4} - \int_0^1 \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2 - 2}dx - \int_0^1 \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2 + 2}dx$$

$$\textit{substitute } x + \frac{1}{x} = t \textit{ in the first integral and } x - \frac{1}{x} = t \textit{ in the second integral},$$

$$A = \frac{\pi}{4} + \int_2^\infty \frac{1}{t^2-2}dt - \int_{-\infty}^0 \frac{1}{t^2+2}dt, \qquad A = \frac{\pi}{4} + \frac{\ln(1+\sqrt{2})}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}}$$

$$B = \int_0^1 \arctan(x)\,dx$$

Applying Integration by parts,

$$B = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx, \quad B = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx$$

$$x^2 = t \textit{ and replace } t \textit{ with } x \textit{ in the last integral, to get}:$$

$$B = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{1}{1+x}dx, \quad B = \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

Now,

$$I = 2(A) - B$$

$$I = 2\left(\frac{\pi}{4} + \frac{\ln(1+\sqrt{2})}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}}\right) - \frac{\pi}{4} + \frac{1}{2}\ln(2),$$

$$I = \frac{\pi}{2} + \sqrt{2}\ln(1+\sqrt{2}) - \frac{\pi}{\sqrt{2}} - \frac{\pi}{4} + \frac{1}{2}\ln(2)$$

$$I = \frac{\pi}{4} + \sqrt{2}\ln(1+\sqrt{2}) - \frac{\pi}{\sqrt{2}} + \frac{1}{2}\ln(2), \quad I = \frac{1}{2}\ln(2) + \sqrt{2}\ln(1+\sqrt{2}) + \frac{\pi}{4}(1-\sqrt{8})$$

$$\int_0^1\int_0^1 \arctan(x^2+y^2-2xy)\,dxdy = \frac{1}{2}\ln(2) + \sqrt{2}\ln(1+\sqrt{2}) + \frac{\pi}{4}(1-\sqrt{8})$$

2556. Prove that:

$$\Omega=\int_0^{\frac{1}{2}} x^3\psi(1+x^2)dx=\frac{1}{16}\left(2+ln\left(\frac{8\Gamma^2\left(\frac{5}{4}\right)}{\pi A^9}\right)-\frac{2G}{\pi}\right)$$

$A=Glaisher-Kinkelin's\ constant, G=Catalan's\ constant,$
$\psi(x)=digamma\ function$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Shobhit Jain-India

$$\Omega=\int_0^{\frac{1}{2}} x^3\psi(1+x^2)dx=\int_{x=0}^{\frac{1}{2}} x^2\psi(1+x^2)xdx\underbrace{=}_{\substack{x\to\sqrt{u}\\ u=x^2}}\frac{1}{2}\int_{u=0}^{\frac{1}{4}} u\,\psi(1+u)du$$

$$=\frac{1}{2}\int_{u=0}^{\frac{1}{4}} u\frac{\Gamma'(1+u)}{\Gamma(1+u)}du$$

$$=\left[\frac{u}{2}ln\Gamma(1+u)\right]_0^{\frac{1}{4}}-\frac{1}{2}\int_{u=0}^{\frac{1}{4}} ln\Gamma(1+u)du=\frac{1}{8}ln\Gamma\left(\frac{5}{4}\right)-\frac{1}{2}I$$

$Here,\ I=\int_0^{\frac{1}{4}} ln\Gamma(1+x)dx$. $Now\ we\ can\ use\ Kummer's\ series\ for\ the\ function\ ln\Gamma(x)$

$$\Rightarrow\frac{1}{2}ln\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)=\left(\frac{1}{2}-x\right)(\gamma+ln2\pi)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log_e n}{n}sin(2n\pi x)\quad for\ 0<x<1$$

$(\gamma=Euler-Mascheroni\ \ Constant)$

$$\frac{\Gamma(x)}{\Gamma(1-x)}=\frac{\Gamma^2(x+1)2\,sin(\pi x)}{2\pi x^2}$$

$$\Rightarrow\frac{1}{2}ln\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right)=ln\Gamma(1+x)+\frac{1}{2}ln(2\,sin(\pi x))-\frac{1}{2}ln(2\pi)-lnx$$

$$\Rightarrow ln\Gamma(1+x)=\frac{1}{2}ln(2\pi)+\left(\frac{1}{2}-x\right)(\gamma+ln2\pi)-\frac{1}{2}ln(2\,sin(\pi x))+lnx+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log_e n}{n}sin(2n\pi x)$$

$$\Rightarrow I=\frac{ln(2\pi)}{8}+(\gamma+ln2\pi)\underbrace{\int_0^{\frac{1}{4}}\left(\frac{1}{2}-x\right)dx}_{P}-\frac{1}{2}\underbrace{\int_0^{\frac{1}{4}} ln(2\,sin(\pi x))dx}_{Q}+\underbrace{\int_0^{\frac{1}{4}} lnxdx}_{R}$$

$$+\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{log_e n}{n^2}\left(1-cos\left(\frac{n\pi}{2}\right)\right)$$

$$P=\int_0^{\frac{1}{4}}\left(\frac{1}{2}-x\right)dx\underbrace{=}_{x\to\frac{1}{4}-x}\int_0^{\frac{1}{4}}\left(x+\frac{1}{4}\right)dx=\frac{1}{2}\times\left(\frac{1}{2}+\frac{1}{4}\right)\times\frac{1}{4}=\frac{3}{32}$$

$$Q=\int_0^{\frac{1}{4}}\ln(2\sin(\pi x))dx\underbrace{=}_{x\to\frac{x}{2\pi}}\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\ln\left(2\sin\left(\frac{x}{2}\right)\right)dx=\frac{-1}{2\pi}\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}}\frac{\cos(nx)}{n}dx=$$

$$=\frac{-1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}=-\frac{G}{2\pi}$$

$$R=\int_0^{\frac{1}{4}}\ln x dx=[x\ln x-x]_0^{\frac{1}{4}}=-\frac{1}{4}-\frac{\ln 2}{2}$$

$$\Rightarrow I=\frac{\ln(2\pi)}{8}+\frac{3}{32}(\gamma+\ln 2\pi)+\frac{G}{4\pi}-\frac{1}{4}-\frac{\ln 2}{2}+\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\log_e n}{n^2}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$$Let,\ N=\sum_{n=1}^{\infty}\frac{\log_e n}{n^2}=O+E\ \ where,\ O=\sum_{n=1}^{\infty}\frac{\log_e(2n-1)}{(2n-1)^2}\ and\ E=\sum_{n=1}^{\infty}\frac{\log_e(2n)}{(2n)^2}$$

$$\Rightarrow\ E=\frac{\ln 2}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{\log_e n}{n^2}=\frac{\pi^2\ln 2}{24}+\frac{N}{4}\Rightarrow\ O=N-E=\frac{3N}{4}-\frac{\pi^2\ln 2}{24}$$

$$\Rightarrow O-E=\left(\frac{3N}{4}-\frac{\pi^2\ln 2}{24}\right)-\left(\frac{\pi^2\ln 2}{24}+\frac{N}{4}\right)=\frac{N}{2}-\frac{\pi^2\ln 2}{12}$$

$$Now,\ \sum_{n=1}^{\infty}\frac{\log_e n}{n^2}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$$=\left(\frac{\ln 1}{1^2}+\frac{\ln 3}{3^2}+\frac{\ln 5}{5^2}\dots\dots\right)+2\left(\frac{\ln 2}{2^2}+\frac{\ln 6}{6^2}+\frac{\ln 10}{10^2}\dots\dots\right)$$

$$=\underbrace{\left(\frac{\ln 1}{1^2}+\frac{\ln 3}{3^2}+\frac{\ln 5}{5^2}\dots\dots\right)}_{O}+\underbrace{\left(\frac{\ln 2}{2^2}+\frac{\ln 4}{4^2}+\frac{\ln 6}{6^2}\dots\dots\right)}_{E}+\left(\frac{\ln 2}{2^2}-\frac{\ln 4}{4^2}+\frac{\ln 6}{6^2}-\ \dots\dots\right)$$

$$=N+\frac{\ln 2}{2^2}\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ \dots\dots\right)+\frac{1}{2^2}\left(\frac{\ln 1}{1^2}-\frac{\ln 2}{2^2}+\frac{\ln 3}{3^2}-\ \dots\dots\right)$$

$$=N+\frac{\pi^2\ln 2}{48}+\frac{1}{4}(O-E)=$$

$$=N+\frac{\pi^2\ln 2}{48}+\frac{1}{4}\left(\frac{N}{2}-\frac{\pi^2\ln 2}{12}\right)=\frac{9}{8}N$$

$$\Rightarrow\ I=\frac{\ln(2\pi)}{8}+\frac{3}{32}(\gamma+\ln 2\pi)+\frac{G}{4\pi}-\frac{1}{4}-\frac{\ln 2}{2}+\frac{9}{8}\left(\frac{N}{2\pi^2}\right)$$

$$Now,\ \zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\ \Rightarrow\zeta'(s)=-\sum_{n=1}^{\infty}\frac{\log_e n}{n^s}\ \Rightarrow\zeta'(2)=-\sum_{n=1}^{\infty}\frac{\log_e n}{n^2}=-N$$

$$Now,\ -\frac{\zeta'(2)}{\zeta(2)}=12lnA-\gamma-ln(2\pi)$$

$$\Rightarrow N=-\zeta'(2)=\frac{\pi^2}{6}(12lnA-\gamma-ln(2\pi))\Rightarrow\frac{N}{2\pi^2}=lnA-\frac{\gamma}{12}-\frac{ln(2\pi)}{12}$$

$$\Rightarrow I=\frac{ln(2\pi)}{8}+\frac{3}{32}(\gamma+ln2\pi)+\frac{G}{4\pi}-\frac{1}{4}-\frac{ln2}{2}+\frac{9}{8}\left(lnA-\frac{\gamma}{12}-\frac{ln(2\pi)}{12}\right)$$

$$=\frac{ln(2\pi)}{8}+\frac{G}{4\pi}-\frac{1}{4}-\frac{ln2}{2}+\frac{9}{8}lnA\Rightarrow I=\frac{ln\pi}{8}+\frac{G}{4\pi}-\frac{1}{4}-\frac{3ln2}{8}+\frac{9}{8}lnA$$

$$\Rightarrow \Omega=\frac{1}{8}ln\Gamma\left(\frac{5}{4}\right)-\frac{1}{2}\left(\frac{ln\pi}{8}+\frac{G}{4\pi}-\frac{1}{4}-\frac{3ln2}{8}+\frac{9}{8}lnA\right)=$$

$$=\frac{1}{16}\left(2-\frac{2G}{\pi}+2ln\Gamma\left(\frac{5}{4}\right)+3ln2-ln\pi-9lnA\right)\Rightarrow\Omega=\frac{1}{16}\left(2+ln\left(\frac{8\Gamma^2\left(\frac{5}{4}\right)}{\pi A^9}\right)-\frac{2G}{\pi}\right)$$

2557. Find:

$$S=\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{(2k^2+k-1)(2k-1)}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pratham Prasad-India

By Partial Fractions:

$$S=\frac{5}{18}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}+\frac{1}{6}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k}$$

$$S=\frac{5}{18}\sum_{k=1}^{\infty}(-1)^k\int_0^1 x^{2k-2}dx+\frac{1}{6}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k}$$

changing order of summation and integration,

$$S=-\frac{5}{18}\int_0^1\sum_{k=1}^{\infty}(-x^2)^{k-1}dx-\frac{1}{6}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k}$$

$$S=-\frac{5}{18}\int_0^1\frac{1}{1+x^2}dx-\frac{1}{6}G+\frac{1}{9}\ln(2)-\frac{1}{9}$$

$$S=-\frac{5\pi}{72}-\frac{1}{6}G+\frac{1}{9}\ln(2)-\frac{1}{9}$$

2558. Prove that:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{e^{(m+n)\pi}+e^{(m-n)\pi}}=\frac{1-G\sqrt{2}}{4(1-e^{\pi})}$$

where, G is Gauss's constant

Proposed by Ankush Kumar Parcha-India

Solution by Shobhit Jain-India

$$I=\sum_{m=1}^{\infty}\frac{1}{e^{m\pi}}\sum_{n=1}^{\infty}\frac{1}{e^{n\pi}+e^{-n\pi}}=M\times N$$

$$Now,\ \ M=\sum_{m=1}^{\infty}\frac{1}{e^{m\pi}}=\sum_{m=1}^{\infty}e^{-m\pi}=\frac{e^{-\pi}}{1-e^{-\pi}}=\frac{1}{e^{\pi}-1}$$

$$N=\sum_{n=1}^{\infty}\frac{1}{q^{-n}+q^{n}}=\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k q^{(2k+1)n}=\sum_{k=0}^{\infty}(-1)^k\sum_{n=1}^{\infty}q^{(2k+1)n}=\sum_{k=0}^{\infty}\frac{(-1)^k q^{2k+1}}{1-q^{2k+1}}$$

$here,\ \ q=e^{-\pi}.\ Consider,\ \ \varphi(q)=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}$

$$here\ \ f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}},$$

$$|ab|<1\qquad (Ramanujan\ general\ theta\ function)$$

$$Let\ \ g(x)=\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1,x\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(1)_n n!}x^n=\sum_{n=0}^{\infty}\frac{\left({}^{2n}C_n\right)^2}{16}x^n\qquad |x|<1$$

$Consider\ complete\ elliptic\ integral\ of\ first\ kind,$

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{\pi}{2}g(k^2)\ \ and\ \ K(k')=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}=\frac{\pi}{2}g\left(k'^2\right)$$

$$=\frac{\pi}{2}g(1-k^2)$$

$$k'=\sqrt{1-k^2}$$

$$If\ \ k=\sqrt{1-\frac{\varphi^4(-q)}{\varphi^4(q)}}\qquad then,\ \ \varphi^2\left(e^{-\pi\frac{K(k')}{K(k)}}\right)=g(k^2)=\frac{2}{\pi}K(k)$$

$$Now,\ \ k=k'=\frac{1}{\sqrt{2}}\ \Rightarrow,\ \ \varphi^2(e^{-\pi})=g\left(\frac{1}{2}\right)=\frac{2}{\pi}K\left(\frac{1}{\sqrt{2}}\right)$$

$$K\left(\frac{1}{\sqrt{2}}\right)=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{2-\sin^2\theta}}=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+\cos^2\theta}}\underbrace{=}_{\theta=\cos^{-1}x}\sqrt{2}\int_0^{1}\frac{dx}{\sqrt{1-x^4}}=\frac{\sqrt{2}}{4}B\left(\frac{1}{4},\frac{1}{2}\right)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}$$

$$\Rightarrow\varphi^2(e^{-\pi})=g\left(\frac{1}{2}\right)=\frac{2}{\pi}K\left(\frac{1}{\sqrt{2}}\right)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{\pi}}=G\sqrt{2}$$

$$\left(G=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2\pi}}=Gauss's\ Constant\right)$$

$Now\ we\ can\ use\ the\ identity\ given\ in\ \ Ramanujan\ Notebook\ (Part-3)\ page-114,\ \ entry\ 8$

$$\varphi^2(q)=1+4\sum_{k=0}^{\infty}\frac{(-1)^k q^{2k+1}}{1-q^{2k+1}}$$

$$\text{Consider} \sum_{n=-\infty}^{\infty}\frac{2}{q^{-n}+q^n}=1+4\sum_{n=1}^{\infty}\frac{1}{q^{-n}+q^n}=1+4\sum_{k=0}^{\infty}\frac{(-1)^k q^{2k+1}}{1-q^{2k+1}}=\varphi^2(q)$$

$$\underbrace{\Rightarrow}_{q=e^{-\pi}}\sum_{n=-\infty}^{\infty}\frac{2}{e^{n\pi}+e^{-n\pi}}=1+4\sum_{n=1}^{\infty}\frac{1}{e^{n\pi}+e^{-n\pi}}=1+4\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)\pi}}{1-e^{-(2k+1)\pi}}=\varphi^2(e^{-\pi})$$
$$=G\sqrt{2}$$

$$\Rightarrow N=\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)\pi}}{1-e^{-(2k+1)\pi}}=\frac{G\sqrt{2}-1}{4}\Rightarrow I=M\times N=\left(\frac{1}{e^{\pi}-1}\right)\left(\frac{G\sqrt{2}-1}{4}\right)$$

$$\Rightarrow I=\frac{1-G\sqrt{2}}{4(1-e^{\pi})}$$

2559. Prove that:

$$\int_0^1\frac{(1-x)(1-x^2)(1-x^3)}{(1+x^2)lnx}dx=ln\left(\frac{18\pi^2}{5\varpi^4}\right)$$

where ϖ is Lemniscate Constant

Proposed by Ankush Kumar Parcha-India

Solution by Shobhit Jain-India

$$I=\int_0^1\frac{(1-x)(1-x^2)(1-x^3)}{(1+x^2)lnx}dx=\int_0^1\frac{(1-x)}{lnx}\left(x^3-2x-1+\frac{2(1+x)}{1+x^2}\right)dx=$$

$$=\int_0^1\frac{-x^4+x^3+2x^2-x-1}{lnx}dx-2\int_0^1\frac{x^2-1}{(1+x^2)lnx}dx$$

$$=-\int_0^1\frac{x^4-1}{lnx}dx+\int_0^1\frac{x^3-1}{lnx}dx+2\int_0^1\frac{x^2-1}{lnx}dx-\int_0^1\frac{x-1}{lnx}dx-2\int_0^1\frac{x^2-1}{(1+x^2)lnx}dx$$
$$=-H(5)+H(4)+2H(3)-H(2)-2K(2)$$

$$\text{here, }\ H(a)=\int_0^1\frac{x^{a-1}-1}{lnx}dx=\int_{x=0}^{1}\int_{t=1}^{a}x^{t-1}\,dtdx=\int_{t=1}^{a}\int_{x=0}^{1}x^{t-1}\,dxdt=\int_1^a\frac{dt}{t}=ln(a)$$

$$\Rightarrow I=-ln5+ln4+2ln3-ln2-2K(2)=ln\left(\frac{18}{5}\right)-2K(2)$$

$$\text{And, }\ K(a)=\int_0^1\frac{x^a-1}{(1+x^2)lnx}dx\ \Rightarrow K(0)=0$$

$$\Rightarrow K'(a)=\int_0^1\frac{x^a}{1+x^2}dx=\int_0^1\frac{x^a-x^{a+2}}{1-x^4}dx\underbrace{=}_{x\to x^{\frac{1}{4}}}\frac{1}{4}\int_0^1\frac{x^{\frac{a-3}{4}}-x^{\frac{a-1}{4}}}{1-x}dx=$$

$$= \frac{1}{4}\left(\psi\left(\frac{a+3}{4}\right) - \psi\left(\frac{a+1}{4}\right)\right)$$

$$\Rightarrow K(a) = \frac{1}{4}\int_0^a \left(\psi\left(\frac{a+3}{4}\right) - \psi\left(\frac{a+1}{4}\right)\right) da = \left[ln\left(\frac{\Gamma\left(\frac{a+3}{4}\right)}{\Gamma\left(\frac{a+1}{4}\right)}\right)\right]_0^a$$

$$= ln\left(\frac{\Gamma\left(\frac{a+3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{a+1}{4}\right)\Gamma\left(\frac{3}{4}\right)}\right)$$

$$\Rightarrow K(2) = ln\left(\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{4}\right)}\right) = ln\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\Gamma^2\left(\frac{3}{4}\right)}\right) = ln\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(\frac{1}{4}\right)}\right) = ln\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\left(\pi\sqrt{2}\right)^2}\right) = ln\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi^2}\right)$$

$$Now,\ \varpi = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} \Rightarrow \varpi^2 = \frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi} \Rightarrow K(2) = ln\left(\frac{\varpi^2}{\pi}\right) \Rightarrow I = ln\left(\frac{18}{5}\right) - 2ln\left(\frac{\varpi^2}{\pi}\right) = ln\left(\frac{18\pi^2}{5\varpi^4}\right)$$

