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1601.

If $a,b,c>0$ *such that* : $a+b+c=3$ *and* $n,k\in\mathbb{N}$ with $k+1\ge n$, then

$$\sum_{cyc}\frac{ab}{a^2+nb^2+ka}\le\frac{3}{n+k+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$a^2+nb^2+ka\overset{\text{weighted AM}\ge\text{weighted GM}}{\ge}(n+k+1).\sqrt[n+k+1]{a^2b^{2n}.a^k}$$

$$\Rightarrow\frac{ab}{a^2+nb^2+ka}\le\frac{1}{n+k+1}.\frac{a}{a^{\frac{k+2}{n+k+1}}}.\frac{b}{a^{\frac{2n}{n+k+1}}}=\frac{1}{n+k+1}.\left(a^{n-1}.b^{k+1-n}\right)^{\frac{1}{n+k+1}}$$

$$=\frac{1}{n+k+1}.\left(\sqrt[k]{a^{n-1}.b^{k+1-n}}\right)^{\frac{k}{n+k+1}}\underset{\text{since }k+1-n\ge0}{\overset{\text{weighted GM}\ge\text{weighted AM}}{\le}}$$

$$\frac{1}{n+k+1}.\left(\frac{(n-1)a+(k+1-n)b}{k}\right)^{\frac{k}{n+k+1}}\overset{\text{Bernoulli}}{\le}$$

$$\frac{1}{n+k+1}.\left(1+\left(\frac{(n-1)a+(k+1-n)b-k}{k}\right).\frac{k}{n+k+1}\right)$$

$$\therefore\frac{ab}{a^2+nb^2+ka}\le\frac{1}{n+k+1}.\left(1+\frac{(n-1)(a-b)+bk-k}{n+k+1}\right)\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{ab}{a^2+nb^2+ka}\le\frac{3}{n+k+1}$$

$$+\frac{(n-1)(a-b+b-c+c-a)+k(a+b+c)-3k}{(n+k+1)^2}$$

$$\overset{a+b+c=3}{=}\frac{3}{n+k+1}+\frac{3k-3k}{(n+k+1)^2}\therefore\sum_{cyc}\frac{ab}{a^2+nb^2+ka}\le\frac{3}{n+k+1}$$

$\forall\, a,b,c>0\,\middle|\,a+b+c=3$ and $n,k\in\mathbb{N}$ with $k+1\ge n$,
$''=''$ iff $a=b=c=1$ (QED)

1602. If $a, b, c > 0$ such that : $abc = 1$ and $\lambda \le 3$, then :

$$(a+b)(b+c)(c+a) - \lambda(a+b+c) \ge 8 - 3\lambda$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(a+b)(b+c)(c+a) - \lambda(a+b+c) \ge 8 - 3\lambda$$

$$\Leftrightarrow \prod_{cyc}(b+c) - 8 \ge \lambda\left(\sum_{cyc} a - 3\right) \overset{abc=1}{\Leftrightarrow} \frac{\prod_{cyc}(b+c) - 8abc}{abc} \overset{(*)}{\ge} \lambda\left(\sum_{cyc} a - 3\right)$$

$$\text{Now, } \sum_{cyc} a - 3 \overset{A-G}{\ge} 3\sqrt[3]{abc} - 3 \overset{abc=1}{=} 3 - 3 = 0 \Rightarrow \sum_{cyc} a - 3 \ge 0$$

$$\Rightarrow (\lambda - 3)\left(\sum_{cyc} a - 3\right) \le 0 \ (\because \lambda - 3 \le 0) \Rightarrow \lambda\left(\sum_{cyc} a - 3\right) \overset{(i)}{\le} 3\left(\sum_{cyc} a - 3\right)$$

$\therefore$ (i) $\Rightarrow$ in order to prove $(*)$, it suffices to prove :

$$\frac{\prod_{cyc}(b+c) - 8abc}{abc} \ge 3\left(\sum_{cyc} a - 3\right) \overset{abc=1}{\Leftrightarrow} \frac{\prod_{cyc}(b+c) - 8abc}{abc} + 9 \ge \frac{3\sum_{cyc} a}{\sqrt[3]{abc}}$$

$$\Leftrightarrow \frac{\prod_{cyc}(b+c) + abc}{abc} \overset{(**)}{\ge} \frac{3\sum_{cyc} a}{\sqrt[3]{abc}}$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $=$ s, R, r (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{ and } (1) \text{ and } (2) \Rightarrow (**) \Leftrightarrow \frac{xyz + r^2 s}{r^2 s} \ge \frac{3s}{\sqrt[3]{r^2 s}}$$

$$\Leftrightarrow \frac{4Rrs + r^2 s}{r^2 s} \ge \frac{3s}{\sqrt[3]{r^2 s}} \Leftrightarrow \frac{(4R+r)^3}{r^3} \ge \frac{27s^3}{r^2 s} \Leftrightarrow (4R+r)^3 \overset{(***)}{\ge} 27rs^2$$

$$\text{Now, } 27rs^2 \overset{Gerretsen}{\le} 27r(4R^2 + 4Rr + 3r^2) \overset{?}{\le} (4R+r)^3$$

$$\Leftrightarrow 16t^3 - 15t^2 - 24t - 20 \overset{?}{\ge} 0 \ \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(16t^2 + 17t + 10) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t \overset{Euler}{\ge} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (a+b)(b+c)(c+a) - \lambda(a+b+c) \ge 8 - 3\lambda$$

$$\forall a, b, c > 0 \,|\, abc = 1, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1603. $If\ a,b,c>0\ ,ab+bc+ca=3abc,n\in N\ then$

$$\frac{1}{na+b}+\frac{1}{nb+c}+\frac{1}{nc+a}\le\frac{3}{n+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{na+b}+\frac{1}{nb+c}+\frac{1}{nc+a}\overset{AM-HM}{\le}\frac{1}{(n+1)^2}\sum\left(\frac{n}{a}+\frac{1}{a}\right)=$$

$$=\frac{1}{(n+1)^2}\left[\sum\frac{n}{a}+\sum\frac{1}{a}\right]=\frac{n+1}{(n+1)^2}\cdot\frac{ab+bc+ca}{abc}=\frac{3}{n+1}$$

$$(since\ ab+bc+ca=3abc)$$

Equality holds for $a=b=c=1$.

1604. If $a,b\ge\frac{1}{6}$ then:

$$\sqrt{8a^2+1}+\sqrt{8b^2+1}+\frac{32}{3}\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\ge\frac{50}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$Lemma1: For\ x>0\ ,\sqrt{8x^2+1}\ge\frac{8x+1}{3},$

$Proof: \sqrt{8x^2+1}\ge\frac{8x+1}{3}\ or\ ,$

$$9(8x^2+1)\ge(8x+1)^2\ or,$$

$$8(x^2-2x+1)\ge0\ or,(x-1)^2\ge0\ (true)$$

$Lemma\ 2:\ for\ x>0\ ,\frac{1+8x}{3}+\frac{32}{3}\frac{1}{x+1}\ge\frac{25}{3}$

$Proof: \frac{1+8x}{3}+\frac{32}{3}\frac{1}{x+1}\ge\frac{25}{3}\ or,$

$$(1+8x)(x+1)+32\ge25(x+1)or,$$

$$8(x^2-2x+1)\ge0\ or,(x-1)^2\ge0\ True$$

$$\sqrt{8a^2+1}+\sqrt{8b^2+1}+\frac{32}{3}\left(\frac{1}{a+1}+\frac{1}{b+1}\right)=$$

$$=\left[\sqrt{8a^2+1}+\frac{32}{3}\left(\frac{1}{a+1}\right)\right]+\left[\sqrt{8b^2+1}+\frac{32}{3}\left(\frac{1}{b+1}\right)\right]\overset{lemma1}{\ge}$$