2560. Find a closed form:

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan(x)\ln(1-tanx)\,dx$$

Proposed by Fao Ler-Iraq

Solution by Pratham Prasad-India

$$I = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan(x)\ln(1-tanx)\,dx$$

$$I = \int_{\sqrt{2}-1}^{1} \frac{tln(1-t)}{1+t^2}dt = \int_{\sqrt{2}-1}^{1} \frac{tln(1-t)}{(t+i)(t-i)}dt$$

$$= \frac{1}{2}\int_{\sqrt{2}-1}^{1} \frac{ln(1-t)}{(t+i)}dt + \frac{1}{2}\int_{\sqrt{2}-1}^{1} \frac{ln(1-t)}{(t-i)}dt$$

$$= \frac{1}{2}\int_{\sqrt{2}-1+i}^{1+i} \frac{ln(1+i-u)}{u}du + \frac{1}{2}\int_{\sqrt{2}-1-i}^{1-i} \frac{\ln(1-i-u)}{u}du$$

$$= \frac{1}{2}\int_{\sqrt{2}-1+i}^{1+i} \frac{ln\left(1-\frac{u}{1+i}\right)}{u}du + \frac{1}{2}\ln(1+i)\ln\left(\frac{1+i}{\sqrt{2}-1+i}\right)$$

$$+ \frac{1}{2}\int_{\sqrt{2}-1-i}^{1-i} \frac{\ln\left(1-\frac{u}{1-i}\right)}{u}du + \frac{1}{2}\ln(1-i)\ln\left(\frac{1-i}{\sqrt{2}-1-i}\right)$$

$$= \frac{1}{2}\int_{\frac{\sqrt{2}-1+i}{1+i}}^{1} \frac{ln(1-u)}{u}du + \frac{1}{2}\ln(1+i)\ln\left(\frac{1+i}{\sqrt{2}-1+i}\right) + \frac{1}{2}\int_{\frac{\sqrt{2}-1-i}{1-i}}^{1} \frac{\ln(1-u)}{u}du$$

$$+ \frac{1}{2}\ln(1-i)\ln\left(\frac{1-i}{\sqrt{2}-1-i}\right) =$$

$$=\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1+i}{1+i}\right)+\frac{1}{2}\ln(1+i)\ln\left(\frac{1+i}{\sqrt{2}-1+i}\right)+\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1-i}{1-i}\right)$$

$$+\frac{1}{2}\ln(1-i)\ln\left(\frac{1-i}{\sqrt{2}-1-i}\right)-Li_2(1)=$$

$$=\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1+i}{1+i}\right)-\frac{1}{2}\ln(1+i)\ln\left(\frac{\sqrt{2}-1+i}{1+i}\right)+\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1-i}{1-i}\right)$$

$$-\frac{1}{2}\ln(1-i)\ln\left(\frac{\sqrt{2}-1-i}{1-i}\right)-Li_2(1)=$$

$$=\frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}}+i\left(1-\frac{1}{\sqrt{2}}\right)\right)-\frac{1}{2}\ln(1+i)\ln\left(\frac{1}{\sqrt{2}}+i\left(1-\frac{1}{\sqrt{2}}\right)\right)$$

$$+\frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}}+i\left(\frac{1}{\sqrt{2}}-1\right)\right)-\frac{1}{2}\ln(1-i)\ln\left(\frac{1}{\sqrt{2}}+i\left(\frac{1}{\sqrt{2}}-1\right)\right)-\zeta(2)$$

where:

$$\ln(1+i)=\frac{1}{2}\ln(2)+\ln\left(cos\left(\frac{\pi}{4}\right)+isin\left(\frac{\pi}{4}\right)\right)=\frac{1}{2}\ln(2)+\frac{i\pi}{4}$$

$$\ln(1+i)=\frac{1}{2}\ln(2)+\ln\left(cos\left(\frac{\pi}{4}\right)-isin\left(\frac{\pi}{4}\right)\right)=\frac{1}{2}\ln(2)-\frac{i\pi}{4}$$

$$\ln\left(\frac{1}{\sqrt{2}}+i\left(1-\frac{1}{\sqrt{2}}\right)\right)=\ln\left(2-\sqrt{2}\right)+iarctan\left(\sqrt{2}-1\right)$$

$$\ln\left(\frac{1}{\sqrt{2}}+i\left(\frac{1}{\sqrt{2}}-1\right)\right)=\ln\left(2-\sqrt{2}\right)-iarctan\left(\sqrt{2}-1\right)$$

Thus,

$$I=\frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}}+i\left(1-\frac{1}{\sqrt{2}}\right)\right)-\frac{1}{2}\ln(1+i)\ln\left(\frac{1}{\sqrt{2}}+i\left(1-\frac{1}{\sqrt{2}}\right)\right)$$

$$+\frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}}+i\left(\frac{1}{\sqrt{2}}-1\right)\right)-\frac{1}{2}\ln(1-i)\ln\left(\frac{1}{\sqrt{2}}+i\left(\frac{1}{\sqrt{2}}-1\right)\right)-\zeta(2)$$

$$I=\frac{\pi}{4}\arctan\left(\sqrt{2}-1\right)-\frac{1}{2}\ln(2)\ln\left(2-\sqrt{2}\right)+Re\left(Li_2\left(\frac{1}{\sqrt{2}}+i\left(1-\frac{1}{\sqrt{2}}\right)\right)\right)-\zeta(2)$$

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\tan(x)\ln(1-tanx)\,dx=$$

$$=\frac{\pi}{4}\arctan\left(\sqrt{2}-1\right)-\frac{1}{2}\ln(2)\ln\left(2-\sqrt{2}\right)+Re\left(Li_2\left(\frac{1}{\sqrt{2}}+i\left(1-\frac{1}{\sqrt{2}}\right)\right)\right)-\zeta(2)$$

2561. Prove that:

$$\int_1^\infty \frac{\ln(\ln x)}{1-x+x^2}dx = \frac{2\pi}{\sqrt{3}}\ln\Gamma\left(\frac{5}{6}\right) - \frac{\pi}{3\sqrt{3}}\ln(2\pi)$$

Proposed by Lunjapao Baite-India

Solution by Shohbit Jain-India

$$I = \int_1^\infty \frac{\ln(\ln x)}{1-x+x^2}dx \underbrace{=}_{x\to\frac{1}{x}} \int_0^1 \frac{\ln\left(\ln\frac{1}{x}\right)}{x^2-x+1}dx = f'(1)$$

$$define,\ f(s) = \int_0^1 \frac{\left(\ln\frac{1}{x}\right)^{s-1}}{x^2-x+1}dx = \int_0^1 \frac{\left(\ln\frac{1}{x}\right)^{s-1}}{x^2-2x\cos\left(\frac{\pi}{3}\right)+1}dx$$

$$\sum_{n=1}^\infty x^n\sin(n\theta) = Im\left(\sum_{n=0}^\infty x^n e^{in\theta}\right) = Im\left(\frac{1}{1-xe^{i\theta}}\right) = Im\left(\frac{1}{1-x\cos\theta-ix\sin\theta}\right) = \frac{x\sin\theta}{1-2x\cos\theta+x^2}$$

$$\Rightarrow \frac{1}{1-2x\cos\theta+x^2} = \sum_{n=1}^\infty x^{n-1}\frac{\sin(n\theta)}{\sin\theta}$$

$$\Rightarrow \frac{1}{x^2-2x\cos\left(\frac{\pi}{3}\right)+1} = \sum_{n=1}^\infty x^{n-1}\frac{\sin\left(\frac{n\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}\sum_{n=1}^\infty x^{n-1}\sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow f(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^\infty \sin\left(\frac{n\pi}{3}\right)\int_0^1\left(\ln\frac{1}{x}\right)^{s-1}x^{n-1}dx = \frac{2}{\sqrt{3}}\Gamma(s)\sum_{n=1}^\infty \frac{\sin\left(\frac{n\pi}{3}\right)}{n^s}$$

$$\Rightarrow f'(s) = \frac{2}{\sqrt{3}}\Gamma'(s)\sum_{n=1}^\infty \frac{\sin\left(\frac{n\pi}{3}\right)}{n^s} - \frac{2}{\sqrt{3}}\Gamma(s)\sum_{n=1}^\infty \frac{\log_e n}{n^s}\sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow f'(1) = \frac{2}{\sqrt{3}}\Gamma'(1)\sum_{n=1}^\infty \frac{\sin\left(\frac{n\pi}{3}\right)}{n} - \frac{2}{\sqrt{3}}\Gamma(1)\sum_{n=1}^\infty \frac{\log_e n}{n}\sin\left(\frac{n\pi}{3}\right)$$

$$here,\ \Gamma'(1) = \Gamma(1)\psi(1) = -\gamma \qquad (\gamma \text{ is Euler} - \text{Mascheroni's constant})$$

$$And, \sum_{n=1}^\infty \frac{\sin\left(\frac{n\pi}{3}\right)}{n} = Im\left(-\ln\left(1-e^{\frac{i\pi}{3}}\right)\right) = -Im\left\{\ln\left(1-\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)\right)\right\}$$

$$= \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{3}\right)}{1-\cos\left(\frac{\pi}{3}\right)}\right) = \tan^{-1}\left(\cot\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{2}-\frac{\pi}{6} = \frac{\pi}{3}$$

$Now,\ from\ Kummer's\ series:$

$$\frac{1}{2}\ln\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) = \left(\frac{1}{2}-x\right)(\gamma+\ln(2\pi)) + \frac{1}{\pi}\sum_{n=1}^\infty \frac{\log_e n}{n}\sin(2n\pi x)$$

$$\underset{x=\frac{1}{6}}{\underbrace{\Rightarrow}}\frac{1}{2}ln\left(\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)}\right)=\frac{1}{3}(\gamma+ln(2\pi))+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{log_e n}{n}\sin\left(\frac{n\pi}{3}\right)$$

$$\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)}=\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma^2\left(\frac{5}{6}\right)}=\frac{2\pi}{\Gamma^2\left(\frac{5}{6}\right)}\Rightarrow\frac{1}{2}ln\left(\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)}\right)=\frac{1}{2}\ln(2\pi)-ln\Gamma\left(\frac{5}{6}\right)$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{log_e n}{n}\sin\left(\frac{n\pi}{3}\right)=\frac{\pi}{2}\ln(2\pi)-\pi ln\Gamma\left(\frac{5}{6}\right)-\frac{\pi}{3}(\gamma+ln(2\pi))=-\pi ln\Gamma\left(\frac{5}{6}\right)-\frac{\pi}{6}\ln(2\pi)-\frac{\pi\gamma}{3}$$

$$\Rightarrow f'(1)=-\frac{2\gamma\pi}{3\sqrt{3}}-\frac{2}{\sqrt{3}}\left(-\pi ln\Gamma\left(\frac{5}{6}\right)-\frac{\pi}{6}\ln(2\pi)-\frac{\pi\gamma}{3}\right)\Rightarrow I$$

$$=\frac{2\pi}{\sqrt{3}}ln\Gamma\left(\frac{5}{6}\right)-\frac{\pi}{3\sqrt{3}}ln(2\pi)$$

2562. Find:

$$J=\int_0^{\frac{\pi}{4}}x\tan x\ln(\cos x)\,dx$$

Proposed by Naren Bhandari-Nepal

Solution by Pratham Prasad-India

$$J=\int_0^{\frac{\pi}{4}}x\tan x\ln(\cos x)\,dx=-\frac{1}{2}\int_0^{\frac{\pi}{4}}x\ \mathrm{d}(\ln^2(\cos x))$$

$$=-\frac{\pi}{32}ln^2(2)+\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln^2(\cos x)dx$$

$$I=\int_0^{\frac{\pi}{4}}\ln^2(\cos x)dx\ ,I=\frac{1}{4}\int_0^1\frac{\ln^2(1+x^2)}{1+x^2}dx\ .\ By\ Weierestras\ substitution:$$

$$I=\frac{1}{4}\int_0^1\frac{\ln^2\left(\frac{2(1+x^2)}{(1+x)^2}\right)}{1+x^2}dx\,,$$

$$I=\int_0^1\frac{\ln^2(1+x)}{1+x^2}dx$$

$$+\frac{1}{4}\int_0^1\frac{\ln^2(1+x^2)}{1+x^2}dx$$

$$+\frac{\ln^2(2)}{4}\int_0^1\frac{1}{1+x^2}dx-\int_0^1\frac{ln(1+x)\ln(1+x^2)}{1+x^2}dx\ -ln2\int_0^1\frac{\ln(1+x)}{1+x^2}dx$$

$$+\frac{ln2}{2}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx$$

By putting results,

$$I=\left(-2G\ln(2)-4\,Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{7\pi^3}{64}+\frac{3}{16}\pi\ln^2(2)\right)$$
$$+\frac{1}{4}\left(-2G\ln(2)+4\,Im\left(Li_3\left(\frac{1+i}{2}\right)\right)-\frac{7\pi^3}{96}+\frac{7}{8}\pi\ln^2(2)\right)+\frac{\ln^2(2)}{4}\left(\frac{\pi}{4}\right)$$
$$-\left(-\frac{5}{2}G\ln(2)-4\,Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{7\pi^3}{64}+\frac{3}{8}\pi\ln^2(2)\right)-\ln(2)\left(\frac{\pi}{8}\ln(2)\right)$$
$$+\frac{\ln(2)}{2}\left(\frac{\pi}{2}\ln(2)-C\right)$$
$$I=\frac{7\pi^3}{192}-\frac{G\ln(2)}{2}+\frac{5\pi\ln^2(2)}{16}-Im(Li_3(1-i))$$
$$J=-\frac{\pi}{32}ln^2(2)+\frac{1}{2}\left(\frac{7\pi^3}{192}-\frac{G\ln(2)}{2}+\frac{5\pi\ln^2(2)}{16}-Im\big(Li_3(1-i)\big)\right)$$
$$J=\frac{7\pi^3}{384}-\frac{G\ln(2)}{4}+\frac{\pi\ln^2(2)}{8}-\frac{1}{2}Im(Li_3(1-i))$$

2563. Find:

$$I=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx-3\int_0^1\frac{\ln(1-x)\,Li_2^2(1-x)}{1-x}dx$$

Proposed by Naren Bhandari-Nepal

Solution by Pratham Prasad-India

Replacing $1-x$ by x in the second integral,

$$=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx-3\int_0^1\frac{\ln(x)\,Li_2^2(x)}{x}dx$$

Let,

$$A=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx\ ,B=\int_0^1\frac{\ln(x)Li_2^2(x)}{x}dx.\ Thus\ \boxed{I=A-3B}$$
$$A=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx$$

Applying Integration By parts, we get $-A=\frac{1}{2}\int_0^1\frac{\ln^4(1-x)\,ln(x)}{x}dx$

Replacing $1-x$ by x, $A=\frac{1}{2}\int_0^1\frac{\ln^4(x)\,ln(1-x)}{1-x}dx$, $A=-\frac{1}{2}\int_0^1\ln^4(x)\sum_{n=1}^{\infty}x^nH_n\,dx$

$$A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\int_0^1 x^n\ln^4(x)\,dx,\ \ A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\int_0^1\frac{\partial^4}{\partial n^4}x^n dx$$
$$A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\frac{d^4}{dn^4}\int_0^1x^n\,dx,\qquad A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\frac{d^4}{dn^4}\left(\frac{1}{n+1}\right)$$
$$A=-12\sum_{n=1}^{\infty}H_n\left(\frac{1}{n+1}\right)^5,A=-12\left(\sum_{n=1}^{\infty}\frac{H_n}{n^5}-\zeta(6)\right),A=-12S_1+12\zeta(6)$$

$$S_1=\sum_{n=1}^{\infty}\frac{H_n}{n^5}, define, \quad J(q)=\int_0^1 x^{n-1}Li_q(x)dx,$$

$$J(q)=\frac{\zeta(q)}{n}-\frac{1}{n}\int_0^1 x^{n-1}Li_{q-1}(x)dx$$

$$Applying\ Integration\ By\ parts, we\ get-J(q)=\frac{\zeta(q)}{n}-\frac{1}{n}\int_0^1 x^{n-1}Li_{q-1}(x)dx$$

$$J(q)=\frac{\zeta(q)}{n}-\frac{1}{n}J(q-1), \quad J(2)=\int_0^1 x^{n-1}Li_2(x)dx$$

$$J(2)=\frac{\zeta(2)}{n}+\frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx, \quad J(2)=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}$$

$$Thus\ by\ recursion: J(q)=\frac{(-1)^{q-1}H_n}{n^q}-\sum_{k=1}^{q-1}\frac{(-1)^k}{n^k}\zeta(q-k+1)$$

$$\int_0^1 x^{n-1}Li_q(x)dx=\frac{(-1)^{q-1}H_n}{n^q}-\sum_{k=1}^{q-1}\frac{(-1)^k}{n^k}\zeta(q-k+1)$$

$$Setting\ q=4\ and\ rearranging,$$

$$\frac{H_n}{n^4}=-\int_0^1 x^{n-1}Li_4(x)dx-\sum_{k=1}^{3}\frac{(-1)^k}{n^k}\zeta(4-k+1)$$

$$\frac{H_n}{n^5}=-\int_0^1\frac{x^{n-1}Li_4(x)}{n}dx-\sum_{k=1}^{3}\frac{(-1)^k}{n^{k+1}}\zeta(5-k)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=-\int_0^1\frac{Li_4(x)}{x}\sum_{n=1}^{\infty}\frac{x^n}{n}dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\int_0^1\frac{Li_4(x)}{x}\ln(1-x)dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\int_0^1\frac{\ln(1-x)}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^4}dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^1 x^{n-1}\ln(1-x)dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=-\sum_{n=1}^{\infty}\frac{H_n}{n^5}-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1),$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\zeta(2)\zeta(4)-\frac{1}{2}\zeta^2(3)+\zeta(2)\zeta(4)$$

$$\boxed{S_1=\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)},\quad A=-12\left(\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)\right)+12\zeta(6)$$

$$A=-21\zeta(6)+6\zeta^2(3)+12\zeta(6),\quad \boxed{A=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx=6\zeta^2(3)-9\zeta(6)}$$

$$B=\int_0^1\frac{\ln(x)Li_2^2(x)}{x}dx,\quad \textit{Applying Integration By parts, we get} -$$

$$B=\int_0^1\frac{\ln^2(x)\ln(1-x)Li_2(x)}{x}dx,B=\int_0^1\frac{\ln^2(x)\ln(1-x)}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}dx$$

$$B=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1\ln^2(x)\ln(1-x)x^{n-1}dx,\quad B=\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{d^2}{dn^2}\int_0^1\ln(1-x)x^{n-1}dx$$

$$B=-\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{d^2}{dn^2}\left(\frac{H_n}{n}\right),\quad B=-\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{d^2}{dn^2}\left(\frac{\gamma+\psi(n+1)}{n}\right)$$

$$B=-\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\gamma}{n^3}+\psi_2(n+1)-\frac{2}{n}\psi_1(n+1)+\frac{2}{n^3}\psi(n+1)\right)$$

$$B=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\zeta(3)}{n}+\frac{2\zeta(2)}{n^2}-\frac{2H_n}{n^3}-\frac{2H_n^{(2)}}{n^2}-\frac{2H_n^{(3)}}{n}\right),$$

$$B=\sum_{n=1}^{\infty}\left(\frac{2\zeta(3)}{n^3}+\frac{2\zeta(2)}{n^4}-\frac{2H_n}{n^5}-\frac{2H_n^{(2)}}{n^4}-\frac{2H_n^{(3)}}{n^3}\right)$$

$$B=2\zeta^2(3)+2\zeta(2)\zeta(4)-2S_1-2S_2-2S_3$$

$(S_1$ *is calculated above*$)$

$$S_2=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}$$

By Cauchy product on $Li_2(x)$ *and* $Li_3(x)$,

$$Li_2(x)Li_3(x)=6\sum_{n=1}^{\infty}\frac{H_n}{n^4}x^n+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}x^n+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}x^n-10Li_5(x)$$

Divide both sides by x *and integrate from* 0 *to* y,

$$\int_0^y\frac{Li_2(x)Li_3(x)}{x}dx=6\sum_{n=1}^{\infty}\frac{H_n}{n^5}y^n+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}y^n+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}y^n-10Li_6(y)$$

$$\frac{1}{2}Li_3^2(y)=6\sum_{n=1}^{\infty}\frac{H_n}{n^5}y^n+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}y^n+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}y^n-10Li_6(y)$$