$$\geq\left[\frac{8a+1}{3}+\frac{32}{3}\left(\frac{1}{a+1}\right)\right]+\left[\frac{8b+1}{3}+\frac{32}{3}\left(\frac{1}{b+1}\right)\right]\overset{lemma2}{\geq}$$

$$\geq\frac{25}{3}+\frac{25}{3}=\frac{50}{3}\,(Equality\ for\ a=b=1)$$

1605. **If** $a_k>0\ (k=1,2,\dots,n)$ **, then**

$$\sum_{cyclic}\frac{a_1{}^2}{a_1+a_2}\geq\frac{1}{2}\sum_{k=1}^{n}a_k\geq\sum_{cyclic}\frac{a_1a_2{}^2}{a_1{}^2+a_2{}^2}$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$LHS:\frac{a_1{}^2}{a_1+a_2}+\frac{a_2{}^2}{a_2+a_3}+\frac{a_3{}^2}{a_3+a_4}+\ \dots\ +\frac{a_n{}^2}{a_n+a_1}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a_1+a_2+\dots+a_n)^2}{2(a_1+a_2+\cdots+a_n)}=\frac{1}{2}\sum_{k=1}^{n}a_k$$

$$RHS:\sum_{cyc}\frac{a_1a_2{}^2}{a_1{}^2+a_2{}^2}=\sum_{cyc}\frac{a_1}{\left(\frac{a_1}{a_2}\right)^2+1}\overset{A-G}{\leq}\sum_{cyc}\frac{a_1}{2\cdot\frac{a_1}{a_2}}=\frac{1}{2}\sum_{cyc}a_k=\frac{1}{2}\sum_{k=1}^{n}a_k$$

1606. **If** $x,y,z>0$**, then prove that:**

$$\prod\frac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}\geq\frac{343}{125}\left(\sum xy\right)^3$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

Solution by Rovsen Pirguliyev-Azerbaijan

To prove that $(x+y)^7-x^7-y^7=7xy(x+y)(x^2+xy+y^2)$ **(1)**

denote $A=(x+y)^7-x^7-y^7$

$$x^7+y^7=(x+y)\left(x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6\right)$$

and $(x+y)^6=x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$

then we have:

$$A=(x+y)\left(7x^5y+14x^4y^2+21x^3y^3+14x^2y^4+7xy^5\right)=$$

$$=7xy(x+y)(x^4+2x^3y+3x^2y^2+2xy^3+y^4)=$$

$$=7xy(x+y)(x^2+xy+y^2)^2\quad(1)$$

Similarly to prove $(x+y)^5-x^5-y^5=5xy(x+y)(x^2+yx+y^2)\quad$ **(2)**

Using (1) and (2) $\Rightarrow$

$$\prod\frac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}=\prod\frac{7xy(x+y)(x^2+xy+y^2)^2}{5xy(x+y)(x^2+xy+y^2)}=$$

$$=\frac{343}{125}(x^2+xy+y^2)\cdot(y^2+yz+z^2)(z^2+zx+x^2)$$

further applying Holder's inequality we have:

$$\prod(x^2+xy+y^2)\ge\left(\sum xy\right)^3$$

1607. *If* $a_k>0\ (k=1,2,\dots,n)$ *and* $\sum_{k=1}^n a_k^2=n$ *then prove that*:

$$\sum_{k=1}^{n}\frac{1}{n+1-ak}\le 1$$

Proposed by Neculai Stanciu-Romania

Solution by Khaled Abd Imouti-Syria

$$by\ using\ AM-GM:$$

$$1=\sqrt{\frac{n}{n}}=\sqrt{\frac{a_1^2+a_2^2+\cdots..+a_n^2}{n}}\le\frac{a_1+a_2+\cdots.+a_n}{n}$$

$$1\le\frac{a_1+a_2+\cdots.+a_n}{n}\Rightarrow n\le a_1+a_2+\cdots.+a_n\ (\text{I})$$

$$but:\sum_{k=1}^{n}a_k^2=n\Rightarrow\sum_{k=1}^{n}\frac{a_k^2}{n}=1$$

$$1=\frac{a_1^2}{n}+\frac{a_2^2}{n}+\cdots+\frac{a_n^2}{n}\overset{BERGSTROM}{\ge}\frac{(a_1+a_2+\cdots.+a_n)^2}{n^2}$$

$$(a_1+a_2+\cdots.+a_n)^2\le n^2\Rightarrow a_1+a_2+\cdots.+a_n\le n\ (\Pi)$$

$$from\ (\text{I}),(\Pi):\ a_1+a_2+\cdots+a_n=n$$

$$but: \sqrt{\frac{a_1^2+a_2^2+\cdots..+a_n^2}{n}} \le \frac{a_1+a_2+\cdots.+a_n}{n} \ holds\ when:$$

$$a_1=a_2=\cdots=a_n, so: a_1=a_2=\cdots=a_n=\frac{1}{n}$$

$$so: \sum_{k=1}^{n}\frac{1}{n+1-a_k}=\frac{n}{n+1-\frac{1}{n}}=\frac{n^2}{n^2+n-1}\le 1 \Leftrightarrow n\ge 1\ \ (true)$$

1608. If $a,b>0$ then:

$$\frac{a^3}{b}+\frac{b^3}{a}+8\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\ge 10$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^4+b^4 \overset{CEBYSHEV}{\geq} \frac{(a^3+b^3)(a+b)}{2}=$$

$$\frac{(a+b)^2(a^2+b^2-ab)}{2}\overset{AM-GM}{\geq}\frac{(a+b)^2(2ab-ab)}{2}=\frac{ab(a+b)^2}{2}\ and$$

$$8\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\overset{Bergstrom}{\geq} 8.\frac{(1+1)^2}{a+b+2}=\frac{32}{a+b+2}$$

Now we need to show

$$\frac{a^3}{b}+\frac{b^3}{a}+8\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\ge 10\ or$$

$$\frac{a^4+b^4}{ab}+8\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\ge 10\ or$$

$$\frac{ab(a+b)^2}{2ab}+\frac{32}{a+b+2}\ge 10\ or$$

$$\frac{x^2}{2}+\frac{32}{x+2}\overset{a+b=x>0}{\geq} 10\ or$$

$$x^3+2x^2-20x+24\ge 0\ or$$

$$(x+6)(x-2)^2\ge 0\ true$$

$$Equality\ for\ x=a+b=2\ or\ a=b=1$$

1609. If $a, b > 0$ then:

$$\frac{a^3}{a^2+1}+\frac{b^3}{b^2+1}+\frac{4}{a+b}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a^3}{a^2+1}+\frac{b^3}{b^2+1}+\frac{4}{a+b}\geq 3$$

$$\left(a-\frac{a}{a^2+1}\right)+\left(b-\frac{b}{b^2+1}\right)+\frac{4}{a+b}\geq 3$$

$$(a+b)+\frac{4}{a+b}-\left(\frac{a}{a^2+1}+\frac{b}{b^2+1}\right)\geq 3$$

$$(a+b)+\frac{4}{a+b}-\left(\frac{a}{2a}+\frac{b}{2b}\right)\geq 3(am-gm)$$

$$(a+b)+\frac{4}{a+b}\geq 4$$

$$(a+b)^2-4(a+b)+4\geq 0 \text{ or } (a+b-2)^2\geq 0 \ (true)$$

$$Equality\ for\ a+b=2 \text{ or } a=b=1$$

1610. If $a, \mathrm{b}\in\mathbb{R}$ and $(a+1)(\mathrm{b}+1)=4$, then prove that :

$$a^4+\mathrm{b}^4\geq a^3+\mathrm{b}^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