Putting $y=1$,

$$\frac{1}{2}Li_3^2(1)=6\sum_{n=1}^{\infty}\frac{H_n}{n^5}+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}-10Li_6(1)$$

$$\frac{1}{2}Li_3^2(1)=6S_1+3S_2+S_3-10Li_6(1) \qquad ---(1)$$

Now,

$$S_3=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}, \qquad S_3=\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{k^3}$$

Splitting the inner Sum,

$$S_3=\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=n}^{\infty}\frac{1}{k^3}+\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=n}^{n}\frac{1}{k^3}$$

Switching the order of sum in the middle term,

$$S_3=\zeta^2(3)-\sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{k}\frac{1}{n^3}+\sum_{n=1}^{\infty}\frac{1}{n^6}, \qquad \textit{Replacing k and n by each other,}$$

$$S_3=\zeta^2(3)-\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{k^3}+\zeta(6), \qquad S_3=\zeta^2(3)-S_3+\zeta(6)$$

$$\boxed{S_3=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}=\frac{1}{2}\left(\zeta^2(3)+\zeta(6)\right)}$$

Now Putting the Calculated Values Of S_1 and S_3 in equaltion (1) we get S_2 as –

$$\boxed{S_2=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\zeta^2(3)-\frac{1}{3}\zeta(6)}$$

$$B=2\zeta^2(3)+2\zeta(2)\zeta(4)-2S_1-2S_2-2S_3$$

$$B=2\zeta^2(3)+2\zeta(2)\zeta(4)-2\left(\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)\right)-2\left(\zeta^2(3)-\frac{1}{3}\zeta(6)\right)-2\left(\frac{1}{2}\left(\zeta^2(3)+\zeta(6)\right)\right)$$

$$B=2\zeta(2)\zeta(4)-\frac{7}{2}\zeta(6)+\frac{2}{3}\zeta(6)-2\zeta(6), \qquad B=\frac{7}{2}\zeta(6)-\frac{7}{2}\zeta(6)+\frac{2}{3}\zeta(6)-2\zeta(6)$$

$$\boxed{B=\int_0^1\frac{\ln(x)\,Li_2^2(x)}{x}dx=\frac{-1}{3}\zeta(6)}, \qquad I=A-3B$$

$$I=\left(6\zeta^2(3)-9\zeta(6)\right)-3\left(\frac{-1}{3}\zeta(6)\right), \quad I=6\zeta^2(3)-8\zeta(6)$$

$$\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx-3\int_0^1\frac{\ln(1-x)\,Li_2^2(1-x)}{1-x}dx=6\zeta^2(3)-8\zeta(6)$$

2564. Find a closed form:

$$J=\int_0^{\frac{\pi}{4}} x\tan x\ln(\cos x)\,dx$$

Proposed by Naren Bhandari-Nepal

Solution by Pratham Prasad-India

$$J=\int_0^{\frac{\pi}{4}} x\tan x\ln(\cos x)\,dx=-\frac{1}{2}\int_0^{\frac{\pi}{4}} x\ d(\ln^2(\cos x))$$

$$=-\frac{\pi}{32}\ln^2(2)+\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln^2(\cos x)dx$$

$$I=\int_0^{\frac{\pi}{4}}\ln^2(\cos x)dx$$

$$I=\int_0^{\frac{\pi}{4}}\ln^2(2\cos x)dx-2\ln(2)\int_0^{\frac{\pi}{4}}\ln(\cos x)\,dx-\frac{\pi}{4}\ln^2(2)$$

$$I=\int_0^{\frac{\pi}{4}}x^2dx+2\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$I=\frac{\pi^3}{192}+\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n^2}\sin\left(\frac{n\pi}{2}\right)-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{\pi^3}{192}+Im\left(\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n^2}i^n\right)-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{\pi^3}{192}+Im\left(\sum_{n=1}^{\infty}\frac{(-i)^nH_{n-1}}{n^2}\right)-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{\pi^3}{192}+Im\left(\zeta(3)-Li_3(1+i)+\ln(1+i)Li_2(1+i)+\frac{1}{2}\ln(-i)\ln^2(1+i)\right)$$

$$-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{7\pi^3}{192}+\frac{5\pi}{16}\ln^2(2)-\frac{1}{2}G\ln(2)+Im\big(Li_3(1-i)\big)$$

$$J=-\frac{\pi}{32}\ln^2(2)+\frac{7\pi^3}{384}+\frac{5\pi}{32}\ln^2(2)-\frac{1}{4}G\ln(2)+\frac{1}{2}Im\big(Li_3(1-i)\big)$$

$$J=\frac{7\pi^3}{384}+\frac{\pi}{8}\ln^2(2)-\frac{1}{4}G\ln(2)+\frac{1}{2}Im\big(Li_3(1-i)\big)$$

2565. Find a closed form:

$$I=\int_0^1 Li_3(-x^2)(1-x^2)x+\frac{x\ln(1+x)}{x^2+1}\,dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pratham Prasad-India

$$I=\frac{1}{2}\int_0^1 Li_3(-x)(1-x)dx+\int_0^1\frac{x\ln(1+x)}{x^2+1}\,dx$$

$$I=\frac{1}{2}\int_0^1 Li_3(-x)\,dx-\frac{1}{2}\int_0^1 xLi_3(-x)\,dx+\int_0^1\frac{x\ln(1+x)}{1+x^2}dx$$

$$\boxed{I=\frac{1}{2}A-\frac{1}{2}B+C}$$

Where,

$$A=\int_0^1 Li_3(-x)\,dx,\qquad A=Li_3(-1)-\int_0^1 x\left(\frac{Li_2(-x)}{x}\right)dx$$

$$A=Li_3(-1)-\int_0^1 Li_2(-x)dx,\qquad A=Li_3(-1)-Li_2(-1)+1-2\ln(2)$$

$$\boxed{A=-\frac{3}{4}\zeta(3)+\frac{\pi^2}{12}+1-2\ln(2)}$$

$$B=\int_0^1 xLi_3(-x)\,dx$$

$$B=\frac{1}{2}Li_3(-1)-\int_0^1\frac{x^2}{2}\left(\frac{Li_2(-x)}{x}\right)dx,\qquad B=\frac{1}{2}Li_3(-1)-\frac{1}{2}\int_0^1 xLi_2(-x)\,dx$$

$$B=\frac{1}{2}Li_3(-1)-\frac{1}{2}\left(\frac{1}{24}(3-\pi^2)\right),\qquad \boxed{B=-\frac{3}{8}\zeta(3)-\frac{1}{16}+\frac{\pi^2}{48}}$$

$$C=\int_0^1\frac{x\ln(1+x)}{1+x^2}dx,\qquad C=\int_0^1\frac{x^2}{1+x^2}\int_0^1\frac{1}{1+xy}dy\,dx$$

$$C=\int_0^1\int_0^1\frac{x^2}{(1+x^2)(1+xy)}dx\,dy$$

$$C=\int_0^1\frac{y}{2(1+y^2)}\ln(2)+\frac{\ln(1+y)}{y(y^2+1)}-\frac{\pi}{4(1+y^2)}dy$$

$$C=\frac{1}{4}\ln^2(2)+\int_0^1\frac{\ln(1+y)}{y(y^2+1)}dy-\frac{\pi^2}{16}$$

$$C=\frac{1}{4}\ln^2(2)+\int_0^1\frac{\ln(1+y)}{y}dy-\int_0^1\frac{y\ln(1+y)}{(y^2+1)}dy-\frac{\pi^2}{16}$$

$$C=\frac{1}{4}\ln^2(2)+\int_0^1\frac{\ln(1+y)}{y}dy-C-\frac{\pi^2}{16}$$

$$2C=\frac{1}{4}\ln^2(2)+\frac{\pi^2}{12}-\frac{\pi^2}{16},\qquad 2C=\frac{1}{4}\ln^2(2)+\frac{\pi^2}{48},\qquad \boxed{C=\frac{1}{8}\ln^2(2)+\frac{\pi^2}{96}}$$

Putting everything back in,

$$I=\frac{1}{2}A-\frac{1}{2}B+C$$

$$I=\frac{1}{2}\left(-\frac{3}{4}\zeta(3)+\frac{\pi^2}{12}+1-2\ln(2)\right)-\frac{1}{2}\left(-\frac{3}{8}\zeta(3)-\frac{1}{16}+\frac{\pi^2}{48}\right)+\left(\frac{1}{8}\ln^2(2)+\frac{\pi^2}{96}\right)$$

Simplifying,

$$I=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{24}+\frac{17}{32}-\ln(2)+\frac{1}{8}\ln^2(2)$$

$$\boxed{\int_0^1 Li_3(-x^2)(1-x^2)x+\frac{x\ln(1+x)}{x^2+1}\,dx=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{24}+\frac{17}{32}-\ln(2)+\frac{1}{8}\ln^2(2)}$$

2566. Prove that:

$$\int_0^{\frac{\pi}{2}}\big(x\ln(1+\cos(x))+x\ln(\sin(x))\big)dx=\pi G-\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}\ln(2)$$

$G\to$ *Catalan's constant* , $\zeta(3)\to$ *Apery's constant*

Proposed by Shirvan Tahirov, Elsen Kerimov-Azerbaijan

Solution 1 by Quadri Faruk Temitope-Nigeria

$$I=\underbrace{\int_0^{\frac{\pi}{2}}x\ln(1+\cos(x))dx}_{A}+\underbrace{\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx}_{B}=A+B$$

$$A=\int_0^{\frac{\pi}{2}}x\ln(1+\cos(x))dx=\int_0^{\frac{\pi}{2}}x\ln\left(2\cos^2\left(\frac{x}{2}\right)\right)dx=\int_0^{\frac{\pi}{2}}x\ln(2)dx+\int_0^{\frac{\pi}{2}}x\ln\left(\cos^2\left(\frac{x}{2}\right)\right)=$$

$$=\ln(2)\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2\right]+2\int_0^{\frac{\pi}{2}}x\ln\left(\cos\left(\frac{x}{2}\right)\right)dx=\frac{\pi^2}{8}\ln(2)+2\int_0^{\frac{\pi}{2}}x\left[-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n}\right]dx=$$

$$\frac{\pi^2}{8}\ln(2)-2\ln(2)\frac{x^2}{8}\Bigg|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x\left[\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n}\right]dx=\frac{\pi^2}{8}\ln(2)-\frac{\pi^2}{4}\ln(2)-$$

$$2\Re\int_0^{\frac{\pi}{2}}x\left[\sum_{n=1}^{\infty}\frac{(-1)^ne^{-inx}}{n}\right]dx=\frac{\pi^2}{8}\ln(2)-\frac{\pi^2}{4}\ln(2)-2\Re\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{2}}x\,e^{-inx}dx=$$

$$-\frac{\pi^2}{8}\ln(2)-2\Re\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[-\frac{1}{n^2}+\frac{e^{-\frac{i\pi n}{2}}}{n^2}+\frac{i\pi ne^{-\frac{i\pi n}{2}}}{2n^2}\right]=-\frac{\pi^2}{8}\ln(2)-2\left[-\frac{\pi G}{2}+\frac{21}{32}\zeta(3)\right]$$

$$A=-\frac{\pi^2}{8}\ln(2)+\pi G-\frac{21}{16}\zeta(3)$$

Working on B

$$B=\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx=\int_0^{\frac{\pi}{2}}x\left[-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right]dx=-\ln(2)\int_0^{\frac{\pi}{2}}xdx-$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}}x\cos(2nx)dx=-\ln(2)\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2\right]-\sum_{n=1}^{\infty}\frac{1}{n}\Re\int_0^{\frac{\pi}{2}}xe^{-2inx}\,dx=-\frac{\pi^2}{8}\ln(2)-$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\Re\left[\frac{-1+e^{-i\pi n}+i\pi n e^{-i\pi n}}{4n^2}\right]=-\frac{\pi^2}{8}\ln(2)-\frac{1}{4}\Re\left[\sum_{n=1}^{\infty}-\frac{1}{n^3}+\sum_{n=1}^{\infty}\frac{e^{-i\pi n}}{n^3}+i\pi\sum_{n=1}^{\infty}\frac{e^{-i\pi n}}{n^2}\right]=$$

$$-\frac{\pi^2}{8}\ln(2)-\frac{1}{4}\Re\left[-\zeta(3)-\frac{3}{4}\zeta(3)-i\pi\frac{\zeta(2)}{2}\right]=-\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)$$

$$B=-\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)$$

$$I=\underbrace{\int_0^{\frac{\pi}{2}}x\ln(1+\cos(x))dx}_{A}+\underbrace{\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx}_{B}=-\frac{\pi^2}{8}\ln(2)+\pi G-\frac{21}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)$$

$$I=\pi G-\frac{7}{8}\zeta(3)-\frac{\pi^2}{4}ln(2)$$

Solution 2 by Pratham Prasad-India

$$\psi=\int_0^{\frac{\pi}{2}}xln(1+cosx)+xlnsinx\,dx=\int_0^{\frac{\pi}{2}}xln\left(2\cos^2\left(\frac{x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}}xln(sinx)\,dx$$

$$=4\int_0^{\frac{\pi}{4}}uln(2\cos^2(u))\,du+\int_0^{\frac{\pi}{2}}xln(sinx)\,dx$$

$$=4\int_0^{\frac{\pi}{4}}uln(2)\,du+8\int_0^{\frac{\pi}{4}}uln(cosu)\,du+\int_0^{\frac{\pi}{2}}xln(sinx)\,dx$$

By expanding and evaluating using Fourier series of the second and third integral

$$\int_0^{\frac{\pi}{4}}uln(cosu)\,du=\int_0^{\frac{\pi}{4}}u\left(-\ln(2)+\sum_{K=1}^{\infty}\frac{(-1)^{k-1}\cos(2ku)}{k}\right)du=$$

$$=-\frac{\pi^2}{32}\ln(2)+\int_0^{\frac{\pi}{4}}u\left(\sum_{K=1}^{\infty}\frac{(-1)^{k-1}\cos(2ku)}{k}\right)du$$

$$=-\frac{\pi^2}{32}\ln(2)+\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^{\frac{\pi}{4}}(u\cos(2ku))\,du$$

$$=-\frac{\pi^2}{32}\ln(2)+\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{\pi}{8k}\sin\left(\frac{\pi k}{2}\right)+\frac{1}{4k^2}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{4k^2}\right)$$

$$=-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{8}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\left(\sin\left(\frac{\pi k}{2}\right)\right)+\frac{1}{4}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}\left(\cos\left(\frac{\pi k}{2}\right)\right)-\frac{1}{4}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$$=-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{8}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{32}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}-\frac{1}{4}\sum_{K=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$$=\frac{1}{128}(16\pi G-21\zeta(3)-4\pi^2\ln(2))$$

$$\int_0^{\frac{\pi}{2}} x\ln(\sin x)\,dx = \int_0^{\frac{\pi}{2}} x\left(-\ln(2) - \sum_{K=1}^{\infty}\frac{\cos(2ku)}{k}\right)dx =$$

$$= -\frac{\pi^2}{8}\ln(2) - \int_0^{\frac{\pi}{2}} x\left(\sum_{K=1}^{\infty}\frac{\cos(2ku)}{k}\right)dx = -\frac{\pi^2}{8}\ln(2) - \sum_{K=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}} x(\cos(2ku))\,dx =$$

$$= -\frac{\pi^2}{8}\ln(2) - \sum_{K=1}^{\infty}\frac{1}{k}\left(\frac{(-1)^k}{4k^2} - \frac{1}{4k^2}\right) = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}ln(2)$$

Putting everything back :

$$\psi = \frac{\pi^2}{8}\ln(2) + 8\left(\frac{1}{128}(16\pi G - 21\zeta(3) - 4\pi^2\ln(2))\right) + \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}ln(2)$$

$$= \frac{\pi^2}{8}\ln(2) + \pi G - \frac{21}{16}\zeta(3) - \frac{\pi^2}{4}\ln(2) + \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}ln(2)$$

$$\psi = \pi G - \frac{7}{8}\zeta(3) - \frac{\pi^2}{4}ln(2)$$

Solution 3 by Exodo Halcalias-Angola

$$\Omega = \int_0^{\frac{\pi}{2}}(xln(1+\cos(x)) + xln(\sin(x)))dx = \int_0^{\frac{\pi}{2}} xln\left(\sin(x) + \frac{1}{2}\sin(x)\cos(x)\right)dx =$$

$$= \int_0^{\frac{\pi}{2}} xln\left(4\sin\left(\frac{x}{2}\right)cos^3\left(\frac{x}{2}\right)\right)dx = 4\int_0^{\frac{\pi}{4}} xln(\sin(2x))dx + 4\ln(2)\int_0^{\frac{\pi}{4}} xdx + 8\int_0^{\frac{\pi}{4}} xln(\cos(x))dx =$$

$$\int_0^{\frac{\pi}{2}} x\left(\ln\left(\frac{1}{2}\right) - \sum_{k\in N}\frac{\cos(2kx)}{k}\right)dx - \frac{\pi^2}{8}ln\left(\frac{1}{2}\right) + 8\int_0^{\frac{\pi}{4}} x\left(\ln\left(\frac{1}{2}\right) + \sum_{k\in N}\frac{(-1)^{k-1}\cos(2kx)}{k}\right)dx = \frac{\pi^2}{4}ln\left(\frac{1}{2}\right)$$

$$-\sum_{k\in N}\frac{1}{k}\left(\frac{\pi\sin(\pi k)}{4k} + \frac{cos(\pi k)}{4k^2} - \frac{1}{k^2}\right) + \sum_{k\in N}\frac{(-1)^{k-1}}{k}\left(\frac{\pi}{k}\sin\left(\frac{k\pi}{2}\right) + \frac{1}{2k^2}\cos\left(\frac{k\pi}{2}\right) - \frac{1}{2k^2}\right) =$$

$$\frac{\pi^2}{4}ln\left(\frac{1}{2}\right) - \frac{1}{4}\sum_{k\in N}\frac{(-1)^k}{k^3} + \frac{1}{4}\sum_{k\in N}\frac{1}{k^3} + \pi\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^2} + \frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k)^3} - \frac{1}{2}\sum_{k\in N}\frac{1}{k^3} =$$

$$\frac{\pi^2}{4}ln\left(\frac{1}{2}\right) + \pi\beta(2) - \frac{7}{8}\sum_{k\in N}\frac{1}{k^3} = \pi G - \frac{7}{8}\zeta(3) - \frac{\pi^2}{4}ln(2)$$

$$\Omega = \pi G - \frac{7}{8}\zeta(3) - \frac{\pi^2}{4}ln(2)$$

2567. Find:

$$\psi = \int_0^{\frac{\pi}{2}}\frac{\ln(sinx)}{1+\cos^2(x)}dx$$

Proposed by Le Thu-Vietnam

Solved by Pratham Prasad-India

$$let, \frac{\pi}{2} - x = y$$

$$\psi=\int_0^{\frac{\pi}{2}}\frac{\ln(cosy)}{1+\sin^2(y)}dy,\qquad \psi=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\ln(1+\tan^2(y))}{1+2\tan^2(y)}\sec^2(y)\,dy$$

$$\psi=-\frac{1}{2}\int_0^{\infty}\frac{\ln(1+\epsilon^2)}{1+2\epsilon^2}d\epsilon,\qquad \psi=-\frac{1}{2}\int_0^{\infty}\frac{1}{1+2\epsilon^2}\int_0^1\frac{\epsilon^2}{1+\alpha\epsilon^2}d\alpha\,d\epsilon$$

$$\psi=-\frac{1}{2}\int_0^1\int_0^{\infty}\frac{1}{1+2\epsilon^2}\frac{\epsilon^2}{1+\alpha\epsilon^2}d\epsilon d\alpha$$

$$\psi=-\frac{1}{2}\int_0^1\frac{1}{2-\alpha}\int_0^{\infty}\frac{1}{1+\alpha\epsilon^2}-\frac{1}{1+2\epsilon^2}d\epsilon d\alpha$$

$$\psi=-\frac{1}{2}\int_0^1\frac{1}{2-\alpha}\int_0^{\infty}\frac{1}{1+\alpha\epsilon^2}d\epsilon d\alpha+\frac{1}{2}\int_0^1\frac{1}{2-\alpha}\int_0^{\infty}\frac{1}{1+2\epsilon^2}d\epsilon d\alpha$$

$$\psi=\frac{1}{2}\int_0^1\frac{1}{\alpha-2}\left(\frac{\pi}{2\sqrt{\alpha}}\right)d\alpha-\frac{1}{2}\int_0^1\frac{1}{\alpha-2}\left(\frac{\pi}{2\sqrt{2}}\right)d\alpha$$

$$\psi=\frac{\pi}{2}\int_0^1\frac{1}{\alpha-2}\left(\frac{1}{2\sqrt{\alpha}}\right)d\alpha-\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\int_0^1\frac{1}{\alpha-2}d\alpha$$

$$\psi=\frac{\pi}{2}\int_0^1\frac{1}{\beta^2-(\sqrt{2})^2}\,d\beta+\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\ln(2)$$

$$\psi=\frac{\pi}{4\sqrt{2}}\int_0^1\frac{1}{\beta-\sqrt{2}}\,d\beta-\frac{\pi}{4\sqrt{2}}\int_0^1\frac{1}{\beta+\sqrt{2}}\,d\beta+\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\ln(2)$$

$$\psi=\frac{\pi}{4\sqrt{2}}\ln\left(1-\frac{1}{\sqrt{2}}\right)-\frac{\pi}{4\sqrt{2}}\ln\left(1+\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\ln(2)=\frac{\pi}{4\sqrt{2}}\ln\left(\frac{2(\sqrt{2}-1)}{\sqrt{2}+1}\right)=$$

$$=\frac{\pi}{4\sqrt{2}}\ln\left(2(\sqrt{2}-1)^2\right)=\frac{\pi}{2\sqrt{2}}\ln(2-\sqrt{2})$$

2568. Find:

$$\int_0^1\frac{\sqrt{x}\ln^2(x)}{1+x+x^2}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Pratham Prasad-India

Lemma: If

$$I(a)=\int_0^1\frac{x^a}{1+x+x^2}dx$$

then:

$$I''(a)=\frac{1}{27}\left(\psi_2\left(\frac{a+1}{3}\right)-\psi_2\left(\frac{a+2}{3}\right)\right)$$

Proof:

$$I(a)=\int_0^1\frac{x^a(1-x)}{1-x^3}dx$$

$$I(a)=\int_0^1 (x^a-x^{a+1})\sum_{r=0}^{\infty} x^{3r}\,dx$$

$$I(a)=\sum_{r=0}^{\infty}\int_0^1 (x^{a+3r}-x^{a+1+3r})\,dx,\quad I(a)=\sum_{r=0}^{\infty}\left(\frac{1}{3r+a+1}-\frac{1}{3r+a+2}\right)$$

$$I(a)=\frac{1}{3}\left(\psi\left(\frac{a+1}{3}\right)-\psi\left(\frac{a+2}{3}\right)\right),\quad I'(a)=\frac{1}{9}\left(\psi_1\left(\frac{a+1}{3}\right)-\psi_1\left(\frac{a+2}{3}\right)\right)$$

$$I''(a)=\frac{1}{27}\left(\psi_2\left(\frac{a+1}{3}\right)-\psi_2\left(\frac{a+2}{3}\right)\right),\qquad I''\left(\frac{1}{2}\right)=\frac{1}{27}\left(\psi_2\left(\frac{1}{2}\right)-\psi_2\left(\frac{5}{6}\right)\right)$$

$$I''\left(\frac{1}{2}\right)=\int_0^1 \frac{\sqrt{x}\ln^2(x)}{1+x+x^2}dx$$

2569. **Find a closed form:**

$$I=\int_0^{\infty}\frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

Proposed by Vincent Nguen-USA

Solution by Pratham Prasad-India

$$I=\int_0^{\infty}\frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$\text{Replace } z \text{ by } -z$$

$$I=\int_0^{-\infty}\frac{e^{-3z}-e^{-z}}{z(e^{-2z}+1)^2}dz,\qquad I=\int_{-\infty}^{0}\frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$2I=I+I=\int_{-\infty}^{0}\frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz+\int_0^{\infty}\frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$2I=\int_{-\infty}^{\infty}\frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz$$

$$\text{Let, } e^z=x$$

$$2I=\int_0^{\infty}\frac{x^2-1}{\ln(x)(1+x^2)^2}dx,\qquad 2I=\int_0^{\infty}\frac{1}{(1+x^2)^2}\int_0^2 x^a\,da\,dx$$

$$2I=\int_0^{\infty}\int_0^2\frac{x^a}{(1+x^2)^2}da\,dx,\qquad 2I=\int_0^2\int_0^{\infty}\frac{x^a}{(1+x^2)^2}dx\,da$$

$$2I=\frac{1}{2}\int_0^2\int_0^{\infty}\frac{x^{\frac{a-1}{2}}}{(1+x)^2}dx\,da,\qquad 2I=\frac{1}{2}\int_0^2 B\left(\frac{a+1}{2},2-\frac{a+1}{2}\right)da$$

$$4I=\int_0^2\frac{\left(\frac{1-a}{2}\right)\pi}{\sin\left(\frac{\pi(a+1)}{2}\right)}da,$$

$$Let\left(\frac{1+a}{2}\right)\pi = u$$

$$4I = -\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\pi - u}{\sin(u)} du$$

Replace $\pi - u$ *by* u

$$4I = -\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{u}{\sin(u)} du$$

Replace u *by* $u+\pi$

$$4I = -\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(u+\pi)}{\sin(u+\pi)} du, \qquad 4I = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(u+\pi)}{\sin(u)} du$$

$$4I = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{u}{\sin(u)} du + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sin(u)} du$$

$$4I = \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{u}{\sin(u)} du + 2(0), \qquad 4I = \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} u\, d\left(\ln\left(tan\left(\frac{u}{2}\right)\right)\right)$$

Apply Integration By parts

$$4I = \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} \ln\left(tan\left(\frac{u}{2}\right)\right) du$$

Replace $\frac{u}{2}$ *by* u

$$4I = \frac{4}{\pi}\left(2\int_{0}^{\frac{\pi}{4}} \ln(tan(u))\, du\right), \qquad 4I = \frac{4}{\pi}(2G), \qquad I = \frac{2G}{\pi}$$

2570. Find a closed form:

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1 - xyz + \sqrt{1-xyz}} dxdydz$$

Proposed by Vincent Nguyen-USA

Solution by Pratham Prasad-India

By the Property:

$$\int_0^1\int_0^1\int_0^1 f(xyz)dxdydz = \frac{1}{2}\int_0^1 f(x)\ln^2(x)dx$$

$$\psi = \int_0^1\int_0^1\int_0^1 \frac{1}{1 - xyz + \sqrt{1-xyz}} dxdydz = \frac{1}{2}\int_0^1 \frac{\ln^2(x)}{1 - x + \sqrt{1-x}} dx$$

$$= \frac{1}{2}\int_0^1 \frac{\ln^2(1-x)}{x+\sqrt{x}} dx = \frac{1}{2}\int_0^1 \frac{(1-\sqrt{x})\ln^2(1-x)}{\sqrt{x}(1-x)} dx =$$

$$=-\frac{1}{6}\int_0^1\frac{(1-\sqrt{x})}{\sqrt{x}}d(\ln^3(1-x))\overset{IBP}{\cong}-\frac{1}{12}\int_0^1\ln^3(1-x)x^{-\frac{3}{2}}dx=$$

$$=-\frac{1}{12}\lim_{\substack{x\to1\\ y\to-\frac{1}{2}}}\frac{\partial^3}{\partial x^3}B(x,y)=-\frac{1}{12}(-24\zeta(3)-16\ln^3(2)+24\zeta(2)\ln(2))=$$

$$=2\zeta(3)+\frac{4}{3}\ln^3(2)-\frac{\pi^2}{3}\ln(2)\,\psi=2\zeta(3)+\frac{1}{6}\ln^3(4)-\frac{\pi^2}{6}\ln(4)$$

$$\int_0^1\int_0^1\int_0^1\frac{1}{1-xyz+\sqrt{1-xyz}}dxdydz=2\zeta(3)+\frac{1}{6}\ln^3(4)-\frac{\pi^2}{6}\ln(4)$$

2571. Find:

$$\int_0^1\ln(1+\tanh(x)+\coth(x))\,dx$$

Proposed by Ahmed Nabeel-Iraq

Solution by Pratham Prasad-India

$$\int_0^1\ln(1+\tanh(x)+\coth(x))\,dx=$$

$$=\int_0^1\ln\left(1+\frac{e^{2x}-1}{e^{2x}+1}+\frac{e^{2x}+1}{e^{2x}-1}\right)dx=\int_0^1\ln\left(\frac{3e^{4x}+1}{e^{4x}-1}\right)dx=$$

$$=\ln(3)+\int_0^1\ln(e^{4x}+1/3)-\ln(e^{4x}-1)\,dx=\ln(3)+\int_0^1\int_{-1}^{1/3}\frac{1}{e^{4x}+a}da\,dx=$$

$$=\ln(3)+\int_{-1}^{\frac{1}{3}}\int_0^1\frac{1}{e^{4x}+a}dx\,da=\ln(3)-\int_{-1}^{\frac{1}{3}}\frac{1}{4a}\int_0^1\frac{-4ae^{-4x}}{1+ae^{-4x}}dx\,da=$$

$$=\ln(3)-\frac{1}{4}\int_{-1}^{\frac{1}{3}}\frac{1}{a}(\ln(1+ae^{-4})-\ln(1+a))\,da$$

$$=\ln(3)+\frac{1}{4}\left(Li_2\left(-\frac{a}{e^4}\right)-Li_2(-a)\right)_{-1}^{\frac{1}{3}}=$$

$$=\ln(3)+\frac{1}{4}Li_2\left(-\frac{1}{3e^4}\right)-\frac{1}{4}Li_2\left(\frac{1}{e^4}\right)-\frac{1}{4}Li_2\left(-\frac{1}{3}\right)+\frac{1}{4}Li_2(1)=$$

$$=\ln(3)+\frac{1}{4}Li_2\left(-\frac{1}{3e^4}\right)-\frac{1}{4}Li_2\left(\frac{1}{e^4}\right)-\frac{1}{4}Li_2\left(-\frac{1}{3}\right)+\frac{1}{4}\zeta(2)$$

2572. Find:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{dxdy}{\sin^2(x)+\cos(y)}$$

Proposed by Ankush Kumar Parcha-India

Solution by Mirsadix Muzefferov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^2(x)+\cos(y)}\right)dy=\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\frac{\frac{dx}{\cos^2(x)}}{\frac{\sin^2(x)}{\cos^2(x)}+\frac{\cos(y)}{\cos^2(x)}}\right)dy=$$

$$=\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\frac{d(tanx)}{(tanx)^2+cosy(1+\tan^2(x))}\right)dy\overset{\tan(x)=t}{\cong}\int_0^{\frac{\pi}{2}}\left(\int_0^{\infty}\frac{dx}{cos(y)+(1+\cos(y))t^2}\right)dy$$

$$=$$

$$=\int_0^{\frac{\pi}{2}}\frac{dy}{\sqrt{\cos(y)\,(1+\cos(y))}}\left(\int_0^{\infty}\frac{d\sqrt{\frac{1+\cos(y)}{\cos(y)}}t}{1+\left(\sqrt{\frac{1+\cos(y)}{\cos(y)}}t\right)^2}\right)dx=$$

$$\int_0^{\frac{\pi}{2}}\frac{dy}{\sqrt{\cos(y)\,(1+\cos(y))}}\left(arctan\sqrt{\frac{1+\cos(y)}{\cos(y)}}t\right)\Bigg|_0^{\infty}$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sqrt{\cos(y)(1+\cos(y))}}\right)dy=$$

$$\overset{\tan\left(\frac{y}{2}\right)=t}{\cong}\frac{\pi}{2}\int_0^1\frac{\left(\frac{2}{1+t^2}\right)}{\sqrt{\frac{1-t^2}{1+t^2}\cdot\frac{2}{1+t^2}}}dt==\frac{\pi}{2}\sqrt{2}\int_0^1\frac{1}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\sqrt{2}\arcsin(1)=\frac{\pi^2}{2\sqrt{2}}$$

2573. Find:

$$\int_0^1\int_0^1\frac{\arctan(x+y)}{x+y}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Quadri Faruk Temitope-Nigeria

Using integration by parts, IBP we write :

$$I=\int_0^1\left[\ln(x+y)\,arctan(x+y)\Big|_0^1-\int_0^1\frac{\ln(x+y)}{(x+y)^2+1}dy\right]dx$$

Let $I=A-B$ where:

$$A=\int_0^1\ln(x+1)\arctan(x+1)\,dx;\quad B=\int_0^1\int_0^1\frac{\ln(x+y)}{(x+y)^2+1}dxdy$$

Applying IBP again to A, we get:

$$A = [(x\ln(x) - x)\arctan(x)|_1^2 - \int_1^2 \frac{x\ln(x)}{x^2+1}dx + \int_1^2 \frac{x}{x^2+1}dx$$

After simplifications:

$$A = 2\ln(2)\arctan(2) - 2\arctan(2) + \frac{\pi}{4}$$

$$-\sum_{n=0}^{\infty}(-1)^n \int_1^2 x^{2n+1}\ln(x)\,dx + \left\{\frac{1}{2}\ln(x^2+1)\right\}_1^2$$

$$A = \pi\ln(2) - 2\arctan\left(\frac{1}{2}\right)\ln(2) - \pi + 2\arctan\left(\frac{1}{2}\right) + \frac{\pi}{4} - \frac{1}{4}Li_2(-4) - \frac{1}{2}\ln(2)\ln(5) + \frac{1}{2}\ln(2) - \frac{\pi^2}{48}$$

Using the identity : $\Im\left|\frac{i}{(x+y)+i}\right| = \frac{i}{(x+y)^2+1}$ *we write*:

$$B = \Im\int_0^1\int_0^1 \frac{\ln(x+y)}{(x+y)+i}dxdy$$

$$B = \Im(Li_2(i(x+1)) - Li_2(ix) + \ln(1+x)\ln(-i(1+x-i)) - \ln(x)\ln(-i(x+i))\,dx$$

$$B = -\left(-\frac{\pi}{2} + 2\arctan(2) - \frac{\ln(5)}{2} - 2G + \pi\ln(2) - \frac{1}{2}\ln(2)\right)$$

$$I = A - B = 2\Im Li_2\left(\frac{1}{2i}\right) - 2G + \pi\ln(2) - \frac{\pi}{2} + 2\arctan\left(\frac{1}{2}\right) + \frac{\ln(5)}{2} - \ln(2)$$

2574. Find:

$$\int_1^2 \frac{dx}{x\sqrt{1+x^3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_1^2 \frac{dx}{x\sqrt{1+x^3}} = \int_1^2 \frac{x^2dx}{x^3\sqrt{1+x^3}} = \frac{1}{3}\int_1^2 \frac{d(1+x^3)}{x^3\sqrt{1+x^3}} \overset{\sqrt{1+x^3}\to t}{=}$$

$$= \frac{1}{3}\int_{\sqrt{2}}^3 \frac{dt^2}{(t^2-1)t} = \frac{2}{3}\int_{\sqrt{2}}^3 \frac{tdt}{t(t^2-1)} = -\frac{2}{3}\int_{\sqrt{2}}^3 \frac{dt}{(1-t^2)} =$$

$$= -\frac{2}{3}ln\left|\frac{1+t}{1-t}\right|\Big|_{\sqrt{2}}^{3} = -\frac{2}{3}\left(\ln(2) - \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right) =$$

$$= -\frac{2}{3}\left(\ln(2) - \ln(\sqrt{2}+1)^2\right) = \frac{2}{3}(\ln(3+2\sqrt{2}) - \ln(2)) =$$

$$= \frac{2}{3}\ln\left(\frac{3+2\sqrt{2}}{2}\right) = \frac{2}{3}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right) = \frac{2}{3}\ln\left(1+\frac{1}{\sqrt{2}}\right)$$

2575. Find a closed form:

$$I = \int_1^{\infty} \frac{Li_2[(1-y)^3]}{(y-1)^3}dy$$

Proposed by Aryan Desai-India

Solution by Quadri Faruk Temitope-Nigeria

$$I=\int_1^\infty \frac{Li_2[(1-y)^3]}{(y-1)^3}dy$$

$$I\overset{y-1=x}{\cong}\int_0^\infty \frac{Li_2[-x^3]}{x^3}dx,\quad I\overset{x^3=p}{\cong}\int_0^\infty \frac{Li_2[-p]}{p}\left(\frac{1}{3}p^{-\frac{2}{3}}\right)dp$$

$$I=\frac{1}{3}\int_0^\infty p^{-1-\frac{2}{3}}Li_2(-p)\,dp$$

$$I\overset{Integration\ By\ Parts}{\cong}-\frac{1}{2}\int_0^\infty p^{-\frac{5}{3}}\ln(1+p)\,dp$$

$$I\overset{Integration\ By\ Parts}{\cong}-\frac{3}{2}\int_0^\infty \frac{p^{-\frac{2}{3}}}{1+p}dp$$

$$I=-\frac{3}{2}B\left(\frac{2}{3},\frac{1}{3}\right),\quad I=-\frac{\frac{3}{2}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}$$

$$I=-\frac{3}{2}\Gamma\left(1-\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)$$

$$I\overset{Euler\ Reflection}{\cong}-\frac{3}{2}\pi\text{cosec}\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}\pi}{2}$$

2576. Find:

$$\int_0^{\frac{\pi}{2}}\frac{\sin(2x)+\sin(x)}{\sqrt{1+5\cos(x)}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\sin(2x)+\sin(x)}{\sqrt{1+5\cos(x)}}dx=\int_0^{\frac{\pi}{2}}\frac{\sin(x)\,(1+2\cos(x))}{\sqrt{1+5\cos(x)}}dx=$$

$$-\int_0^{\frac{\pi}{2}}\frac{(1+2\cos(x))}{\sqrt{1+5\cos(x)}}d(\cos(x))\overset{\cos(x)\to t}{\cong}-\frac{1}{5}\int_1^0\frac{(1+2t)}{\sqrt{1+5t}}d((1+5t)=$$

$$-\frac{1}{5}\int_1^0(1+2t)d(2\sqrt{1+5t})=-\frac{2}{5}\int_1^0(1+2t)d\left(\sqrt{1+5t}\right)\overset{\sqrt{1+5t}\to u}{\cong}$$

$$\frac{2}{5}\int_1^{\sqrt{6}}\frac{(2u^2+3)}{5}du=\frac{2}{25}\int_1^{\sqrt{6}}(2u^2+3)\,du$$

$$\left\{1+5t=u^2\to\ t=\frac{u^2-1}{5}\to\ 1+2t=1+\frac{2u^2-2}{5}=\frac{2u^2+3}{5}\right\}$$

$$\frac{2}{25}\int_1^{\sqrt{6}}(2u^2+3)\,du=\frac{2}{25}\left(\frac{2}{3}u^3+3u\right)\Big|_1^{\sqrt{6}}=\frac{2}{25}\left(7\sqrt{6}-\frac{11}{3}\right)$$