As $(a+1)(\mathrm{b}+1)=4 \therefore$ either $(a+1),(\mathrm{b}+1)>0$ or $(a+1),(\mathrm{b}+1)<0$ and in the latter case, $a,\mathrm{b}<-1<0$ $\Rightarrow a^3+\mathrm{b}^3<0\leq a^4+\mathrm{b}^4 \therefore a^4+\mathrm{b}^4>a^3+\mathrm{b}^3$

When $(a+1),(\mathrm{b}+1)>0, (a+1)(\mathrm{b}+1)=4\Rightarrow\sqrt{(a+1)(\mathrm{b}+1)}=2\Rightarrow 2\overset{A-G}{\leq}\frac{a+1+\mathrm{b}+1}{2}$ $(\because (a+1)(\mathrm{b}+1)=4\Rightarrow(a+1),(\mathrm{b}+1)>0)\Rightarrow a+\mathrm{b}\geq 2$

$$\Rightarrow 3-a\mathrm{b}\geq 2\left(\because (a+1)(\mathrm{b}+1)=4\Rightarrow a+\mathrm{b}+a\mathrm{b}\overset{(*)}{=}3\right)\Rightarrow x=a\mathrm{b}\leq 1\to(1)$$

$$\text{Now, } a^4+\mathrm{b}^4\overset{?}{\geq}a^3+\mathrm{b}^3\Leftrightarrow(a^2+\mathrm{b}^2)^2-2a^2\mathrm{b}^2\overset{?}{\geq}(a+\mathrm{b})(a^2+\mathrm{b}^2-a\mathrm{b})$$

$$\Leftrightarrow ((a+\mathrm{b})^2-2a\mathrm{b})^2-2a^2\mathrm{b}^2 \overset{?}{\geq} (a+\mathrm{b})((a+\mathrm{b})^2-3a\mathrm{b})$$

$$\overset{\text{via } (*)}{\Leftrightarrow} ((3-x)^2-2x)^2-2x^2 \overset{?}{\geq} (3-x)((3-x)^2-2x)$$

$$\Leftrightarrow x^4-15x^3+68x^2-108x+54 \overset{?}{\geq} 0 \Leftrightarrow (x-1)\left(x^2(x-14)+54(x-1)\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because (x-1) \overset{\text{via } (1)}{\leq} 0 \text{and } x^2(x-14) \overset{\text{via } (1)}{<} 0 \text{ and } 54(x-1) \overset{\text{via } (1)}{\leq} 0$$

$$\Rightarrow \left(x^2(x-14)+54(x-1)\right) < 0 \therefore a^4+\mathrm{b}^4 \geq a^3+\mathrm{b}^3 \therefore \text{combining } all \text{ cases,}$$

$$a^4+\mathrm{b}^4 \geq a^3+\mathrm{b}^3 \ \forall\, a,\mathrm{b} \in \mathbb{R} \,\Big|\, (a+1)(\mathrm{b}+1)=4, '' ='' \text{ iff } a=\mathrm{b}=1 \text{ (QED)}$$

1611. If $a, b, c \in [3;5]$ and $a^2+b^2+c^2=50$, then prove that:

$$a+b+c \geq 12$$

Proposed by Hung Nguyen Cuong-Vietnam

Solution by Pham Duc Nam-Vietnam

$$* \ a,b,c \in [3,5] \Rightarrow (a-3)(b-3)(c-3) \geq 0$$

and $(5-a)(5-b)(5-c) \geq 0$

$$\Rightarrow (a-3)(b-3)(c-3)+(5-a)(5-b)(5-c) \geq 0$$

Expand and simplify:

$$\Leftrightarrow 2(ab+bc+ca)-16(a+b+c)+98 \geq 0$$

But: $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$

$$\Leftrightarrow (a+b+c)^2-(a^2+b^2+c^2)-16(a+b+c)+98 \geq 0$$

$$\Leftrightarrow (a+b+c)^2-16(a+b+c)+48 \geq 0$$

$\Leftrightarrow a+b+c \geq 12$ or $a+b+c \leq 4$

$a,b,c \in [3,5] \Rightarrow a+b+c \geq 12$, proved.

Equality holds iff: $a=3, b=4, c=5$ and permutations.

1612. If $a, b > 0$ then:

$$\sqrt{3a^2+1}+\sqrt{3b^2+1}+6\left(\frac{1}{a+1}+\frac{1}{b+1}\right) \geq 10$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sqrt{3a^2+1}+\sqrt{3b^2+1}=\sqrt{\left(\sqrt{3}a\right)^2+(1)^2}+\sqrt{\left(\sqrt{3}b\right)^2+(1)^2}\overset{Minkowski}{\geq}$$
$$\sqrt{\left(\sqrt{3}(a+b)\right)^2+(1+1)^2}=\sqrt{3(a+b)^2+4}\ and$$
$$6\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\overset{Bergstrom}{\geq}6.\frac{(1+1)^2}{a+b+2}=\frac{24}{a+b+2}$$
$$Now\ we\ need\ to\ show\ \sqrt{3a^2+1}+\sqrt{3b^2+1}+6\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\geq 10\ or$$
$$\sqrt{3(a+b)^2+4}+\frac{24}{a+b+2}\geq 10\ or,$$
$$\sqrt{3x^2+4}+\frac{24}{x+2}\overset{a+b=x>0}{\geq}10\ or,$$
$$\sqrt{3x^2+4}\geq 10-\frac{24}{x+2}\ or,$$
$$\sqrt{3x^2+4}\geq\frac{10x-4}{x+2}\ or,$$
$$(3x^2+4)(x+2)^2\geq(10x-4)^2\ or,$$
$$x^3+4x^2-28x+32\geq 0\ or,$$
$$(x-2)^2(x+8)\geq 0\ true\ ,equality\ for\ x=a+b=2\ or, a=b=1$$

1613. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a(b+c)}{b^2.\sqrt{ac}+bc.\sqrt{bc}}+\frac{b(c+a)}{c^2.\sqrt{ab}+ca.\sqrt{ca}}+\frac{c(a+b)}{a^2.\sqrt{bc}+ab.\sqrt{ab}}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a(b+c)}{b^2.\sqrt{ac}+bc.\sqrt{bc}}+\frac{b(c+a)}{c^2.\sqrt{ab}+ca.\sqrt{ca}}+\frac{c(a+b)}{a^2.\sqrt{bc}+ab.\sqrt{ab}}$$
$$=\frac{a(b+c)}{b.\sqrt{bc}\left(\sqrt{ab}+c\right)}+\frac{b(c+a)}{c.\sqrt{ca}\left(\sqrt{bc}+a\right)}+\frac{c(a+b)}{a.\sqrt{ab}\left(\sqrt{ca}+b\right)}$$
$$\overset{abc=1}{=}\frac{a\sqrt{a}(b+c)}{b\left(\sqrt{ab}+\sqrt{c}.\sqrt{c}\right)}+\frac{b\sqrt{b}(c+a)}{c\left(\sqrt{bc}+\sqrt{a}.\sqrt{a}\right)}+\frac{c\sqrt{c}(a+b)}{a\left(\sqrt{ca}+\sqrt{b}.\sqrt{b}\right)}$$
$$\overset{CBS}{\geq}\frac{a\sqrt{a}(b+c)}{b\left(\sqrt{c+a}.\sqrt{b+c}\right)}+\frac{b\sqrt{b}(c+a)}{c\left(\sqrt{a+b}.\sqrt{c+a}\right)}+\frac{c\sqrt{c}(a+b)}{a\left(\sqrt{b+c}.\sqrt{a+b}\right)}$$
$$=\frac{a\sqrt{a}.\sqrt{b+c}}{b.\sqrt{c+a}}+\frac{b\sqrt{b}.\sqrt{c+a}}{c.\sqrt{a+b}}+\frac{c\sqrt{c}.\sqrt{a+b}}{a.\sqrt{b+c}}$$

$$\overset{A-G}{\geq} 3.\sqrt[3]{\frac{a\sqrt{a}.\sqrt{b+c}}{b.\sqrt{c+a}}.\frac{b\sqrt{b}.\sqrt{c+a}}{c.\sqrt{a+b}}.\frac{c\sqrt{c}.\sqrt{a+b}}{a.\sqrt{b+c}}} = 3.\sqrt[3]{\sqrt{abc}} \overset{abc=1}{=} 3$$

$$\therefore \frac{a(b+c)}{b^2.\sqrt{ac}+bc.\sqrt{bc}} + \frac{b(c+a)}{c^2.\sqrt{ab}+ca.\sqrt{ca}} + \frac{c(a+b)}{a^2.\sqrt{bc}+ab.\sqrt{ab}} \geq 3$$

$$\forall a,b,c>0 \,|\, abc=1, '' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1614. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a.\sqrt{b^3+c^3}+b.\sqrt{a^3+c^3}}{ab(a+b)} + \frac{c.\sqrt{a^3+b^3}+b.\sqrt{a^3+c^3}}{bc(b+c)} + \frac{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}{ac(a+c)} \geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kokata-India