2577. Find a closed form:

$$\psi=\int_0^1\frac{x+1}{x}\log(2x+1)\,dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pratham Prasad-India

$$\psi=\int_0^1\frac{x+1}{x}\log(2x+1)\,dx=\int_0^1\log(2x+1)\,dx+\int_0^1\frac{\log(2x+1)}{x}\,dx$$

$$=[x\log(2x+1)]_0^1-\int_0^1\frac{2x}{2x+1}dx+\int_0^1\frac{\log(2x+1)}{x}\,dx$$

$$=\log(3)-\int_0^1 1\,dx+\int_0^1\frac{1}{2x+1}\,dx+\int_0^1\frac{\log(2x+1)}{x}\,dx$$

$$=\log(3)-1+\left[\frac{1}{2}\log(2x+1)\right]_0^1+\int_0^1\frac{\log(2x+1)}{x}\,dx$$

$$=\log(3)-1+\frac{1}{2}\log(3)+\int_0^1\frac{\log(2x+1)}{x}\,dx=\frac{3}{2}\log(3)-1+\int_0^2\frac{\log(x+1)}{x}\,dx=$$

$$=\frac{3}{2}\log(3)-1+\int_0^1\frac{\log(x+1)}{x}\,dx+\int_1^2\frac{\log(x+1)}{x}\,dx$$

$$=\frac{3}{2}\log(3)-1+\frac{1}{2}\zeta(2)+\int_{\frac{1}{2}}^1\frac{\log\left(\frac{1}{x}+1\right)}{x}\,dx$$

$$=\frac{3}{2}\log(3)-1+\frac{1}{2}\zeta(2)+\int_{\frac{1}{2}}^1\frac{\log(1+x)}{x}\,dx-\int_{\frac{1}{2}}^1\frac{\log(x)}{x}\,dx$$

$$=\frac{3}{2}\log(3)-1+\frac{1}{2}\zeta(2)+\int_0^1\frac{\log(1+x)}{x}\,dx-\int_0^{\frac{1}{2}}\frac{\log(1+x)}{x}\,dx-\left[\frac{1}{2}\log^2(x)\right]_{\frac{1}{2}}^1$$

$$=\frac{3}{2}\log(3)-1+\frac{1}{2}\zeta(2)+\frac{1}{2}\zeta(2)+Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\log^2(2)$$

$$=\frac{3}{2}\log(3)-1+\zeta(2)+Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\log^2(2)$$

$$\psi=\frac{3}{2}\log(3)-1+\frac{\pi^2}{6}+Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\log^2(2)$$

$$\int_0^1\frac{x+1}{x}\log(2x+1)\,dx=\frac{3}{2}\log(3)-1+\frac{\pi^2}{6}+Li_2\left(-\frac{1}{2}\right)+\frac{1}{2}\log^2(2)$$

2578. Find a closed form:

$$\psi=\int_0^1\int_0^1\frac{x^2\ln(x+1)\ln(y+2)}{(x+1)(y+1)}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Pratham Prasad-India

$$\psi=\int_0^1\int_0^1\frac{x^2\ln(x+1)\ln(y+2)}{(x+1)(y+1)}dxdy$$

$$=\left(\int_0^1\frac{x^2\ln(x+1)}{(x+1)}dx\right)\left(\int_0^1\frac{\ln(y+2)}{(y+1)}dy\right)$$

$$=\left(\int_0^1\frac{(x+1)^2\ln(x+1)}{(x+1)}dx-2\int_0^1\frac{(x+1)\ln(x+1)}{(x+1)}dx\right.$$

$$\left.+\int_0^1\frac{\ln(x+1)}{(x+1)}dx\right)\left(\int_0^1\frac{\ln(y+2)}{(y+1)}dy\right)$$

$$=\left(\int_0^1(x+1)\ln(x+1)dx-2\int_0^1\ln(x+1)\,dx+\int_0^1\frac{\ln(x+1)}{(x+1)}dx\right)\left(\int_0^1\frac{\ln(y+2)}{(y+1)}dy\right)$$

$$=(I_1-2I_2+I_3)(I_4)$$

$$I_1=\int_0^1(x+1)\ln(x+1)dx$$

$$I_1\overset{IBP}{=}\left[\frac{(x+1)^2}{2}\ln(x+1)\right]_0^1-\frac{1}{2}\int_0^1(x+1)\,dx$$

$$I_1=2\ln(2)-\frac{1}{2}\left(\frac{1}{2}+1\right)$$

$$I_1=2\ln(2)-\frac{3}{4}$$

$$I_2=\int_0^1\ln(x+1)\,dx$$

$$I_2\overset{IBP}{=}[(x+1)\ln(1+x)]_0^1-\int_0^1\frac{x+1}{x+1}dx$$

$$I_2=2\ln(2)-\int_0^1 1\,dx$$

$$I_2=2\ln(2)-1$$

$$I_3=\int_0^1\frac{\ln(x+1)}{(x+1)}dx$$

$$I_3\overset{\ln(1+x)=t}{=}\int_0^{\ln(2)}tdt$$

$$I_3=\frac{1}{2}\ln^2(2)$$

$$I_1-2I_2+I_3=2\ln(2)-\frac{3}{4}-2(2\ln(2)-1)+\frac{1}{2}\ln^2(2)$$

$$I_1-2I_2+I_3=-2\ln(2)+\frac{5}{4}+\frac{1}{2}\ln^2(2)$$

$$I_4=\int_0^1\frac{\ln(y+2)}{(y+1)}dy$$

$$I_4\overset{x=y+1}{\cong}\int_1^2\frac{\ln(y+1)}{y}dy$$

$$I_4=\int_0^2\frac{\ln(y+1)}{y}dy-\int_0^1\frac{\ln(y+1)}{y}dy$$

$$I_4=-Li_2(-2)-\frac{1}{2}\zeta(2)$$

$$\psi=(I_1-2I_2+I_3)(I_4)$$

$$\psi=\left(-2\ln(2)+\frac{5}{4}+\frac{1}{2}\ln^2(2)\right)\left(-Li_2(-2)-\frac{1}{2}\zeta(2)\right)$$

$$\psi=\left(2\ln(2)-\frac{5}{4}-\frac{1}{2}\ln^2(2)\right)\left(Li_2(-2)+\frac{1}{2}\zeta(2)\right)$$

$$\psi=2\ln(2)Li_2(-2)-\frac{5}{4}Li_2(-2)-\frac{1}{2}\ln^2(2)\,Li_2(-2)+\zeta(2)\ln(2)-\frac{5}{4}\zeta(2)$$
$$-\frac{1}{2}\zeta(2)\ln^2(2)$$

$$\boxed{\int_0^1\int_0^1\frac{x^2\ln(x+1)\ln(y+2)}{(x+1)(y+1)}dxdy=2\ln(2)Li_2(-2)-\frac{5}{4}Li_2(-2)-\frac{1}{2}\ln^2(2)\,Li_2(-2)+\zeta(2)\ln(2)-\frac{5}{4}\zeta(2)-\frac{1}{2}\zeta(2)\ln^2(2)}$$

2579. Find a closed form:

$$\int_0^1\frac{(x-1)^2Li_3(-x)}{x+1}dx,\qquad \zeta(3)\to Apery's\ constant$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$$\int_0^1\frac{(x-1)^2Li_3(-x)}{x+1}dx\to\begin{cases}u=Li_3(-x)\\ dv=\dfrac{(x-1)^2}{1+x}dx\end{cases}\to\begin{cases}du=\dfrac{Li_2(-u)}{x}\\ v=\left(\dfrac{x^2}{2}-3x+4\ln(1+x)\right)\end{cases}$$

$$\Omega=\left(\frac{x^2}{2}-3x+4\ln(1+x)\right)Li_3(-x)\Big|_0^1-\int_0^1\frac{Li_2(-u)}{x}\left(\frac{x^2}{2}-3x+4\ln(1+x)\right)dx=$$

$$\frac{15}{8}\zeta(3)-3\ln(2)\zeta(3)$$
$$-\frac{1}{2}\int_0^1 xLi_2(-x)dx+3\int_0^1 Li_2(-x)dx-4\int_0^1\frac{\ln(1+x)Li_2(-x)}{x}dx$$
$$A=\int_0^1 xLi_2(-x)dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 x^{n+1}\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(n+2)}=-\frac{\pi^2}{24}+\frac{1}{8}$$
$$B=\int_0^1 Li_2(-x)dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 x^n dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(n+1)}=-\frac{\pi^2}{12}+2\ln(2)-1$$
$$C=\int_0^1\frac{\ln(1+x)Li_2(-x)}{x}dx=-\int_0^1 Li_2(-x)d\big(Li_2(-x)\big)=-\frac{1}{2}Li_2^2(-x)\Big|_0^1=-\frac{\pi^4}{288}$$
$$\Omega=\frac{15}{8}\zeta(3)-3\ln(2)\zeta(3)-\frac{1}{2}A+3B-4C$$
$$=\frac{15}{8}\zeta(3)-3\ln(2)\zeta(3)-\frac{1}{2}\left(-\frac{\pi^2}{24}+\frac{1}{8}\right)+$$
$$3\left(-\frac{\pi^2}{12}+2\ln(2)-1\right)-4\left(-\frac{\pi^4}{288}\right)$$
$$\int_0^1\frac{(x-1)^2Li_3(-x)}{x+1}dx=\frac{15}{8}\zeta(3)-3\ln(2)\zeta(3)+\frac{\pi^4}{72}-\frac{11}{48}\pi^2+6ln(2)-\frac{49}{16}$$

Solution 2 by Pratham Prasad-India

$$\int_0^1\frac{(x-1)^2Li_3(-x)}{x+1}dx$$
$$=\int_0^1\frac{(x+1)^2Li_3(-x)}{x+1}d\,x-4\int_0^1 Li_3(-x)dx+4\int_0^1\frac{Li_3(-x)}{1+x}dx=$$
$$\int_0^1 xLi_3(-x)dx-3\int_0^1 Li_3(-x)dx+4\int_0^1\frac{Li_3(-x)}{1+x}dx=I_1-3I_2+4I_3$$
$$I_1=\int_0^1 xLi_3(-x)dx=\int_0^1 x\sum_{n=1}^{\infty}\frac{(-x)^n}{n^3}dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^{n+1}\,dx=$$
$$\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{n^2(n+2)}\right\}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n+2}+\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=$$

$$-\frac{3}{8}\zeta(3)+\frac{1}{8}\zeta(2)-\frac{1}{8}\left(\frac{1}{2}-\ln(2)\right)+\frac{1}{8}(-\ln(2))=-\frac{3}{8}\zeta(3)+\frac{1}{8}\zeta(2)-\frac{1}{16}$$

$$I_2=\int_0^1 Li_3(-x)dx=\int_0^1\sum_{n=1}^{\infty}\frac{(-x)^n}{n^3}dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^n\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n+1)}=$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-\ln(2)-\ln(2)+1=$$

$$-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1$$

$$I_3=\int_0^1\frac{Li_3(-x)}{1+x}dx=Li_3(-x)\ln(x+1)\Big|_0^1-\int_0^1\frac{Li_2(-x)}{x}\ln(1+x)\,dx=Li_3(-1)\ln(2)+$$

$$\frac{(Li_2(-x))^2}{2}\Bigg|_0^1=\ln(2)\left(-\frac{3}{4}\zeta(3)\right)+\frac{1}{2}(Li_2(-2))^2=\ln(2)\left(-\frac{3}{4}\zeta(3)\right)+\frac{1}{2}\left(-\frac{1}{2}\zeta(2)\right)^2=$$

$$\frac{\pi^4}{288}-\frac{3}{4}\zeta(3)\ln(2)$$

$$\int_0^1\frac{(x-1)^2Li_3(-x)}{x+1}dx=I_1-3I_2+4I_3=-\frac{3}{8}\zeta(3)+\frac{1}{8}\zeta(2)-\frac{1}{16}-3\left(-\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)-2\ln(2)+1\right)+4\left(\frac{\pi^4}{288}-\frac{3}{4}\zeta(3)\ln(2)\right)$$

$$\int_0^1\frac{(x-1)^2Li_3(-x)}{x+1}dx=\frac{15}{8}\zeta(3)-3\ln(2)\zeta(3)+\frac{\pi^4}{72}-\frac{11}{48}\pi^2+6\ln(2)-\frac{49}{16}$$

2580. Find:

$$\int_0^1\int_0^1\frac{(a-x^2y^2)dxdy}{(1+x)(1+y)(1+x^2y^2)}$$

Proposed by Bui Hong Suc-Vietnam

Solution by Pratham Prasad-India

$$\int_0^1\int_0^1\frac{(a-x^2y^2)dxdy}{(1+x)(1+y)(1+x^2y^2)}=$$

$$=\int_0^1\int_0^1\frac{(a+1)dxdy}{(1+x)(1+y)(1+x^2y^2)}-\int_0^1\int_0^1\frac{dxdy}{(1+x)(1+y)}$$

$$=(a+1)\int_0^1\int_0^1\frac{dxdy}{(1+x)(1+y)(1+x^2y^2)}-\ln^2(2)$$

$$= (a+1)\int_0^1\left(\int_0^1 \frac{dx}{(1+x)(1+y)(1+x^2y^2)}\right)dy - \ln^2(2)$$

$$= (a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\int_0^1 \frac{\left((1+x^2y^2)+y^2(1-x)(1+x)\right)dx}{(1+x)(1+x^2y^2)}\right)dy - \ln^2(2)$$

$$= (a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\int_0^1 \frac{1}{1+x}+\frac{y^2}{1+x^2y^2}-\frac{y^2x}{1+x^2y^2}dx\right)dy - \ln^2(2)$$

$$= (a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\int_0^1 \frac{1}{1+x}dx+\int_0^1\frac{y^2}{1+x^2y^2}dx-\int_0^1\frac{y^2x}{1+x^2y^2}dx\right)dy - \ln^2(2)$$

$$= (a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\ln(2)+y\int_0^1\frac{1}{1+x^2y^2}d(xy) - \frac{1}{2}\int_0^1\frac{1}{1+x^2y^2}d(x^2y^2)\right)dy - \ln^2(2)$$

$$= (a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\ln(2)+y\arctan(y)-\frac{1}{2}\ln(1+y^2)\right)dy - \ln^2(2)$$

$$= (a+1)\int_0^1 \frac{\ln(2)}{(1+y)(1+y^2)}dy + (a+1)\int_0^1\frac{\text{y}\arctan(\text{y})}{(1+y)(1+y^2)}dy - \frac{1}{2}(a+1)\int_0^1\frac{\ln(1+y^2)}{(1+y)(1+y^2)}dy - \ln^2(2)$$

$$= \frac{(a+1)}{2}\ln(2)\int_0^1\frac{1}{(1+y)}dy + \frac{(a+1)}{2}\ln(2)\int_0^1\frac{1}{(1+y^2)}dy - \frac{(a+1)}{4}\ln(2)\int_0^1\frac{2\text{y}}{(1+y^2)}dy + \frac{1}{2}(a+1)\int_0^1\frac{\arctan(\text{y})}{(1+y^2)}dy$$
$$+\frac{1}{2}(a+1)\int_0^1\frac{\text{y}\arctan(\text{y})}{(1+y^2)}dy - \frac{1}{2}(a+1)\int_0^1\frac{\arctan(\text{y})}{(1+y)}dy - \frac{1}{4}(a+1)\int_0^1\frac{\ln(1+y^2)}{(1+y^2)}dy + \frac{1}{4}(a+1)\int_0^1\frac{\text{y}\ln(1+y^2)}{(1+y^2)}dy$$
$$-\frac{1}{4}(a+1)\int_0^1\frac{\ln(1+y^2)}{(1+y)}dy - \ln^2(2)$$

$$= \frac{(a+1)}{2}\ln(2)I_1 + \frac{(a+1)}{2}\ln(2)I_2 - \frac{(a+1)}{4}\ln(2)I_3 + \frac{1}{2}(a+1)I_4 + \frac{1}{2}(a+1)I_5 - \frac{1}{2}(a+1)I_6 - \frac{1}{4}(a+1)I_7 + \frac{1}{4}(a+1)I_8 - \frac{1}{4}(a+1)I_9 - \ln^2(2)$$

$$=\frac{(a+1)}{2}\ln(2)\left(\ln(2)\right)+\frac{(a+1)}{2}\ln(2)\left(\frac{\pi}{4}\right)-\frac{(a+1)}{4}\ln(2)(\ln(2))+\frac{1}{2}(a+1)\left(\frac{\pi^2}{32}\right)$$

$$+\frac{1}{2}(a+1)\left(\frac{1}{8}(4G-\pi\ln(2))\right)-\frac{1}{2}(a+1)\left(\frac{\pi}{8}\ln(2)\right)$$

$$-\frac{1}{4}(a+1)\left(\frac{\pi}{2}\ln(2)-G\right)+\frac{1}{4}(a+1)\left(\frac{1}{4}\ln^2(2)\right)$$

$$-\frac{1}{4}(a+1)\left(\frac{3}{4}\ln^2(2)-\frac{\pi^2}{48}\right)-\ln^2(2)$$

$$=\frac{1}{8}(a+1)\left(\pi\ln(2)+2\ln^2(2)+\frac{\pi^2}{8}+2G-\frac{\pi}{2}\ln(2)-\frac{\pi}{2}\ln(2)-\pi\ln(2)+2G\right.$$

$$\left.+\frac{1}{2}\ln^2(2)-\frac{3}{2}\ln^2(2)+\frac{\pi^2}{24}\right)-\ln^2(2)$$

$$=\frac{1}{8}(a+1)\left(-\pi\ln(2)+\ln^2(2)+\frac{\pi^2}{6}+4G\right)-\ln^2(2)$$

$$I_1=\int_0^1\frac{1}{(1+y)}dy=[\ln(y+1)]_0^1=\ln(2)$$

$$I_2=\int_0^1\frac{1}{(1+y^2)}dy=\arctan(1)=\frac{\pi}{4}$$

$$I_3=\int_0^1\frac{2y}{(1+y^2)}dy\overset{y^2=x}{\triangleq}\int_0^1\frac{1}{(1+x)}dx\overset{x=y}{\triangleq}\int_0^1\frac{1}{(1+y)}dy=I_1=\ln(2)$$

$$I_4=\int_0^1\frac{\arctan(y)}{(1+y^2)}dy\overset{\arctan(y)=t}{\triangleq}\int_0^{\frac{\pi}{4}}t\,dt=\left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}}=\frac{\pi^2}{32}$$

$$I_5=\int_0^1\frac{y\arctan(y)}{(1+y^2)}dy=\left[\frac{1}{2}\arctan(y)\ln(1+y^2)\right]_0^1-\frac{1}{2}\int_0^1\frac{\ln(1+y^2)}{(1+y^2)}dy=$$

$$=\frac{\pi}{8}\ln(2)-\frac{1}{2}\int_0^1\frac{\ln(1+y^2)}{(1+y^2)}dy=\frac{\pi}{8}\ln(2)-\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln(sec^2(x))\,dx=$$

$$=\frac{\pi}{8}\ln(2)+\int_0^{\frac{\pi}{4}}\ln(cosx)dx=\frac{\pi}{8}\ln(2)+\frac{G}{2}-\frac{\pi}{4}\ln(2)=\frac{1}{8}(4G-\pi\ln(2))$$

$$I_5=\frac{\pi}{8}\ln(2)+\int_0^{\frac{\pi}{4}}\ln(cosx)dx=\frac{\pi}{8}\ln(2)+\frac{G}{2}-\frac{\pi}{4}\ln(2)=\frac{1}{8}(4G-\pi\ln(2))$$

$$I_6=\int_0^1\frac{\arctan(y)}{(1+y)}dy=[\ln(1+y)\arctan(y)]_0^1-\int_0^1\frac{\ln(1+y)}{1+y^2}dy$$

$$=\frac{\pi}{4}\ln(2)-\int_0^{\frac{\pi}{4}}\ln(siny+cosy)\,dy+\int_0^{\frac{\pi}{4}}\ln(cosy)\,dy$$

$$=\frac{\pi}{4}\ln(2)-\int_0^{\frac{\pi}{4}}\ln\left(\sqrt{2}\cos\left(\frac{\pi}{4}-y\right)\right)dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy$$

$$=\frac{\pi}{8}\ln(2)-\int_0^{\frac{\pi}{4}}\ln\left(\cos\left(\frac{\pi}{4}-y\right)\right)dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy$$

$$=\frac{\pi}{8}\ln(2)-\int_0^{\frac{\pi}{4}}\ln(\cos(y))\,dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy=\frac{\pi}{8}\ln(2)$$

$$I_6=\frac{\pi}{8}\ln(2)-\int_0^{\frac{\pi}{4}}\ln(\cos(y))\,dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy=\frac{\pi}{8}\ln(2)$$

$$I_7=\int_0^1\frac{\ln(1+y^2)}{(1+y^2)}dy=\int_0^{\frac{\pi}{4}}\ln(sec^2(x))\,dx=-2\int_0^{\frac{\pi}{4}}\ln(cosx)dx=\frac{\pi}{2}\ln(2)-G$$

$$I_8=\int_0^1\frac{y\ln(1+y^2)}{(1+y^2)}dy\overset{y^2=y}{\triangleq}\frac{1}{2}\int_0^1\frac{\ln(1+y)}{(1+y)}dy\overset{y^2=y}{\triangleq}\frac{1}{4}\ln^2(2)$$

$$I_9=\int_0^1\frac{\ln(1+y^2)}{(1+y)}dy$$

$$I_9=\int_0^1\frac{\ln(1+y^2)}{(1+y)}dy=[\ln(1+y)\ln(1+y^2)]_0^1-2\int_0^1\frac{y\ln(1+y)}{(1+y^2)}dy$$

$$=\ln^2(2)-2\int_0^1\frac{y\ln(1+y)}{(1+y^2)}dy=\ln^2(2)-2\left(\frac{\pi^2}{96}+\frac{1}{8}\log^2(2)\right)=\frac{3}{4}\ln^2(2)-\frac{\pi^2}{48}$$

$$I_9=\ln^2(2)-2\int_0^1\frac{y\ln(1+y)}{(1+y^2)}dy=\ln^2(2)-2\left(\frac{\pi^2}{96}+\frac{1}{8}\log^2(2)\right)=\frac{3}{4}\ln^2(2)-\frac{\pi^2}{48}$$