$\forall A,B,C>0, (A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$) \Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right) - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB + 2\sum_{cyc}A^2 - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now, } \forall x,y,z>0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2 + xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have : } \frac{a.\sqrt{b^3+c^3}+b.\sqrt{a^3+c^3}}{ab(a+b)} + \frac{c.\sqrt{a^3+b^3}+b.\sqrt{a^3+c^3}}{bc(b+c)} + \frac{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}{ac(a+c)}$$

$$\overset{abc=1}{=} \frac{ac.\sqrt{b^3+c^3}+bc.\sqrt{a^3+c^3}}{a+b} + \frac{ca.\sqrt{a^3+b^3}+ab.\sqrt{a^3+c^3}}{b+c} + \frac{bc.\sqrt{a^3+b^3}+ab.\sqrt{b^3+c^3}}{c+a}$$

$$= \frac{c}{a+b}.\left(a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}\right) + \frac{a}{b+c}.\left(c.\sqrt{a^3+b^3}+b.\sqrt{c^3+a^3}\right)$$

$$+\frac{b}{c+a}.\left(c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}\right)$$

$$=\frac{a}{b+c}.\left(b.\sqrt{c^3+a^3}+c.\sqrt{a^3+b^3}\right)+\frac{b}{c+a}.\left(c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}\right)$$

$$+\frac{c}{a+b}.\left(a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a,y=b,z=z,A=a.\sqrt{b^3+c^3},B=b.\sqrt{c^3+a^3},C=c.\sqrt{a^3+b^3}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(a.\sqrt{b^3+c^3}.b.\sqrt{c^3+a^3}\right)}\overset{A-G}{\geq}\sqrt{3}.\sqrt{3.\sqrt[3]{a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3)}}$$

$$\overset{\text{Cesaro}}{\geq}3.\sqrt[6]{a^2b^2c^2.8a^3b^3c^3}\overset{abc=1}{=}3\sqrt{2}\therefore\frac{a.\sqrt{b^3+c^3}+b.\sqrt{a^3+c^3}}{ab(a+b)}$$

$$+\frac{c.\sqrt{a^3+b^3}+b.\sqrt{a^3+c^3}}{bc(b+c)}+\frac{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}{ac(a+c)}\geq 3\sqrt{2}$$

$$\forall\, a,b,c>0\,\middle|\,abc=1,''=''\text{ iff } a=b=c=1\ (QED)$$

1615. If $x,y,z>0$ then:

$$\sum_{cyc}\frac{(x+y)^5-x^5-y^5}{5((x+y)^3-x^3-y^3)}\leq\sum_{cyc}x^2$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$(x+y)^5-x^5-y^5=$$

$$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5-x^5-y^5$$

$$=5xy(x+y)\left(x^2+xy+y^2\right)$$

$$(x+y)^3-x^3-y^3=3xy(x+y)$$

$$\sum_{cyc}\frac{(x+y)^5-x^5-y^5}{5((x+y)^3-x^3-y^3)}=\sum_{cyc}\frac{5xy(x+y)\left(x^2+xy+y^2\right)}{15xy(x+y)}=$$

$$=\frac{1}{3}\sum_{cyc}(x^2+xy+y^2)\overset{AM-GM}{\leq}\frac{1}{3}\sum_{cyc}\left(x^2+\frac{x^2+y^2}{2}+y^2\right)=$$

$$=\frac{1}{3}\cdot\frac{3}{2}\sum_{cyc}(x^2+y^2)=\frac{1}{2}\cdot 2\sum_{cyc}x^2=\sum_{cyc}x^2$$

1616. **If $a,b,c>0, abc=1$ then:**

$$(a+b+c)^2\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\geq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a+b+c)^2=\sum a^2+2\sum ab\overset{AM-GM}{\geq}$$

$$\geq\sum a^2+6\sqrt[3]{a^2b^2c^2}=\sum a^2+6(as\ abc=1)=(a^2+2)+(b^2+2)+(c^2+2)$$

$$(a+b+c)^2\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\geq$$

$$\geq[\,(a^2+2)+(b^2+2)+(c^2+2)\,]\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\overset{Cauchy-Schwarz}{\geq}9$$

$$(Equality\ for\ a=b=c=1)$$

1617. **If $a,b>0$ then:**

$$4\left(\frac{a^2}{b+1}+\frac{b^2}{a+1}\right)+3\left(\frac{1}{a}+\frac{1}{b}\right)\geq 10$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$4\left(\frac{a^2}{b+1}+\frac{b^2}{a+1}\right)+3\left(\frac{1}{a}+\frac{1}{b}\right)\geq 10\ or,$$

$$\frac{4(a+b)^2}{a+b+2}+\frac{3(1+1)^2}{a+b}\geq 10\ (Bergstrom)\ or,$$

$$\frac{4x^2}{x+2}+\frac{12}{x}\overset{a+b=x>0}{\geq}10\ or,$$

$$2x^3-5x^2-4x+12\geq 0\ or,$$

$$(x-2)^2(2x+3)\geq 0\ true.$$

$$Equality\ for\ \ a=b=1$$

1618. If $a,b,c\geq 0, a+b+c=3$ then:

$$\frac{a}{b^2+c^2+2a+2}+\frac{b}{c^2+a^2+2b+2}+\frac{c}{a^2+b^2+2c++2}\leq\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum_{cyc}\frac{a}{b^2+c^2+2a+2}=\sum_{cyc}\frac{a}{(b^2+1)+(c^2+1)+2a}\leq$$

$$\overset{AM-GM}{\leq}\sum_{cyc}\frac{a}{2a+2b+2c}=\frac{a+b+c}{2(a+b+c)}=\frac{1}{2}$$

$$Equality\ for\ a=\ b=c=1$$

1619. If $a,b,c>0, a+b+c=\frac{1}{2}$ then:

$$\sqrt[4]{ab-2c+5}+\sqrt[4]{ac-2b+5}+\sqrt[4]{bc-2a+5}<\frac{19}{4}$$

Proposed by Samed Ahmedov-Azerbaijan

Solution by Tapas Das-India

$$a+b+c=\frac{1}{2}\Leftrightarrow\ 2c=1-2a-2b$$

$$ab-2c+5=ab-1+2a+2b+5=$$

$$=ab+2a+2b+4=(a+2)(b+2)$$

$$\sqrt[4]{ab-2c+5}=\sqrt[4]{(a+2)(b+2)}=\sqrt[4]{(a+2)(b+2).1.1}\overset{AM-GM}{\leq}$$

$$\leq\frac{a+2+b+2+1+1}{4}=\frac{a+b+6}{4}$$

$$\sqrt[4]{ab-2c+5}+\sqrt[4]{ac-2b+5}+\sqrt[4]{bc-2a+5}<$$

$$<\frac{a+b+6}{4}+\frac{a+c+6}{4}+\frac{b+c+6}{4}=\frac{2(a+b+c)+18}{4}=\frac{19}{4}\left(as\ a+b+c=\frac{1}{2}\right)$$