2581. Find a closed form:

$$\int_0^1\int_0^1\frac{\ln(x)\ln(xy)}{1-xy}dxdy$$

Proposed by Aryan Desai-India

Solution by Pham Duc Nam-Vietnam

$$I=\int_0^1\int_0^1\frac{\ln(x)\ln(xy)}{1-xy}dxdy\overset{symmetry}{\triangleq}\int_0^1\int_0^1\frac{\ln(y)\ln(xy)}{1-xy}dxdy$$

$$2I=\int_0^1\int_0^1\frac{ln^2(xy)}{1-xy}dxdy\overset{symmetry}{\triangleq}-\int_0^1\frac{ln^3(x)}{1-x}dx=$$

$$=-\sum_{n=0}^{\infty}\int_0^1x^n\,ln^3(x)dx=6\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}=6\zeta(4)=\frac{\pi^4}{30}$$

2582. Find a closed form:

$$\int_0^1\frac{\ln(y)Li_2(y)}{1-y}dy$$

Proposed by Aryan Desai-India

Solution by Pham Duc Nam-Vietnam

$$I=\int_0^1\frac{\ln(y)Li_2(y)}{1-y}dy=-\int_0^1\frac{x\ln(x)}{1-x}dx\left(\int_0^1\frac{\ln(t)}{1-xt}dt\right)$$

$$I=\int_0^1\ln(t)\,dt(\zeta(2)\frac{1}{1-t}-Li_2(t)(\frac{1}{1-t}+\frac{1}{t}))$$

$$I=\ \zeta(2)\int_0^1\frac{\ln(t)}{1-t}dt-I-\int_0^1\frac{\ln(t)Li_2(t)}{t}dt$$

$$I=-\frac{1}{2}\zeta^2(2)-\frac{1}{2}\left(\ln(t)\,Li_3(t)|_0^1-Li_4(t)|_0^1\right)$$

$$I=-\frac{1}{2}\zeta^2(2)+\frac{1}{2}Li_4(1)=\frac{1}{2}\zeta(4)-\frac{1}{2}\zeta^2(2)=-\frac{\pi^4}{120}$$

2583. Find a closed form:

$$\int_0^{\frac{\pi}{2}}\frac{1}{(\sin(x)+1)^2\sqrt{\ln(tan(x)+sec(x))}}dx$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Algeria

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{1}{(\sin(x)+1)^2\sqrt{\ln(tan(x)+sec(x))}}dx\overset{x\to\frac{\pi}{2}-x}{\cong}$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{(\cos(x)+1)^2\sqrt{\ln(ctg(x)+csc(x))}}dx=$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1}{cos^2\left(\frac{x}{2}\right)\sqrt{-\ln(\tan\left(\frac{x}{2}\right)}}\frac{dx}{cos^2\left(\frac{x}{2}\right)}\overset{2x\to x}{\cong}$$

$$=\frac{1}{2}\int_0^\infty\frac{e^{-t}(1+e^{-2t})}{\sqrt{t}}dt=\frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}\,dt+\frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1}e^{-3t}\,dt=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)+\frac{1}{2\sqrt{3}}\Gamma\left(\frac{1}{2}\right)=$$

$$=\frac{\sqrt{\pi}}{2}\left(1+\frac{1}{\sqrt{3}}\right)$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{(\sin(x)+1)^2\sqrt{\ln(tan(x)+sec(x))}}dx=\frac{\sqrt{\pi}}{2}\left(1+\frac{1}{\sqrt{3}}\right)$$

2584. Find:

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))}dx$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Mirsadix Muzefferov-Azerbaijan

Let's simplify the subintegral expression :

$$\frac{\sin(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))}=\frac{\sin(x)\,(1+\cos(x))}{\cos(x)(1+\sin(x))(1+\cos(x))}=\frac{\sin(x)}{\cos(x)\,(+\sin(x))}=$$

$$\frac{\tan(x)}{1+\sin(x)};\ \left\{Let:tan\frac{x}{2}=t\ .\ Then\ tan(x)=\frac{2t}{1-t^2};\ \sin(x)=\frac{2t}{1+t^2},\right.$$

$$\left.dx=\frac{2}{1-t^2}dt\right\}$$

$$\int_0^{\frac{\pi}{4}}\frac{\sin(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))}dx=\int_0^{\frac{\pi}{4}}\frac{\tan(x)}{(1+\sin(x))}dx\overset{t=tan\frac{x}{2}}{\cong}\int_0^{tan\frac{\pi}{8}}\frac{4t}{(1-t)(1+t)^3}dt$$

$$From:\ \frac{4t}{(1-t)(1+t)^3}=\frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{(1+t)^2}+\frac{D}{(1+t)^3}$$

$$A(1+t)^3+B(1-t)(1+t)^2+C(1-t)(1+t)+D(1-t)=4t$$

Let's simplify . We get

$$\begin{cases}A-B=0\\3A-B-C=0\\3A+B-D=0\\A+B+C+D=0\end{cases}\rightarrow\ \begin{cases}A-B=0\\7A+B=4\end{cases}\rightarrow\ A=\frac{1}{2};\ B=\frac{1}{2};\ C=1\,;D=-2$$

$$Then\ :\ \frac{4t}{(1-t)(1+t)^3}=\frac{\frac{1}{2}}{1-t}+\frac{\frac{1}{2}}{1+t}+\frac{1}{(1+t)^2}-\frac{2}{(1+t)^3}$$

$$\int_0^{tan\frac{\pi}{8}}\frac{4t}{(1-t)(1+t)^3}dt$$

$$=\frac{1}{2}\int_0^{tan\frac{\pi}{8}}\frac{dt}{1-t}+\frac{1}{2}\int_0^{tan\frac{\pi}{8}}\frac{dt}{1+t}+\int_0^{tan\frac{\pi}{8}}\frac{dt}{(1+t)^2}-2\int_0^{tan\frac{\pi}{8}}\frac{dt}{(1+t)^3}=$$

$$\frac{1}{2}ln\left|\frac{1+t}{1-t}\right|\Bigg|_0^{tan\frac{\pi}{8}}-\frac{1}{(1+t)}\Bigg|_0^{tan\frac{\pi}{8}}+\frac{1}{(1+t)^3}\Bigg|_0^{tan\frac{\pi}{8}}=$$

$$\frac{1}{2}\left(ln\left|\frac{1+\sqrt{2}-1}{2-\sqrt{2}}\right|-\ln(1)\right)-\left(\frac{1}{1+\sqrt{2}-1}-1\right)+\left(\frac{1}{\left(1+\sqrt{2}-1\right)^2}-1\right)=$$

$$\frac{1}{2}\ln(1+\sqrt{2})-\frac{\sqrt{2}}{2}+1+\frac{1}{2}-1=\frac{1}{2}\left(\ln(1+\sqrt{2})-\sqrt{2}+1\right)$$

2585. Find a closed form:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos(x)-\sin(x)}\,dx$$

Proposed by Sonu Aarnav-India

Solution by Odeyemi Gideon-Nigeria

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos(x)-\sin(x)}\,dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}\,dx$$

$$I=\sqrt{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sqrt{\cos(x)}dx=\sqrt{\sqrt{2}}\int_0^{\frac{\pi}{2}} cos^{2\left(\frac{3}{4}\right)-1}(x)sin^{2\left(\frac{1}{2}\right)-1}(x)dx$$

$$I=\sqrt{\sqrt{2}}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}=\sqrt{\sqrt{2}}\frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{2\Gamma\left(\frac{5}{4}\right)}$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos(x)-\sin(x)}\,dx=\sqrt{\pi\sqrt{2}}\frac{\Gamma\left(\frac{3}{4}\right)}{2\Gamma\left(\frac{5}{4}\right)}$$

2586. Find a closed form:

$$\Omega=\int_1^{e^{\pi}}\int_0^1 \frac{\arcsin(x^a)\arccos(x^a)}{x}\,dxda$$

Proposed by Ankush Kumar Parcha-India

Solution by Ose Favour-Nigeria

$$\Omega=\int_0^1 \frac{\arccos(x)\arcsin(x)}{x}dx=\int_0^1 \frac{\arcsin(x)}{x}\left(\frac{\pi}{2}-\arcsin(x)\right)dx=$$

$$=\frac{\pi}{2}\int_0^1 \frac{\arcsin(x)}{x}dx-\int_0^1 \frac{(\arcsin(x))^2}{x}dx\overset{x=\sin(y)}{=}$$

$$=\frac{\pi}{2}\underbrace{\int_0^{\frac{\pi}{2}} yctg(y)dy}_{\ln(2)\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}} y^2ctg(y)dy=\frac{\pi^2}{4}\ln(2)+2\int_0^{\frac{\pi}{2}} yln(\sin(y))dy=$$

$$=\frac{\pi^2}{4}\ln(2)-2(\frac{\pi^2}{8}\ln(2)-2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}} ycos(2ky)dy=\frac{7}{8}\zeta(3)$$

2587. Find a closed form:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{1}{\ln^2(xyz)(1+x^2y^2z^2)}dxdydz$$

Proposed by Asmat Qatea-Afghanistan

Solution by Obiajunwa Januarius-Nigeria

It is known that:

$$\int_0^1\int_0^1\int_0^1 F(xyz)dxdydz=\frac{1}{2}\int_0^1\ln^2(x)F(x)\,dx$$

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{1}{\ln^2(xyz)(1+x^2y^2z^2)}dxdydz=\frac{1}{2}\int_0^1\frac{\ln^2(x)}{(1+x^2)\ln^2(x)}dx$$

$$=\frac{1}{2}\int_0^1\frac{dx}{1+x^2}=\frac{\pi}{8}$$

2588. Find a closed form:

$$I=\int_0^\infty\frac{(z^2+1)z^{106}\ln(z)}{(z^{100}+1)^2(z^4+\frac{2}{5}z^2+1)^4}dz$$

Proposed by Bui Hong Suc-Vietnam

Solution by Pham Duc Nam-Vietnam

$$I=\int_0^\infty\frac{(z^2+1)z^{106}\ln(z)}{(z^{100}+1)^2(z^4+\frac{2}{5}z^2+1)^4}dz=625\int_0^\infty\frac{(z^2+1)z^{106}\ln(z)}{(z^{100}+1)^2(5z^4+2z^2+5)^4}dz=$$

$$625\left(\int_0^1\frac{(z^2+1)z^{106}\ln(z)}{(z^{100}+1)^2(5z^4+2z^2+5)^4}dz+\int_1^\infty\frac{(z^2+1)z^{106}\ln(z)}{(z^{100}+1)^2(5z^4+2z^2+5)^4}dz\right)$$

$$\textit{Change the variable } z=\frac{1}{t}\textit{ for the latter integral yields:}$$

$$I=625\int_0^1\frac{(z^2+1)z^{106}\ln(z)}{(z^{100}+1)^2(5z^4+2z^2+5)^4}dz+$$

$$625\int_1^\infty\frac{\left((\frac{1}{t})^2+1\right)(\frac{1}{t})^{106}\ln\left(\frac{1}{t}\right)}{t^2\left((\frac{1}{t})^{100}+1\right)^2\left(5(\frac{1}{t})^4+2(\frac{1}{t})^2+5\right)^4}dt$$

$$I=625\left(\int_0^1\frac{(z^2+1)z^{106}\ln(z)}{(z^{100}+1)^2(5z^4+2z^2+5)^4}dz-\int_0^1\frac{(t^2+1)t^{106}\ln(t)}{(t^{100}+1)^2(5t^4+2t^2+5)^4}dt=0\right.$$

2589. **Find a closed form:**

$$I=\int_{-\infty}^{\infty}\frac{x(x^2(x(x(x^2(4x-20)-20)+17)-20)-20)+4}{\left(\left(\left((4x^2+4)x^2+17\right)x^2+17\right)x^2+4\right)x^2+4}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Obiajunwa Januarius-Nigeria

$$I=\int_{-\infty}^{\infty}\frac{x(x^2(x(x(x^2(4x-20)-20)+17)-20)-20)+4}{\left(\left(\left((4x^2+4)x^2+17\right)x^2+17\right)x^2+4\right)x^2+4}dx$$

$$I=\int_{-\infty}^{\infty}\left(\frac{1}{1+x^2}-\frac{1}{2-2x+x^2}+\frac{1}{2+2x+x^2}-\frac{1}{1-2x+2x^2}+\frac{1}{1+2x+2x^2}\right)dx$$

$$I=\int_{-\infty}^{\infty}\left(\frac{1}{1+x^2}-\frac{1}{1+(x-1)^2}+\frac{1}{1+(x+1)^2}-\frac{1}{2}\frac{1}{(x-\frac{1}{2})^2+\frac{1}{4}}+\frac{1}{2}\frac{1}{(x+\frac{1}{2})^2+\frac{1}{4}}\right)dx$$

$$I=\arctan(x)-\arctan(x-1)+\arctan(x+1)-\arctan\left(2\left(x-\frac{1}{2}\right)\right)+\arctan\left(2\left(x+\frac{1}{2}\right)\right)\Big|_{-\infty}^{\infty}=\pi$$

2590. **Prove that:**

$$\sum_{k\in Z}\frac{1}{(1+k^2)(2+k^2)(3+k^2)}=\frac{\pi}{2}\left(\frac{coth(\pi\sqrt{3})}{\sqrt{3}}-\sqrt{2}coth(\pi\sqrt{2})+coth(\pi)\right)$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Shobhit Jain-India

$$I=\sum_{k=-\infty}^{\infty}\frac{1}{(1+k^2)(2+k^2)(3+k^2)}=\sum_{k=-\infty}^{\infty}\left(\frac{\frac{1}{2}}{3+k^2}-\frac{1}{2+k^2}+\frac{\frac{1}{2}}{1+k^2}\right)=$$

$$=\frac{1}{2}f(\sqrt{3})-f(\sqrt{2})+\frac{1}{2}f(1)$$

$$here,\ f(x)=\sum_{k=-\infty}^{\infty}\frac{1}{x^2+k^2}.Now,\ \frac{sin(\pi x)}{\pi x}=\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

$$\Rightarrow ln(sin(\pi x))-ln(\pi x)=\sum_{k=1}^{\infty}ln\left(1-\frac{x^2}{k^2}\right)\ Now\ differentiate\ w.r.t\ x$$

$$\Rightarrow \pi\,cot(\pi x)-\frac{1}{x}=\sum_{k=1}^{\infty}\frac{2x}{x^2-k^2}\Rightarrow\frac{\pi}{tan(\pi x)}=\sum_{k=-\infty}^{\infty}\frac{x}{x^2-k^2}$$

$$Now\ replace\ x\ by\ ix\ \Rightarrow\frac{\pi}{tan(i\pi x)}=\sum_{k=-\infty}^{\infty}\frac{ix}{-x^2-k^2}\qquad use,\ \ tan(i\pi x)$$

$$=i\,tanh(\pi x)$$

$$\Rightarrow \frac{\pi}{i\,tanh(\pi x)} = \sum_{k=-\infty}^{\infty} \frac{ix}{-x^2-k^2} \Rightarrow \frac{\pi}{tanh(\pi x)} = \sum_{k=-\infty}^{\infty} \frac{x}{x^2+k^2}$$

$$\Rightarrow f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{x^2+k^2} = \frac{\pi\,coth(\pi x)}{x} \Rightarrow I = \frac{1}{2}f(\sqrt{3}) - f(\sqrt{2}) + \frac{1}{2}f(1)$$

$$= \frac{1}{2}\left(\frac{\pi\,coth(\pi\sqrt{3})}{\sqrt{3}}\right) - \left(\frac{\pi\,coth(\pi\sqrt{2})}{\sqrt{2}}\right) + \frac{1}{2}(\pi\,coth(\pi))$$

$$\Rightarrow I = \frac{\pi}{2}\left(\frac{coth(\pi\sqrt{3})}{\sqrt{3}} - \sqrt{2}coth(\pi\sqrt{2}) + coth(\pi)\right)$$

Solution 2 by Pratham Prasad-India

$$\sum_{q\epsilon Z} \frac{1}{(1+q^2)(2+q^2)(3+q^2)} = \frac{1}{2}\sum_{q\epsilon Z} \frac{1}{(1+q^2)} - \frac{2}{(2+q^2)} + \frac{1}{(3+q^2)}$$

$$= \frac{1}{2}\pi\left(\coth(\pi) - \sqrt{2}\coth(\pi\sqrt{2}) + \frac{1}{\sqrt{3}}\coth(\pi\sqrt{3})\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{\sqrt{3}}\coth(\pi\sqrt{3}) - \sqrt{2}\coth(\pi\sqrt{2}) + \coth(\pi)\right)$$

As by Fourier series:

$$\sum_{n\epsilon Z} \frac{1}{k^2+n^2} = \frac{\pi}{k}\coth(\pi k)$$

2591. Find:

$$\int_0^1 \frac{x\ln^2(x)}{1+x+x^2}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Pratham Prasad-India

$$Let\ I(a) = \int_0^1 \frac{x^a}{1+x+x^2}dx.\ Thus, I''(1) = \int_0^1 \frac{x\ln^2(x)}{1+x+x^2}dx$$

$$I(a) = \int_0^1 \frac{x^a(1-x)}{1-x^3}dx, \qquad I(a) = \int_0^1 (x^a - x^{a+1})\sum_{r=0}^{\infty} x^{3r}\,dx$$

$$I(a) = \sum_{r=0}^{\infty}\int_0^1 (x^{a+3r} - x^{a+1+3r})\,dx, \qquad I(a) = \sum_{r=0}^{\infty}\left(\frac{1}{3r+a+1} - \frac{1}{3r+a+2}\right)$$

$$I(a) = \frac{1}{3}\left(\psi\left(\frac{a+1}{3}\right) - \psi\left(\frac{a+2}{3}\right)\right), \qquad I'(a) = \frac{1}{9}\left(\psi_1\left(\frac{a+1}{3}\right) - \psi_1\left(\frac{a+2}{3}\right)\right)$$

$$I''(a) = \frac{1}{27}\left(\psi_2\left(\frac{a+1}{3}\right) - \psi_2\left(\frac{a+2}{3}\right)\right), \qquad I''(1) = \frac{1}{27}\left(\psi_2\left(\frac{2}{3}\right) - \psi_2(1)\right)$$

$$I''(1) = \frac{1}{27}\left(\frac{4\pi^3}{3\sqrt{3}} - 26\zeta(3) + 2\zeta(3)\right), \qquad I''(1) = \frac{1}{27}\left(\frac{4\pi^3}{3\sqrt{3}} - 24\zeta(3)\right)$$

$$\int_0^1 \frac{x\ln^2(x)}{1+x+x^2}dx = \frac{4\pi^3}{81\sqrt{3}} - \frac{8}{9}\zeta(3)$$