1620. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{(a+b).\sqrt{a^3+b^3}}{a^2b.\sqrt{b^3+c^3}+b^2a.\sqrt{c^3+a^3}}+\frac{(b+c).\sqrt{b^3+c^3}}{c^2b.\sqrt{a^3+b^3}+b^2c.\sqrt{c^3+a^3}}+\frac{(c+a).\sqrt{c^3+a^3}}{c^2a.\sqrt{a^3+b^3}+a^2c.\sqrt{b^3+c^3}}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\ A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall\ x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have}:\frac{(a+b).\sqrt{a^3+b^3}}{a^2b.\sqrt{b^3+c^3}+b^2a.\sqrt{c^3+a^3}}+\frac{(b+c).\sqrt{b^3+c^3}}{c^2b.\sqrt{a^3+b^3}+b^2c.\sqrt{c^3+a^3}}$$

$$+\frac{(c+a).\sqrt{c^3+a^3}}{c^2a.\sqrt{a^3+b^3}+a^2c.\sqrt{b^3+c^3}}$$

$$=\frac{\left(\frac{a+b}{ab}\right).\sqrt{a^3+b^3}}{a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}}+\frac{\left(\frac{b+c}{bc}\right).\sqrt{b^3+c^3}}{c.\sqrt{a^3+b^3}+b.\sqrt{c^3+a^3}}$$

$$+\frac{\left(\frac{c+a}{ca}\right).\sqrt{c^3+a^3}}{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}\overset{abc=1}{=}\frac{c(a+b).\sqrt{a^3+b^3}}{a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}}$$

$$+\frac{a(b+c).\sqrt{b^3+c^3}}{c.\sqrt{a^3+b^3}+b.\sqrt{c^3+a^3}}+\frac{b(c+a).\sqrt{c^3+a^3}}{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}$$

$$=\frac{a.\sqrt{b^3+c^3}}{b.\sqrt{c^3+a^3}+c.\sqrt{a^3+b^3}}.(b+c)+\frac{b.\sqrt{c^3+a^3}}{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}.(c+a)+$$

$$\frac{c.\sqrt{a^3+b^3}}{a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}}.(a+b)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a.\sqrt{b^3+c^3},y=b.\sqrt{c^3+a^3},z=c.\sqrt{a^3+b^3},A=a,B=b,C=c\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}ab}$$

$$\overset{A-G}{\geq}\sqrt{9.\sqrt[3]{a^2b^2c^2}}\overset{abc=1}{=}3$$

$$\therefore\frac{(a+b).\sqrt{a^3+b^3}}{a^2b.\sqrt{b^3+c^3}+b^2a.\sqrt{c^3+a^3}}+\frac{(b+c).\sqrt{b^3+c^3}}{c^2b.\sqrt{a^3+b^3}+b^2c.\sqrt{c^3+a^3}}$$

$$+\frac{(c+a).\sqrt{c^3+a^3}}{c^2a.\sqrt{a^3+b^3}+a^2c.\sqrt{b^3+c^3}}\geq 3\ \forall\ a,b,c>0\,|\,abc=1,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1621. If $a,b,c>0$, *then prove that* :

$$\sum_{cyc}\sqrt{\frac{a+b}{c}.\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{b(b+c)}+\sqrt{a(c+a)}}}\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\ A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall\ x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \ge \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\ge} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\ge} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\ge} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \sqrt{\frac{a+b}{c}}\cdot\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{b(b+c)}+\sqrt{a(c+a)}} +$$

$$\sqrt{\frac{b+c}{a}}\cdot\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}{\sqrt{b(a+b)}+\sqrt{c(c+a)}} + \sqrt{\frac{a+c}{b}}\cdot\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})}{\sqrt{a(a+b)}+\sqrt{c(b+c)}}$$

$$= \sqrt{\frac{a+b}{c}}\cdot\frac{\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{ab}}}{\sqrt{\frac{b(b+c)}{ab}}+\sqrt{\frac{a(c+a)}{ab}}} + \sqrt{\frac{b+c}{a}}\cdot\frac{\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}{\sqrt{bc}}}{\sqrt{\frac{b(a+b)}{bc}}+\sqrt{\frac{c(c+a)}{bc}}}$$

$$+ \sqrt{\frac{a+c}{b}}\cdot\frac{\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})}{\sqrt{ca}}}{\sqrt{\frac{a(a+b)}{ca}}+\sqrt{\frac{c(b+c)}{ca}}}$$

$$= \frac{\sqrt{\frac{b+c}{a}}}{\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}}}\cdot\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right) + \frac{\sqrt{\frac{c+a}{b}}}{\sqrt{\frac{a+b}{c}}+\sqrt{\frac{b+c}{a}}}\cdot\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)$$

$$+ \frac{\sqrt{\frac{a+b}{c}}}{\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}}\cdot\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)$$

$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{l} x = \sqrt{\frac{b+c}{a}}, y = \sqrt{\frac{c+a}{b}}, z = \sqrt{\frac{a+b}{c}}, \\ A = \frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}, B = \frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}, C = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}} \end{array}\right)$$

$$= \frac{x}{y+z}\cdot\sqrt{B+C}^2 + \frac{y}{z+x}\cdot\sqrt{C+A}^2 + \frac{z}{x+y}\cdot\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\ge}$$

$$\text{4F. } \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\ge} 2\sqrt{\sum_{cyc} AB}\cdot\frac{\sqrt{3}}{2} = \sqrt{3}\cdot\sqrt{\sum_{cyc}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\cdot\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3}.\sqrt{3.\sqrt[3]{\prod_{cyc}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}.\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}}=3.\sqrt[6]{\frac{(\sqrt{b}+\sqrt{c})^2(\sqrt{c}+\sqrt{a})^2(\sqrt{a}+\sqrt{b})^2}{abc}}$$

$$\overset{A-G}{\geq}3.\sqrt[6]{\frac{(4\sqrt{bc})(4\sqrt{ca})(4\sqrt{ab})}{abc}}=3.2=6$$

$$\therefore\sqrt{\frac{a+b}{c}}.\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{b(b+c)}+\sqrt{a(c+a)}}+\sqrt{\frac{b+c}{a}}.\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}{\sqrt{b(a+b)}+\sqrt{c(c+a)}}$$

$$+\sqrt{\frac{a+c}{b}}.\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})}{\sqrt{a(a+b)}+\sqrt{c(b+c)}}\geq 6\ \forall\, a,b,c>0,''=''\ \text{iff}\ a=b=c\ (\text{QED})$$

1622. If $x,y,z>0,x+y+z=1$, then :

$$\prod_{cyc}(x+2y)\leq\frac{2}{3}+3\sum_{cyc}xy^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(x+2y)\leq\frac{2}{3}+3\sum_{cyc}xy^2\overset{\because x+y+z=1}{\Leftrightarrow}$$

$$3\left(2\sum_{cyc}x^2y+4\sum_{cyc}xy^2+9xyz\right)\leq 2\left(\sum_{cyc}x\right)^3+9\sum_{cyc}xy^2$$

$$\Leftrightarrow 3\left(2\sum_{cyc}x^2y+4\sum_{cyc}xy^2+9xyz\right)\leq$$

$$2\left(\sum_{cyc}x^3+3\left(2xyz+\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\right)+9\sum_{cyc}xy^2$$

$$\Leftrightarrow 2\sum_{cyc}x^3+3\sum_{cyc}xy^2\geq 15xyz\to\text{true}\because\sum_{cyc}x^3\overset{A-G}{\geq}3xyz\ \text{and}\ \sum_{cyc}xy^2\overset{A-G}{\geq}3xyz$$

$$\therefore\prod_{cyc}(x+2y)\leq\frac{2}{3}+3\sum_{cyc}xy^2\ \forall\,x,y,z>0\,|\,x+y+z=1,$$

$$''=''\ \text{iff}\ x=y=z=\frac{1}{3}\ (\text{QED})$$

1623. If $a,b>0$ and $ab(a^4+b^4)\ge 2$, then prove that :

$$a^5+b^5\ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall prove that}: \left(a^5+b^5\right)^6\ge 2a^5b^5(a^4+b^4)^5\to(1)$$

$$\text{and}\because a^5+b^5\overset{\text{Chebyshev}}{\ge}\frac{1}{2}(a+b)(a^4+b^4)$$

$$\therefore\left(a^5+b^5\right)^6\ge\frac{1}{32}\left(a^5+b^5\right)(a+b)^5(a^4+b^4)^5$$

$$\overset{\text{Holder}}{\ge}\frac{1}{32}\cdot\frac{1}{16}\cdot(a+b)^5(a+b)^5(a^4+b^4)^5\overset{?}{\ge}2a^5b^5(a^4+b^4)^5$$

$$\Leftrightarrow(a+b)^{10}\overset{?}{\ge}1024a^5b^5\Leftrightarrow(a+b)^2\overset{?}{\ge}4ab\to\text{true via A}-\text{G}\therefore(1)\text{ is true}$$

$$\Rightarrow\left(a^5+b^5\right)^6\overset{ab(a^4+b^4)\ge 2}{\ge}64\Rightarrow a^5+b^5\ge 2$$

$$\forall\, a,b>0\,\big|\,ab(a^4+b^4)\ge 2,''=''\text{ iff }a=b=1\text{ (QED)}$$

1624. If $a,b>0$ and $ab=4$, then prove that :

$$\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}\ge\frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{1}{a},y=\frac{1}{b}\text{ and then}: xy=\frac{1}{4}\text{ and } t=x+y\overset{A-G}{\ge}2\sqrt{xy}=2.\sqrt{\frac{1}{4}}$$

$$\Rightarrow t\ge 1\to(1)\text{ and now},\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}=\frac{1}{\sqrt{\frac{1}{x^3}+1}}+\frac{1}{\sqrt{\frac{1}{y^3}+1}}$$

$$=\frac{x.\sqrt{x}.\sqrt{x}}{\sqrt{x(x+1)}.\sqrt{x^2-x+1}}+\frac{y.\sqrt{y}.\sqrt{y}}{\sqrt{y(y+1)}.\sqrt{y^2-y+1}}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{(x+y)^2}{\sqrt{x^2+y^2+x+y}.\sqrt{x^2+y^2-(x+y)+2}}=\frac{t^2}{\sqrt{t^2-2xy+t}.\sqrt{t^2-2xy-t+2}}$$

$$\overset{xy=\frac{1}{4}}{=} \frac{2t^2}{\sqrt{(2t^2+2t-1)(2t^2-2t+3)}} \overset{?}{\geq} \frac{2}{3} \Leftrightarrow 5t^4-8t+3 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-1)(5t^3+5t^2+5(t-1)+2) \overset{?}{\geq} 0 \to \text{true} \because t \geq 1 \text{ via } (1)$$

$$\therefore \frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{b^3+1}} \geq \frac{2}{3}\ \forall\, a,b > 0 \,|\, ab = 4,'' ='' \text{ iff } a = b = 2 \text{ (QED)}$$

1625. Let $a,b,c \geq 0, ab+bc+ca > 0$. Prove that :

$$\frac{a(b+c)}{ab+ac+2bc} + \frac{b(c+a)}{bc+ba+2ca} + \frac{c(a+b)}{ca+cb+2ab} \leq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2(ab+bc+ca)\sum_{cyc}\frac{a(b+c)}{ab+ac+2bc} = \sum_{cyc}\left(a(b+c) + \frac{a^2(b+c)^2}{ab+ac+2bc}\right) \leq$$

$$\overset{CBS}{\leq} 2\sum_{cyc} bc + \sum_{cyc} a^2\left(\frac{b^2}{ab+bc} + \frac{c^2}{ac+bc}\right) = 2\sum_{cyc} bc + \sum_{cyc}\left(\frac{a^2b}{a+c} + \frac{ca^2}{a+b}\right) =$$

$$= 2\sum_{cyc} bc + \sum_{cyc}\left(\frac{c^2a}{c+b} + \frac{ab^2}{b+c}\right) = 2\sum_{cyc} bc + \sum_{cyc}\frac{a(b^2+c^2)}{b+c} =$$

$$= 2\sum_{cyc} bc + \sum_{cyc}\left(a(b+c) - \frac{2abc}{b+c}\right) \overset{CBS}{\leq} 4\sum_{cyc} bc - 2abc.\frac{9}{\sum_{cyc}(b+c)} =$$

$$= 2\sum_{cyc} bc + \left(2\sum_{cyc} bc - \frac{9abc}{a+b+c}\right) \overset{Schur}{\leq} 2\sum_{cyc} bc + \sum_{cyc} a^2 = (a+b+c)^2,$$

as desired. Equality holds iff $(a=b=c>0)$ and

$(a=0, b=c>0)$ and permutation.

1626. ***If*** $a_k > 0$ $(k = 1,2,3 \dots n)$ ***then :***

$$\sum_{cyclic} \frac{a_1^2}{a_1 + a_2} \ge \frac{1}{2}\sum_{k=1}^{n} a_k \ge \sum_{cyclic} \frac{a_1 a_1^2}{a_1^2 + a_2^2}$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyclic} \frac{a_1^2}{a_1 + a_2} \ge \frac{1}{2}\sum_{k=1}^{n} a_k \ge \sum_{cyclic} \frac{a_1 a_1^2}{a_1^2 + a_2^2}$$

$$\frac{a_1^2}{a_1 + a_2} + \frac{a_2^2}{a_2 + a_3} + \frac{a_3^2}{a_3 + a_4} + \cdots + \frac{a_n^2}{a_n + a_1} \overset{Bergstrom}{\ge}$$

$$\ge \frac{(a_1 + a_2 + a_3 + \cdots a_n)^2}{2(a_1 + a_2 + a_3 + \cdots a_n)} = \frac{1}{2}\sum_{k=1}^{n} a_k$$

$$\sum_{cyclic} \frac{a_1 a_1^2}{a_1^2 + a_2^2} = \sum_{cyc} \frac{a_1}{\left(\frac{a_1}{a_2}\right)^2 + 1} \overset{A-G}{\le} \sum_{cyc} \frac{a_1}{2 \cdot \frac{a_1}{a_2}} = \frac{1}{2}\sum_{cyc} a_2 = \frac{1}{2}\sum_{k=1}^{n} a_k \ (true)$$