2592. Find a closed form:

$$\Omega = \int_0^\pi \ln^2(1+cosx)\,dx$$

Proposed by Bui Hong Suc-Vietnam

Solution by Pratham Prasad-India

$$\Omega = \int_0^\pi \ln^2(1+cosx)\,dx = \int_0^\pi \ln^2\left(2\cos^2\left(\frac{x}{2}\right)\right)dx \overset{y=\frac{x}{2}}{\triangleq} 2\int_0^{\frac{\pi}{2}} \ln^2(2\cos^2(y))\,dy =$$

$$= 2\int_0^{\frac{\pi}{2}} (\ln(2) + 2\ln(cosy))^2 dy$$

$$= \pi\ln^2(2) + 8\int_0^{\frac{\pi}{2}} (\ln(cosy))^2 dy + 8ln(2)\int_0^{\frac{\pi}{2}} \ln(cosy)\,dy$$

$$\overset{y=\frac{\pi}{2}-y}{\triangleq} \pi\ln^2(2) + 4\underbrace{\int_0^{\pi} (\ln(siny))^2 dy}_{I_2} + 8ln(2)\underbrace{\int_0^{\frac{\pi}{2}} \ln(siny)\,dy}_{I_1}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \ln(siny)\,dy, \qquad I_1 \overset{y=\frac{\pi}{2}-y}{\triangleq} \int_0^{\frac{\pi}{2}} \ln(cosy)\,dy$$

$$2I_1 = \int_0^{\frac{\pi}{2}} \ln(sinycosy)\,dy, \qquad 2I_1 = \int_0^{\frac{\pi}{2}} \ln(\sin(2y))\,dy - \frac{\pi}{2}\ln(2)$$

$$2I_1 = \frac{1}{2}\int_0^{\pi} \ln(\sin(x))\,dx - \frac{\pi}{2}\ln(2), \quad 2I_1 = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\,dx - \frac{\pi}{2}\ln(2)$$

$$2I_1 = I_1 - \frac{\pi}{2}\ln(2), \quad I_1 = -\frac{\pi}{2}\ln(2)$$

$$I_2 = Re\int_0^{\pi} (\ln(siny))^2 dy, \quad I_2 \overset{y=\frac{\theta}{2}}{\triangleq} \frac{1}{2}Re\int_0^{2\pi} \left(\ln\left(\sin\left(\frac{\theta}{2}\right)\right)\right)^2 dy$$

$$I_2 = \frac{1}{2}Re\int_0^{2\pi} \left[\log(1-e^{i\theta}) - \ln(2) - \frac{i}{2}(\theta-\pi)\right]^2 dy$$

$$I_2 = \frac{1}{2}Re\int_0^{2\pi} \log^2(2) - \frac{1}{4}(\theta-\pi)^2\,d\theta + \frac{1}{2}Re\int_0^{2\pi} -ilog(1-e^{i\theta})(\theta-\pi)d\theta$$

$$+\frac{1}{2}Re\int_0^{2\pi} \log^2(1-e^{i\theta}) - 2\log(2)\log(1-e^{i\theta})\,d\theta$$

$I_2=\frac{1}{2}\left(2\pi\log^2(2)-\frac{\pi^3}{6}\right)+\frac{1}{2}\left(\frac{\pi^3}{3}\right)+\frac{1}{2}Re\oint_{|z|=1}\left(-\frac{i}{z}\right)(\log^2(1-z)-2\log(2)\log(1-z))\ dz$

$$I_2=\frac{1}{2}\left(2\pi\log^2(2)+\frac{\pi^3}{6}\right)$$

$$\boxed{\int_0^\pi \ln^2(1+cosx)\,dx=\pi\ln^2(2)+2\pi\zeta(2)}$$

2593. Find a closed form:

$$\int_0^\infty \frac{\sqrt{x}}{1+e^{2x}}dx$$

Proposed by Manuel Suka-Angola

Solution by Pratham Prasad-India

$$\int_0^\infty \frac{\sqrt{x}}{1+e^{2x}}dx=\int_0^\infty \frac{\sqrt{x}e^{-2x}}{1+e^{-2x}}dx=\int_0^\infty \sqrt{x}\sum_{r=1}^\infty(-1)^{r-1}e^{-2rx}\,dx=$$

$$=\sum_{r=1}^\infty(-1)^{r-1}\int_0^\infty\sqrt{x}\,e^{-2rx}dx=\sum_{r=1}^\infty\frac{(-1)^{r-1}}{(2r)^{\frac{3}{2}}}\int_0^\infty\sqrt{t}\,e^{-t}dt=\frac{1}{2^{\frac{3}{2}}}\sum_{r=1}^\infty\frac{(-1)^{r-1}}{(r)^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)=$$

$$=\frac{1}{2^{\frac{3}{2}}}\left(1-\frac{1}{\sqrt{2}}\right)\zeta\left(\frac{3}{2}\right)\left(\frac{\sqrt{\pi}}{2}\right)=\frac{\pi}{8}(\sqrt{2}-1)\zeta\left(\frac{3}{2}\right)$$

2594. Find a closed form:

$$\sum_{m=3}^\infty\sum_{n=2}^{m-1}\frac{H_{n-1}2^{-m}}{n(m-n)},\qquad \text{where } H_k \text{ is the k\_th harmonic number}$$

Proposed by Vincenzo Dima-Italy

Solution by Shobhit Jain-India

$$I=\sum_{1<n<}\sum_{m<\infty}\frac{H_{n-1}2^{-m}}{n(m-n)}=\sum_{n=2}^\infty\frac{H_{n-1}}{n}\sum_{m=n+1}^\infty\frac{2^{-m}}{(m-n)}\underbrace{=}_{m=k+n}\left(\sum_{n=2}^\infty\frac{H_{n-1}}{n2^n}\right)\left(\sum_{k=1}^\infty\frac{2^{-k}}{k}\right)$$

$$Use,\ \sum_{k=1}^\infty\frac{2^{-k}}{k}=-ln(1-2^{-1})=ln2$$

$$Now,\quad -ln(1-y)=\sum_{k=1}^\infty\frac{y^k}{k}\Rightarrow-\frac{ln(1-y)}{1-y}=\sum_{n=2}^\infty H_{n-1}y^{n-1}$$

$$Integrate\ w.r.t\ y\ from\ y=0\ to\ y=x$$

$$\Rightarrow\frac{ln^2(1-x)}{2}=\sum_{n=2}^\infty\frac{H_{n-1}}{n}x^n\underbrace{\Rightarrow}_{x=\frac{1}{2}}\frac{ln^2 2}{2}=\sum_{n=2}^\infty\frac{H_{n-1}}{n2^n}\Rightarrow I=\left(\frac{ln^2 2}{2}\right)(ln2)=\frac{1}{2}(ln2)^3$$

2595. Prove that:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{y\sin(x)-1}{1+y\sin(x)}dxdy=2\left(Li_2\left(\frac{2-\pi-\sqrt{\pi^2-4}}{2}\right)+Li_2\left(\frac{2-\pi+\sqrt{\pi^2-4}}{2}\right)\right)$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Shobhit Jain-India

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{y\sin(x)-1}{1+y\sin(x)}dxdy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}1-\frac{2}{1+y\sin(x)}dydx=$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\frac{2}{\sin(x)}ln\left(1+\frac{\pi}{2}\sin(x)\right)\right)dx=$$

$$=\frac{\pi^2}{4}-2\int_0^{\frac{\pi}{2}}\frac{ln\left(1+\frac{\pi}{2}\sin(x)\right)}{\sin(x)}dx=\frac{\pi^2}{4}-2f\left(\frac{\pi}{2}\right)$$

$$Now,\ consider\ f(a)=\int_0^{\frac{\pi}{2}}\frac{ln(1+a\sin(x))}{\sin(x)}dx\quad a\geq -1\Rightarrow\ f(0)=0$$

$$\Rightarrow f(1)=\int_0^{\frac{\pi}{2}}\frac{ln(1+\sin(x))}{\sin(x)}dx\underbrace{=}_{\substack{x\to 2\tan^{-1}t\\ t=\tan\left(\frac{x}{2}\right)}}\int_0^1\frac{ln\left(1+\frac{2t}{1+t^2}\right)}{\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt$$

$$=\int_0^1\frac{ln\left(\frac{(1+t)^2}{1+t^2}\right)}{t}dt=2\int_0^1\frac{ln(1+t)}{t}dt-\int_0^1\frac{ln(1+t^2)}{t}dt=$$

$$=2\frac{\pi^2}{12}-\frac{1}{2}\int_0^1\frac{ln(1+y)}{y}dy\ \ (by\ substituting\ y=t^2)$$

$$\Rightarrow f(1)=\frac{\pi^2}{6}-\frac{1}{2}\times\frac{\pi^2}{12}=\frac{\pi^2}{8}$$

$$f'(a)=\int_0^{\frac{\pi}{2}}\frac{1}{1+a\sin(x)}dx\underbrace{=}_{x\to\frac{\pi}{2}-x}\int_0^{\frac{\pi}{2}}\frac{1}{1+a\cos(x)}dx\underbrace{=}_{\substack{x\to 2\tan^{-1}t\\ t=\tan\left(\frac{x}{2}\right)}}\int_0^1\frac{2}{(1+a)+(1-a)t^2}dt$$

$$\Rightarrow f'(a)=\frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\sqrt{\frac{1-a}{1+a}}\right)=\frac{\cos^{-1}a}{\sqrt{1-a^2}}\quad for\ \ -1\leq a\leq 1$$

$$\Rightarrow f'(a)=\frac{2}{\sqrt{a^2-1}}\tanh^{-1}\left(\sqrt{\frac{a-1}{a+1}}\right)=\frac{\cosh^{-1}a}{\sqrt{a^2-1}}\quad for\quad a\ge 1$$

$$\Rightarrow f(a)=\frac{\pi^2}{8}-\frac{(\cos^{-1}a)^2}{2}\quad -1\le a\le 1\Rightarrow f(a)=\frac{\pi^2}{8}+\frac{(\cosh^{-1}a)^2}{2}\quad a\ge 1$$

$$\Rightarrow I=\frac{\pi^2}{4}-2f\left(\frac{\pi}{2}\right)=\frac{\pi^2}{4}-2\left(\frac{\pi^2}{8}+\frac{\left(\cosh^{-1}\left(\frac{\pi}{2}\right)\right)^2}{2}\right)=-\left(\cosh^{-1}\left(\frac{\pi}{2}\right)\right)^2=$$

$$=-\ln^2\left(\frac{\pi}{2}+\sqrt{\frac{\pi^2}{4}-1}\right)=-\ln^2\left(\frac{\pi+\sqrt{\pi^2-4}}{2}\right)$$

$$Let,\ z=\frac{\pi+\sqrt{\pi^2-4}}{2}\Rightarrow\frac{1}{z}=\frac{\pi-\sqrt{\pi^2-4}}{2}$$

$$Now,\ Li_2(1-z)+Li_2\left(1-\frac{1}{z}\right)=-\frac{\ln^2(z)}{2}\Rightarrow -\ln^2(z)$$

$$=2\left(Li_2(1-z)+Li_2\left(1-\frac{1}{z}\right)\right)$$

$$\Rightarrow -\ln^2\left(\frac{\pi+\sqrt{\pi^2-4}}{2}\right)=2\left(Li_2\left(1-\frac{\pi+\sqrt{\pi^2-4}}{2}\right)+Li_2\left(1-\frac{\pi-\sqrt{\pi^2-4}}{2}\right)\right)$$

$$\Rightarrow I=2\left(Li_2\left(\frac{2-\pi-\sqrt{\pi^2-4}}{2}\right)+Li_2\left(\frac{2-\pi+\sqrt{\pi^2-4}}{2}\right)\right)$$

2596. Prove that:

$$\iint_{[0,1]^2}\left(\frac{x+y}{\ln(xy)}-\tan^{-1}\left(\frac{y^2}{x}\right)\right)dxdy=\frac{1}{3\sqrt{2}}\sinh^{-1}(1)-\frac{1}{\sqrt{2}}\coth^{-1}(\sqrt{2})-\frac{13}{6}\ln(2)+\pi\left(\frac{1}{3\sqrt{2}}-\frac{1}{4}\right)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shobhit Jain-India

$$\Omega=\underbrace{\iint_{[0,1]^2}\frac{x+y}{\ln(xy)}dxdy}_{I_1}-\underbrace{\iint_{[0,1]^2}\tan^{-1}\left(\frac{y^2}{x}\right)dxdy}_{I_2}=$$

$$I_1\underbrace{=}_{\substack{x=e^{-u}\\y=e^{-v}}}-\int_0^\infty\int_0^\infty\frac{(e^{-u}+e^{-v})}{u+v}e^{-u-v}dudv=-2\int_0^\infty\int_0^\infty\frac{e^{-2u-v}}{u+v}dudv=$$

$$=-2\int_0^\infty\left(\int_0^\infty\frac{e^{-2u-v}}{u+v}dv\right)du\underbrace{=}_{v\to ut}-2\int_0^\infty\left(\int_0^\infty\frac{e^{-(t+2)u}}{1+t}dt\right)du=$$

$$=-2\int_0^\infty\int_0^\infty \frac{e^{-(t+2)u}}{1+t}dudt=-2\int_0^\infty \frac{1}{(1+t)(2+t)}dt=-2ln2$$

$$I_2=\int_0^1\left(\int_0^1 tan^{-1}\left(\frac{y^2}{x}\right)dx\right)dy\underbrace{=}_{x\to Xy^2}\int_0^1 y^2\left(\int_0^{\frac{1}{y^2}} tan^{-1}\left(\frac{1}{X}\right)dX\right)dy=$$

$$\underbrace{=}_{y\to\frac{1}{Y}}\int_1^\infty \frac{1}{Y^4}\left(\int_0^{Y^2} tan^{-1}\left(\frac{1}{X}\right)dX\right)dY=\int_1^\infty \frac{1}{Y^4}\left(\int_0^{Y^2}\frac{\pi}{2}-tan^{-1}(X)dX\right)dY=$$

$$=\frac{\pi}{2}\int_1^\infty \frac{1}{Y^2}dY-\int_1^\infty \frac{F(Y^2)}{Y^4}dY=\frac{\pi}{2}-\int_1^\infty \frac{F(Y^2)}{Y^4}dY$$

$$here,\ \ F(Y^2)=\int_0^{Y^2} tan^{-1}(X)dX\ \ \Rightarrow F(Y)=\int_0^Y tan^{-1}(X)dX$$

$$=Ytan^{-1}(Y)-\frac{1}{2}ln(1+Y^2)$$

$$\int_1^\infty \frac{F(Y^2)}{Y^4}dY\underbrace{=}_{IBP}\frac{1}{3}F(1)+\frac{1}{3}\int_1^\infty \frac{2Y}{Y^3}F'(Y^2)dY=\frac{1}{3}\left(\frac{\pi}{4}-\frac{ln2}{2}\right)+\frac{2}{3}\int_1^\infty \frac{tan^{-1}(Y^2)}{Y^2}dY$$

$$\Rightarrow I_2=\frac{\pi}{2}-\frac{1}{3}\left(\frac{\pi}{4}-\frac{ln2}{2}\right)-\frac{2}{3}\int_1^\infty \frac{tan^{-1}(Y^2)}{Y^2}dY$$

$$Now,\ \int_1^\infty \frac{tan^{-1}(Y^2)}{Y^2}dY\underbrace{=}_{IBP}\frac{\pi}{4}+\int_1^\infty \frac{2}{1+Y^4}dY$$

$$\Rightarrow I_2=\frac{\pi}{2}-\frac{1}{3}\left(\frac{\pi}{4}-\frac{ln2}{2}\right)-\frac{2}{3}\left(\frac{\pi}{4}\right)-\frac{2}{3}\int_1^\infty \frac{2}{1+Y^4}dY=\frac{\pi}{4}+\frac{ln2}{6}-\frac{2}{3}\int_1^\infty \frac{2}{1+Y^4}dY$$

$$\int_1^\infty \frac{2}{1+Y^4}dY=\int_1^\infty \frac{1+\frac{1}{Y^2}-\left(1-\frac{1}{Y^2}\right)}{Y^2+\frac{1}{Y^2}}dY=\int_1^\infty \frac{1+\frac{1}{Y^2}}{Y^2+\frac{1}{Y^2}}dY-\int_1^\infty \frac{1-\frac{1}{Y^2}}{Y^2+\frac{1}{Y^2}}dY=$$

$$\underbrace{=}_{\substack{u=Y-\frac{1}{Y}\\ v=Y+\frac{1}{Y}}}\int_0^\infty \frac{du}{u^2+2}-\int_2^\infty \frac{dv}{v^2-2}=\frac{\pi}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\left[ln\left(\frac{v-\sqrt{2}}{v+\sqrt{2}}\right)\right]_2^\infty=\frac{\pi}{2\sqrt{2}}-\frac{1}{\sqrt{2}}ln(1+\sqrt{2})$$

$$\Rightarrow I_2=\frac{\pi}{4}+\frac{ln2}{6}-\frac{2}{3}\left(\frac{\pi}{2\sqrt{2}}-\frac{1}{\sqrt{2}}ln(1+\sqrt{2})\right)=\frac{\pi}{4}-\frac{\pi}{3\sqrt{2}}+\frac{2}{3\sqrt{2}}ln(1+\sqrt{2})+\frac{ln2}{6}$$

$$\Rightarrow \Omega=I_1-I_2=-2ln2-\left(\frac{\pi}{4}-\frac{\pi}{3\sqrt{2}}+\frac{2}{3\sqrt{2}}ln(1+\sqrt{2})+\frac{ln2}{6}\right)=$$

$$= -\frac{2}{3\sqrt{2}}ln(1+\sqrt{2}) - \frac{13}{6}ln(2) + \pi\left(\frac{1}{3\sqrt{2}} - \frac{1}{4}\right) =$$

$$= \frac{1}{3\sqrt{2}}ln(1+\sqrt{2}) - \frac{1}{\sqrt{2}}ln(1+\sqrt{2}) - \frac{13}{6}ln(2) + \pi\left(\frac{1}{3\sqrt{2}} - \frac{1}{4}\right)$$

$$Now,\ \ sinh^{-1}x = ln\left(x+\sqrt{x^2+1}\right) \quad and \quad coth^{-1}x = \frac{1}{2}ln\left(\frac{x+1}{x-1}\right)$$

$$\Rightarrow sinh^{-1}(1) = ln(1+\sqrt{2}) = coth^{-1}\sqrt{2}$$

$$\Rightarrow \Omega = \frac{1}{3\sqrt{2}}sinh^{-1}(1) - \frac{1}{\sqrt{2}}coth^{-1}\sqrt{2} - \frac{13}{6}ln(2) + \pi\left(\frac{1}{3\sqrt{2}} - \frac{1}{4}\right)$$

2597. Find a closed form:

$$\Omega = \int_1^\infty \frac{x^2}{x^4+x^2+1}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\Omega = \int_1^\infty \frac{x^2}{x^4+x^2+1}dx = \int_1^\infty \frac{x^2}{x^2\left(x^2+\frac{1}{x^2}+1\right)}dx = \int_1^\infty \frac{1}{x^2+\frac{1}{x^2}+1}dx =$$

$$\overset{x=\frac{1}{y}}{=} \int_1^\infty \frac{-\frac{1}{y^2}}{y^2+\frac{1}{y^2}+1}dy = \int_1^\infty \frac{-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx$$

$$2\Omega = \int_1^\infty \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx$$

$$\Omega = \frac{1}{2}\int_1^\infty \frac{\left(x+\frac{1}{x}\right)' dx}{\left(x+\frac{1}{x}\right)^2 - 3} \overset{y=x+\frac{1}{x}}{=} \frac{1}{2}\int_2^\infty \frac{1}{y^2-3}dy =$$

$$\Omega = \frac{1}{4\sqrt{3}}\lim_{y\to\infty}\left(ln\left|\frac{y-\sqrt{3}}{y+\sqrt{3}}\right|\right) - \frac{1}{4\sqrt{3}}ln\left|\frac{2-\sqrt{3}}{2+\sqrt{3}}\right|$$

$$\Omega = \frac{1}{4\sqrt{3}}ln1 - \frac{1}{4\sqrt{3}}ln\left|\frac{2-\sqrt{3}}{2+\sqrt{3}}\right| = \frac{1}{4\sqrt{3}}ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