1627. If $a, b, c > 0$ **then:**

$$\sum \frac{a^3}{a + b} \ge \frac{1}{2}\sum a^2 \ge \sum \frac{ab^2}{a + b}$$

Proposed by Neculai Stanciu, Mihaly Bencze-Romania

Solution by Tapas Das-India

$$\sum \frac{a^3}{a + b} = \sum \left(a^2 - \frac{a^2 b}{a + b}\right) = \sum a^2 - \sum \frac{a^2 b}{a + b} \overset{AM-HM}{\ge}$$

$$\ge \sum a^2 - \frac{1}{4}\sum \left(\frac{a^2 b}{a} + \frac{a^2 b}{b}\right) = \sum a^2 - \frac{1}{4}\sum ab - \frac{1}{4}\sum a^2 =$$

$$= \frac{3}{4}\sum a^2 - \frac{1}{4}\sum ab \overset{AM-GM}{\ge} \frac{3}{4}\sum a^2 - \frac{1}{4}\sum a^2 = \frac{1}{2}\sum a^2 \ (A)$$

$$\sum \frac{ab^2}{a + b} \overset{AM-HM}{\le} \frac{1}{4}\sum \left(\frac{ab^2}{a} + \frac{ab^2}{b}\right) = \frac{1}{4}\sum b^2 + \frac{1}{4}\sum ab \le$$

$$\le \frac{1}{4}\sum b^2 + \frac{1}{4}\sum b^2 = \frac{1}{2}\sum a^2 \ (B)$$

$$\textit{From } (A) \textit{ and } (B) \textit{we get } \sum \frac{a^3}{a + b} \ge \frac{1}{2}\sum a^2 \ge \sum \frac{ab^2}{a + b}$$

$$\textit{Equality holds for } a = b = c$$

1628. If $a,b,c>0$ then:

$$\sum\frac{a^2+2bc}{(a+b+1)(a+b+c^2)}\le 1$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$(a+b+1)(a+b+c^2)=$$

$$=\left(\left(\sqrt{a+b}\right)^2+(1)^2\right)\left(\left(\sqrt{a+b}\right)^2+(c)^2\right)\overset{C-S}{\ge}(a+b+c)^2\ (1)$$

$$\sum\frac{a^2+2bc}{(a+b+1)(a+b+c^2)}\overset{(1)}{\le}\sum\frac{a^2+2bc}{(a+b+c)^2.}=$$

$$=\frac{a^2+b^2+c^2+2bc+2ca+2ab}{(a+b+c)^2}=\frac{(a+b+c)^2}{(a+b+c)^2}=1$$

$$\textit{Equality holds for } a=b=c=1$$

1629. If $a,\mathrm{b}\in\mathbb{R}$ and $a\mathrm{b}(a^4+a^2\mathrm{b}^2+\mathrm{b}^4)\ge 3$, then prove that :

$$a^2+\mathrm{b}^2\ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

It's clear that $a\mathrm{b}>0$ and so, $\mathrm{t}=\frac{a}{\mathrm{b}}=\frac{a\mathrm{b}}{\mathrm{b}^2}>0$ and now,
we shall prove that : $3(a^2+\mathrm{b}^2)^3\ge 8a\mathrm{b}(a^4+a^2\mathrm{b}^2+\mathrm{b}^4)$

$$\Leftrightarrow 3(\mathrm{t}^2+1)^3\ge 8\mathrm{t}(\mathrm{t}^4+\mathrm{t}^2+1)\Leftrightarrow 3\mathrm{t}^6-8\mathrm{t}^5+9\mathrm{t}^4-8\mathrm{t}^3+9\mathrm{t}^2-8\mathrm{t}+3\ge 0$$

$$\Leftrightarrow\frac{1}{16}(\mathrm{t}-1)^2\left((12\mathrm{t}^2+4\mathrm{t}+9)(2\mathrm{t}-1)^2+39\right)\ge 0\rightarrow \text{true}$$

$$\therefore 3(a^2+\mathrm{b}^2)^3\ge 8a\mathrm{b}(a^4+a^2\mathrm{b}^2+\mathrm{b}^4)\ge 24\Rightarrow(a^2+\mathrm{b}^2)^3\ge 8$$

$$\Rightarrow a^2+\mathrm{b}^2\ge 2,''='' \text{ iff } a=\mathrm{b}=1 \text{ or } a=\mathrm{b}=-1\ (\text{QED})$$

1630. **If $a, b > 0$ then:**

$$8\left(\frac{a^2}{b+1}+\frac{b^2}{a+1}\right)+3\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq 14$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$8\left(\frac{a^2}{b+1}+\frac{b^2}{a+1}\right)+3\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=$$

$$=8\left(\frac{a^2}{b+1}+\frac{b^2}{a+1}\right)+3\left(\frac{1^3}{a^2}+\frac{1^3}{b^2}\right)\overset{Bergstrom\ \&\ Radon}{\geq}$$

$$\geq\frac{8(a+b)^2}{a+b+2}+\frac{3(1+1)^3}{(a+b)^2}\overset{a+b=t>0}{=}\frac{8t^2}{t+2}+\frac{24}{t^2}$$

$$We\ need\ to\ show\ \frac{8t^2}{t+2}+\frac{24}{t^2}\geq 14\ or$$

$$8t^4+24t+48\geq 14t^3+28t^2\ or$$

$$8t^4-14t^3-28t^2+24t+48\geq 0$$

$$or\ (t-2)^2(8t^2+18t+12)\geq 0\ true$$

$$Equality\ holds\ for\ t=a+b=2\ or\ a=b=1$$

1631. **If $a, b, c > 0$ then:**

$$4\left(\frac{a^3}{b+1}+\frac{b^3}{a+1}\right)+5\left(\frac{1}{a}+\frac{1}{b}\right)\geq 14$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$4\left(\frac{a^3}{b+1}+\frac{b^3}{a+1}\right)+5\left(\frac{1}{a}+\frac{1}{b}\right)=$$

$$=4\left(\frac{a^3}{b+1}+\frac{b^3}{a+1}\right)+5\left(\frac{1^2}{a}+\frac{1^2}{b}\right)\overset{Bergstrom\ \&\ Holder}{\geq}$$

$$\geq \frac{4(a+b)^3}{2(a+b+2)} + \frac{5(1+1)^2}{a+b} \overset{a+b=t>0}{=} \frac{2t^3}{t+2} + \frac{20}{t^2}$$

$$We\ \ need\ to\ show \frac{2t^3}{t+2} + \frac{20}{t^2} \geq 14\ or,$$

$$\frac{t^3}{t+2} + \frac{10}{t^2} \geq 7\ \ or\ t^4 + 10t + 20 \geq 7t^2 + 14t\ or,$$

$$t^4 - 7t^2 - 4t + 20 \geq 0\ or\ (t-2)^2(t^2+4t+5) \geq 0, true$$

$$Equality\ holds\ for\ t = a+b = 2\ or\ \ a = b = 1$$

1632. Let $a, b, c \geq 0, ab+bc+ca > 0$ and $a+b+c = 3.$ Prove that :

$$\frac{a}{\sqrt{bc}+3} + \frac{b}{\sqrt{ca}+3} + \frac{c}{\sqrt{ab}+3} \leq \frac{9}{4(ab+bc+ca)}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$4(ab+bc+ca)\sum_{cyc}\frac{a}{\sqrt{bc}+3} = (ab+bc+ca)\sum_{cyc}\frac{4a}{\sqrt{bc}+a+b+c} \leq$$

$$\overset{CBS}{\leq} (ab+bc+ca)\sum_{cyc} a\left(\frac{1}{b+c} + \frac{1}{a+\sqrt{bc}}\right) =$$

$$= \sum_{cyc}\frac{a(ab+bc+ca)}{b+c} + (ab+bc+ca)\sum_{cyc}\frac{a}{a+\sqrt{bc}} =$$

$$= \sum_{cyc} a^2 + \sum_{cyc}\frac{abc}{b+c} + \left(3 - \sum_{cyc}\frac{\sqrt{bc}}{a+\sqrt{bc}}\right).\sum_{cyc} bc \leq$$

$$\overset{\substack{HM-GM \\ CBS}}{\leq} \sum_{cyc} a^2 + \sum_{cyc}\frac{a\sqrt{bc}}{2} + \left(3 - \frac{\left(\sum_{cyc}\sqrt{bc}\right)^2}{\sum_{cyc}\sqrt{bc}\left(a+\sqrt{bc}\right)}\right).\sum_{cyc} bc =$$

$$= \sum_{cyc} a^2 + \frac{1}{2}\sum_{cyc} a\sqrt{bc} + \left(2 - \frac{\sum_{cyc} a\sqrt{bc}}{\sum_{cyc} bc + \sum_{cyc} a\sqrt{bc}}\right).\sum_{cyc} bc \leq$$

$$\leq \sum_{cyc} a^2 + \frac{1}{2}\sum_{cyc} a\sqrt{bc} + \left(2 - \frac{\sum_{cyc} a\sqrt{bc}}{2\sum_{cyc} bc}\right).\sum_{cyc} bc = (a+b+c)^2 = 9,$$

Equality holds iff $(a=b=c>0)$ and $(a=0, b=c>0)$ and permutation.

1633. Let $a,b,c>0.$ Prove that :

$$(6ab+6bc+6ca-a^2-b^2-c^2)\left(\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right)\le\frac{45}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc.$ WLOG, we assume that $p=1$.

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}=\frac{(a^2+b^2+c^2)^2+a^2b^2+b^2c^2+c^2a^2}{(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)-a^2b^2c^2}=$$

$$=\frac{(p^2-2q)^2+q^2-2pr}{(p^2-2q)(q^2-2pr)-r^2}=\frac{(1-2q)^2+q^2-2r}{(1-2q)(q^2-2r)-r^2}=\frac{1-4q+5q^2-2r}{q^2-2q^3-2(1-2q)r-r^2}.$$

The desired inequality is equivalent to

$$(8q-1).\frac{1-4q+5q^2-2r}{q^2-2q^3-2(1-2q)r-r^2}\le\frac{45}{2}$$

$$\Leftrightarrow f(r)=2-24q+119q^2-170q^3-(94-212q)r-45r^2\ge 0.$$

From the identity

$$0\le(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$

It follows

$$r\le\frac{-2p^3+9pq+2\sqrt{(p^2-3q)^3}}{27}=\frac{-2+9q+2\sqrt{(1-3q)^3}}{27}=r_0.$$

Since $3q\le p^2=1$, then $94-212q>0$, and let $x^2=1-3q$.

We have $r_0=\dfrac{1-3x^2+2x^3}{27}$, and

$$f(r)\ge f(r_0)=$$

$$=2-8(1-x^2)+119\left(\frac{1-x^2}{3}\right)^2-170\left(\frac{1-x^2}{3}\right)^3-\left(94-212\left(\frac{1-x^2}{3}\right)\right)r_0-45{r_0}^2$$

$$=\frac{2x^2(32-80x+66x^2-182x^3+245x^4)}{81}=\frac{2x^2(4-7x)^2(2+2x+5x^2)}{81}\geq 0,$$

which completes the proof.

Equality holds iff $\left(x=0\ \Leftrightarrow q=\frac{1}{3}\Leftrightarrow a=b=c\right)$ and

$\left(x=\frac{4}{7}\ \Leftrightarrow q=\frac{11}{49}\Leftrightarrow a=b=\frac{c}{5}\text{ and permutation}\right)$.

1634. **Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that**

$$\frac{2a^2+bc}{b+c}+\frac{2b^2+ca}{c+a}+\frac{2c^2+ab}{a+b}\geq\frac{9}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c,\ \ q:=ab+bc+ca,\ \ r:=abc$. WLOG, we assume that $p=1$.

We have

$$\sum_{cyc}\frac{2a^2+bc}{b+c}=\frac{\sum_{cyc}(2a^2+bc)(a+b)(c+a)}{(a+b)(b+c)(c+a)}=\frac{2\sum_{cyc}a^2(ap+bc)+\sum_{cyc}bc(a^2+q)}{pq-r}$$

$$=\frac{2p(p^3-3pq+3r)+3pr+q^2}{pq-r}=\frac{2-6q+q^2+9r}{q-r}\overset{?}{\geq}\frac{9}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}=\frac{9(1-2q)}{2}$$

$$\Leftrightarrow 4-21q+20q^2+9(3-2q)r\geq 0.\quad (1)$$

We have $q\leq\frac{p^2}{3}=\frac{1}{3}$, and by the fourth degree Schur's inequality, we have

$$r\geq\frac{(4q-p^2)(p^2-q)}{6p}=\frac{(4q-1)(1-q)}{6}.$$

If $0\leq q\leq\frac{1}{4}$, we have: $LHS_{(1)}\geq 4-21q+20q^2=(1-4q)(4-5q)\geq 0$.

If $\frac{1}{4}\leq q\leq\frac{1}{3}$, we have: $LHS_{(1)}\geq 4-21q+20q^2+9(3-2q)\cdot\frac{(4q-1)(1-q)}{6}=$

$$=\frac{1}{2}(4q-1)(1-3q)(1-2q)\geq 0.$$

So the proof is complete. Equality holds iff $(a=b=c)$ and

$(a=0,b=c)$ and permutation.

1635. If $x,y,z>0$, $x^2+y^2+z^2=3$ then:

$$\frac{yz}{4-x^2}+\frac{zx}{4-y^2}+\frac{xy}{4-z^2}\leq 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Tapas Das-India

$$4-x^2\overset{x^2+y^2+z^2=3}{=}=1+x^2+y^2+z^2-x^2=1+y^2+z^2\overset{AM-GM}{\geq}1+2yz\ (1)$$
$$\frac{yz}{4-x^2}+\frac{zx}{4-y^2}+\frac{xy}{4-z^2}=\sum\frac{yz}{4-x^2}\overset{(1)}{\leq}\sum\frac{yz}{1+2yz}=$$
$$=\frac{1}{2}\sum\left(1-\frac{1}{1+2yz}\right)=\frac{3}{2}-\frac{1}{2}\sum\frac{1^2}{1+2yz}\overset{Bergstrom}{\leq}$$
$$\leq\frac{3}{2}-\frac{1}{2}\frac{(1+1+1)^2}{3+2(xy+yz+zx)}=\frac{3}{2}-\frac{1}{2}\frac{9}{3+2\sum xy}\leq\frac{3}{2}-\frac{1}{2}\frac{9}{3+2\sum x^2}=$$
$$=\frac{3}{2}-\frac{1}{2}\cdot\frac{9}{3+2.3}(as\ \ x^2+y^2+z^2=3)=\frac{3}{2}-\frac{1}{2}=1$$
Equality holds for $x=y=z=1$

Solution 2 by Lamiye Quliyeva-Azerbaijan

$$\frac{yz}{4-x^2}+\frac{xz}{4-y^2}+\frac{xy}{4-z^2}\leq\frac{yz}{4-(3-y^2-z^2)}+\frac{xz}{4-(3-x^2-z^2)}+$$
$$+\frac{xy}{4-(3-x^2-y^2)}\leq\frac{yz}{1+y^2+z^2}+\frac{xz}{1+x^2+z^2}+\frac{xy}{1+x^2+y^2}$$
$$\left.\begin{array}{l}\dfrac{yz}{1+y^2+z^2}\leq\dfrac{x^2}{3}\\ \dfrac{xz}{1+x^2+z^2}\leq\dfrac{y^2}{3}\\ \dfrac{xy}{1+x^2+y^2}\leq\dfrac{z^2}{3}\end{array}\right\}\Rightarrow\frac{yz}{1+y^2+z^2}+\frac{xz}{1+x^2+z^2}+\frac{xy}{1+x^2+y^2}\leq\frac{x^2}{3}+\frac{y^2}{3}+\frac{z^2}{3}$$
$$\frac{x^2}{3}+\frac{y^2}{3}+\frac{z^2}{3}\leq\frac{x^2+y^2+z^2}{3}\leq 1$$

$$\frac{yz}{4-x^2}+\frac{xz}{4-y^2}+\frac{xy}{4-z^2}\le 1$$

$$Equality\ holds\ for\ x=y=z=1$$

Solution 3 by Ertan Yildirim-Turkiye

$$\frac{yz}{4-x^2}+\frac{xz}{4-y^2}+\frac{xy}{4-z^2}\le 1$$

$$\frac{2yz}{4-x^2}+\frac{2xz}{4-