2598. Find:

$$I = \int_1^\infty \frac{x}{\sqrt{x^4+x^2+1}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Shirvan Tahirov-Azerbaijan, Kartick Chandra Betal-India

$$I=\int_1^\infty \frac{x}{\sqrt{x^4+x^2+1}}dx=\int_1^\infty \frac{dx}{\sqrt{x^2+\frac{1}{x^2}+1}}=$$

$$x=\frac{1}{y}\to dx=-\frac{1}{y^2}dy\to=\int_1^\infty \frac{-\frac{1}{y^2}dy}{\sqrt{y^2+\frac{1}{y^2}+1}}$$

$$I=\int_1^\infty \frac{dx}{\sqrt{x^2+\frac{1}{x^2}+1}},I=\int_1^\infty \frac{-\frac{1}{x^2}dx}{\sqrt{x^2+\frac{1}{x^2}+1}}\Rightarrow 2I=\int_1^\infty \frac{(1-\frac{1}{x^2})dx}{\sqrt{x^2+\frac{1}{x^2}+1}}$$

$$x+\frac{1}{x}=y\to\left(x+\frac{1}{x}\right)'dx=(1-\frac{1}{x^2})dx=dy$$

$$2I=\int_1^\infty \frac{\left(x+\frac{1}{x}\right)'dx}{\sqrt{\left(x+\frac{1}{x}\right)^2-1}}=\int_2^\infty \frac{dy}{\sqrt{y^2-1}}=log\left|y+\sqrt{y^2-1}\right|\Big|_2^\infty=\infty$$

$$I=\infty$$

2599. **Find a closed form:**

$$\Omega=\int_0^1 \frac{xln^2(x)}{(1-x)(2-x)(3-x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\Omega=\int_0^1 \frac{xln^2(x)}{(1-x)(2-x)(3-x)}dx=-2\int_0^1 \frac{ln^2(x)}{2-x}dx-\frac{1}{2}\int_0^1 \frac{ln^2(x)}{x-1}dx+\frac{3}{2}\int_0^1 \frac{ln^2(x)}{3-x}dx=$$

$$-2\int_0^1 \frac{\frac{1}{2}ln^2(x)}{1-\frac{1}{2}x}dx+\frac{1}{2}\int_0^1 \frac{ln^2(x)}{1-x}dx+\frac{3}{2}\int_0^1 \frac{\frac{1}{3}ln^2(x)}{1-\frac{1}{3}x}dx$$

$$=-4Li_3\left(\frac{1}{2}\right)+Li_3(1)+3Li_3\left(\frac{1}{3}\right)=$$

$$-\frac{1}{6}\left(21\zeta(3)+4ln^3(2)+12\zeta(2)\ln\left(\frac{1}{2}\right)\right)+\zeta(3)+3Li_3\left(\frac{1}{3}\right)$$

Note :

$$\therefore\int_0^1 \frac{yln^n(z)}{1-yz}dz=(-1)^n n!\,Li_{n+1}(y)$$

$$Li_{s+1}(z)=\int_0^z \frac{Li_s(t)}{t}dt$$

Solution 2 by Pratham Prasad-India

$$\Omega=\int_0^1\frac{x\ln^2(x)}{(1-x)(2-x)(3-x)}dx=$$

$$=\frac{3}{2}\int_0^1\frac{\ln^2(x)}{3-x}dx+\frac{1}{2}\int_0^1\frac{\ln^2(x)}{1-x}dx-2\int_0^1\frac{\ln^2(x)}{2-x}dx=\frac{3}{2}I(3)+\frac{1}{2}I(1)-2I(2)$$

$$Define:$$

$$I(a)=\int_0^1\frac{\ln^2(x)}{a-x}dx=\frac{1}{a}\int_0^1\frac{\ln^2(x)}{1-\frac{x}{a}}dx$$

$$=\frac{1}{a}\int_0^1\sum_{r=0}^{\infty}\left(\frac{x}{a}\right)^r\ln^2(x)\,dx=\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\int_0^1 x^r\ln^2(x)\,dx=$$

$$\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\frac{d^2}{dr^2}\left(\int_0^1 x^r\,dx\right)=\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\frac{d^2}{dr^2}\left(\frac{1}{r+1}\right)=\frac{1}{a}\sum_{r=0}^{\infty}\left(\frac{1}{a}\right)^r\left(\frac{2}{(r+1)^3}\right)=2\sum_{r=1}^{\infty}\frac{\left(\frac{1}{a}\right)^r}{r^3}$$

$$I(a)=2Li_3\left(\frac{1}{a}\right)$$

$$\psi=3Li_3\left(\frac{1}{3}\right)+Li_3(1)-4Li_2\left(\frac{1}{2}\right)$$

$$\int_0^1\frac{x\ln^2(x)}{(1-x)(2-x)(3-x)}dx=3Li_3\left(\frac{1}{3}\right)+\zeta(3)-4Li_2\left(\frac{1}{2}\right)$$

Solution 3 by Shobhit Jain-India

$$\Delta=\int_0^1\frac{x\ln^2x}{(1-x)(2-x)(3-x)}dx=\int_0^1 ln^2x\left(\frac{\frac{1}{2}}{1-x}-\frac{2}{2-x}+\frac{\frac{3}{2}}{3-x}\right)dx$$

$$=\int_0^1 ln^2x\left(\frac{1}{2}(1-x)^{-1}-\left(1-\frac{x}{2}\right)^{-1}+\frac{1}{2}\left(1-\frac{x}{3}\right)^{-1}\right)dx=\frac{1}{2}F(1)-F\left(\frac{1}{2}\right)+\frac{1}{2}F\left(\frac{1}{3}\right)$$

$$here,\ \ F(a)=\int_0^1(ln^2x)(1-ax)^{-1}dx=\sum_{n=1}^{\infty}\int_0^1(ln^2x)a^{n-1}x^{n-1}dx=\Gamma3\sum_{n=1}^{\infty}\frac{a^{n-1}}{n^3}=\frac{2}{a}Li_3(a)$$

$$\Rightarrow\Delta=\frac{1}{2}F(1)-F\left(\frac{1}{2}\right)+\frac{1}{2}F\left(\frac{1}{3}\right)=Li_3(1)-4Li_3\left(\frac{1}{2}\right)+3Li_3\left(\frac{1}{3}\right)$$

By Landen's Trilogarithmic identity,

$$Li_3(z)+Li_3(1-z)+Li_3\left(1-\frac{1}{z}\right)=\zeta(3)+\frac{1}{6}ln^3z+\zeta(2)lnz-\frac{1}{2}(ln^2z)ln(1-z)$$

$$\underbrace{\Rightarrow}_{z=1}Li_3(1)=\zeta(3)\underbrace{\Rightarrow}_{z=\frac{1}{2}}2Li_3\left(\frac{1}{2}\right)+Li_3(-1)=\zeta(3)+\frac{1}{3}ln^32-\zeta(2)ln2$$

$$\Rightarrow 2Li_3\left(\frac{1}{2}\right)-\eta(3)=\zeta(3)+\frac{1}{3}ln^3 2-\frac{\pi^2}{6}ln2 \Rightarrow 2Li_3\left(\frac{1}{2}\right)-\frac{3}{4}\zeta(3)$$
$$=\zeta(3)+\frac{1}{3}ln^3 2-\frac{\pi^2}{6}ln2$$
$$\Rightarrow Li_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}ln2+\frac{1}{6}ln^3 2 \Rightarrow \Delta=\zeta(3)-4\left(\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}ln2+\frac{1}{6}ln^3 2\right)+3Li_3\left(\frac{1}{3}\right)$$
$$\Rightarrow \Delta=3Li_3\left(\frac{1}{3}\right)+\frac{\pi^2}{3}ln2-\frac{5}{2}\zeta(3)-\frac{2}{3}ln^3 2$$

2600. Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^2\ln\left(sin^2(x)\right)+x^2(1+\sin(x^2))\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\therefore \ln(sin(x))=-\ln(2)-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k},\qquad for\ \ 0<x<\pi$$
$$\therefore \int x^{n-1}\cos(ax)\,dx=\sum_{j=0}^{n-1}j!\binom{n-1}{j}\frac{x^{n-1-j}}{a^{j+1}}\sin\left(ax+\frac{j\pi}{2}\right)$$
$$For\ \ n,\ \ m,\qquad k\in Z^+;\ \ m\in R;\ a,b\ \in R$$
$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^n\ln\left(sin^k(x)\right)+x^m(a+b\sin(x^m))\right)dx=\underbrace{\int_0^{\frac{\pi}{2}}x^n\ln\left(sin^k(x)\right)dx}_{A}+$$
$$\underbrace{a\int_0^{\frac{\pi}{2}}x^m dx}_{B}+\underbrace{b\int_0^{\frac{\pi}{2}}x^m\left(sin(x^m)\right)dx}_{C}$$
$$A=\int_0^{\frac{\pi}{2}}x^n\ln\left(sin^k(x)\right)dx=k\int_0^{\frac{\pi}{2}}x^n\ln\left(sin(x)\right)dx$$
$$=-k\int_0^{\frac{\pi}{2}}x^n\left(\ln(2)+\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}\right)dx=$$
$$-k(\ln(2)\int_0^{\frac{\pi}{2}}x^{n-1}dx+\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}}x^{n-1}\cos(2kx))=-k(\frac{\pi^n\ln(2)}{2^n.n}+$$
$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=0}^{\infty}j!\binom{n-1}{j}\frac{x^{n-1-j}}{(2k)^{j+1}}\sin\left(2kx+\frac{j\pi}{2}\right)\Bigg|_0^{\frac{\pi}{2}})=-k(\frac{\pi^{n+1}\ln(2)}{2^{n+1}.(n+1)}-$$
$$\frac{n!}{2^{n+1}}\sin\left(\frac{\pi x}{2}\right)\sum_{k=1}^{\infty}\frac{1}{k^{n+2}}+\sum_{j=0}^{n}j!\binom{n}{j}\frac{\pi^{n-j}}{2^{n+1}}\sum_{k=1}^{\infty}\frac{\sin\left(k\pi+\frac{j\pi}{2}\right)}{k^{j+1}})=-k(\frac{\pi^{n+1}\ln(2)}{2^{n+1}.(n+1)}-$$

$$\sum_{j=0}^{n} j!\binom{n}{j}\frac{\pi^{n-j}}{2^{n+1}}\sin\left(\frac{j\pi}{2}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{j+1}}-\frac{n!}{2^{n+1}}\sin\left(\frac{n\pi}{2}\right)\zeta(n+2))=-(\frac{\pi^{n+1}\ln(2)}{2^{n+1}.(n+1)}-$$

$$\sum_{j=0}^{n} j!\binom{n}{j}\frac{\pi^{n-j}}{2^{n+1}}\sin\left(\frac{j\pi}{2}\right)(1-2^{-j})\zeta(n+1)-\frac{n!}{2^{n+1}}\sin\left(\frac{n\pi}{2}\right)\zeta(n+2))$$

$$B=\int_0^{\frac{\pi}{2}} x^m dx=\frac{x^{m+1}}{m+1}\Big|_0^{\frac{\pi}{2}}=\frac{1}{m+1}\left(\frac{\pi}{2}\right)^{m+1}$$

$$C=\int_0^{\frac{\pi}{2}} x^m(\sin(x^m))dx=-\frac{1}{m}\int_0^{\frac{\pi}{2}} xd(\cos(x^m))\overset{I.B.P}{=}-\frac{1}{m}\left(x\cos(x^m)|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\cos(x^m)\right)=$$

$$-\frac{1}{m}\left(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-\int_0^{\infty}\cos(x^m)dx+\int_{\frac{\pi}{2}}^{\infty}\cos(x^m)dx\right)=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)\cos\frac{\pi}{2m}+\frac{1}{2m}\int_{\frac{\pi}{2}}^{\infty}\frac{e^{ix^m}+e^{-ix^m}}{(x^m)^{\frac{m-1}{m}}}d(x^m))=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)\cos\frac{\pi}{2m}+\frac{1}{2m}\int_1^{\infty}\frac{e^{i\left(\frac{\pi}{2}\right)^m t}+e^{-i\left(\frac{\pi}{2}\right)^m t}}{\left(\left(\frac{\pi}{2}\right)^n t\right)^{\frac{m-1}{m}}}\left(\frac{\pi}{2}\right)dt)=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)\cos\frac{\pi}{2m}+\frac{1}{4m}\left(\int_1^{\infty}\frac{e^{i\left(\frac{\pi}{2}\right)^m t}}{t^{\frac{m-1}{m}}}dt+\int_1^{\infty}\frac{e^{-i\left(\frac{\pi}{2}\right)^m t}}{t^{\frac{m-1}{m}}}dt\right))=-\frac{1}{m}(\frac{\pi}{2}\cos\left(\left(\frac{\pi}{2}\right)^m\right)-$$

$$\Gamma\left(1+\frac{1}{m}\right)\cos\frac{\pi}{2m}+\frac{1}{4m}\left(E_{\frac{m-1}{m}}\left(-i\left(\frac{\pi}{2}\right)^m\right)+E_{\frac{m-1}{m}}\left(i\left(\frac{\pi}{2}\right)^m\right)\right)$$

Note :

$$\therefore\int_0^{\infty}\cos(x^m)\,dx=\Re\left(\int_0^{\infty}e^{ix^m}\,dx\right)=\Re\left(\Gamma\left(1+\frac{1}{m}\right)\left(\cos\frac{\pi}{2m}+i\sin\frac{\pi}{2m}\right)\right)$$

$$=\Gamma\left(1+\frac{1}{m}\right)\cos\frac{\pi}{2m}$$

$$\therefore E_n(x)=\int_1^{\infty}\frac{e^{-xt}}{t^n}dt:\ \textit{The exponential integral}$$

$$n=2,\quad k=2,\quad m=2,\ a=1,b=1$$

$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^2\ln\left(\sin^2(x)\right)+x^2(1+\sin(x^2))\right)dx$$

$$\Omega=-\frac{\ln(2)\pi^3}{12}+\frac{3\pi}{8}\zeta(3)+\frac{\pi^3}{24}-\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right)+\frac{1}{2}\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{\pi}{2}}\right)$$

Solution 2 by Pratham Prasad-India

$$\Omega=\int_0^{\frac{\pi}{2}}\left(x^2\ln(\sin^2(x))+x^2(1+\sin(x^2))\right)$$

$$\int_0^{\frac{\pi}{2}}\left(x^2\ln(sin^2(x))+x^2(1+\sin(x^2))\right)$$

$$=\int_0^{\frac{\pi}{2}}x^2\ln(sin^2(x))\,dx+\int_0^{\frac{\pi}{2}}x^2\,dx+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$2\int_0^{\frac{\pi}{2}}x^2\ln(sinx)\,dx+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$2\int_0^{\frac{\pi}{2}}x^2\left(-ln(2)-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}\right)dx+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$-2\ln(2)\int_0^{\frac{\pi}{2}}x^2\,dx-2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\frac{\pi}{2}}x^2\cos(2x)\,dx+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$-\frac{\ln(2)\,\pi^3}{12}-2\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\pi(-1)^k}{4k^2}\right)dx+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$-\frac{\ln(2)\,\pi^3}{12}-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=$$

$$-\frac{\ln(2)\,\pi^3}{12}+\frac{3\pi}{8}\zeta(3)+\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx$$

Now,

$$\int_0^{\frac{\pi}{2}}x^2\sin(x^2)\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}x(2x\sin(x^2))\,dx=\left[-\frac{1}{2}xcos(x^2)\right]_0^{\frac{\pi}{2}}+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(x^2)\,dx=$$

$$=-\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right)+\frac{1}{2}\left[\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}x\right)\right]_0^{\frac{\pi}{2}}=-\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right)+\frac{1}{2}\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{\pi}{2}}\right)$$

$$\psi=\int_0^{\frac{\pi}{2}}\left(x^2\ln(sin^2(x))+x^2(1+\sin(x^2))\right)$$

$$\Omega=-\frac{\ln(2)\,\pi^3}{12}+\frac{3\pi}{8}\zeta(3)+\frac{\pi^3}{24}-\frac{\pi}{4}\cos\left(\frac{\pi^2}{4}\right)+\frac{1}{2}\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{\pi}{2}}\right)$$

Solution 3 by Shobhit Jain-India

$$\Omega=\int_0^{\frac{\pi}{2}}x^2\,ln(sin^2x)\,dx+\int_0^{\frac{\pi}{2}}x^2dx+\int_0^{\frac{\pi}{2}}x^2sin(x^2)dx$$

$$=\int_0^{\frac{\pi}{2}}2x^2\,ln(2sinx)\,dx+(1-2ln2)\int_0^{\frac{\pi}{2}}x^2dx+\frac{1}{2}\int_0^{\frac{\pi}{2}}xsin(x^2)2xdx$$

$$\underset{\theta=2x}{\overset{}{=}}\frac{1}{4}\int_0^{\pi}\theta^2\,ln\left(2sin\left(\frac{\theta}{2}\right)\right)d\theta+(1-2ln2)\frac{\pi^3}{24}+\frac{1}{2}\int_0^{\frac{\pi}{2}}xsin(x^2)d(x^2)$$

$$=-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\int_0^{\pi}\theta^2\cos(n\theta)\,d\theta}_{I_1}+(1-2ln2)\frac{\pi^3}{24}-\frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}}xd[cos(x^2)]}_{I_2}$$

$$I_1=\int_0^{\pi}\theta^2\cos(n\theta)\,d\theta=\left[\frac{\theta^2\sin(n\theta)}{n}+2\frac{\theta\cos(n\theta)}{n^2}-2\frac{\sin(n\theta)}{n^3}\right]_0^{\pi}=\frac{2\pi\cos(n\pi)}{n^2}=\frac{2\pi(-1)^n}{n^2}$$

$$I_2=\int_0^{\frac{\pi}{2}}xd[cos(x^2)]=\frac{\pi}{2}cos\left(\frac{\pi^2}{4}\right)-\int_0^{\frac{\pi}{2}}cos(x^2)dx=\frac{\pi}{2}cos\left(\frac{\pi^2}{4}\right)-\sqrt{\frac{\pi}{2}}C\left(\frac{\pi}{2}\right)$$

$$\Rightarrow\Omega=-\frac{1}{4}\sum_{n=1}^{\infty}\frac{2\pi(-1)^n}{n^3}+(1-2ln2)\frac{\pi^3}{24}-\frac{1}{2}\left(\frac{\pi}{2}cos\left(\frac{\pi^2}{4}\right)-\sqrt{\frac{\pi}{2}}C\left(\frac{\pi}{2}\right)\right)$$

$$=\frac{\pi}{2}\eta(3)+(1-2ln2)\frac{\pi^3}{24}-\frac{\pi}{4}cos\left(\frac{\pi^2}{4}\right)+\sqrt{\frac{\pi}{8}}C\left(\frac{\pi}{2}\right)$$

$$\Omega=\frac{3\pi}{8}\zeta(3)+(1-2ln2)\frac{\pi^3}{24}-\frac{\pi}{4}cos\left(\frac{\pi^2}{4}\right)+\sqrt{\frac{\pi}{8}}C\left(\frac{\pi}{2}\right)$$

NOTES:

$$\eta(x)-Dirichlet's\ Eta\ Function$$

$$\eta(x)=(1-2^{1-x})\zeta(x)$$

$$\zeta(x)-Riemann's\ Zeta\ Function$$

$$C(x)-Fresenel's\ cosine\ integral\ function$$

$$C(x)=\sqrt{\frac{2}{\pi}}\int_0^{x}cos(t^2)dt=\frac{1}{2}\int_0^{x^2}J_{-\frac{1}{2}}(t)dt$$

$$J_{-\frac{1}{2}}(t)=\sqrt{\frac{2}{\pi t}}\cos(t)-Bessel's\ cosine\ function$$

It's nice to be important but more important it's to be nice.
At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru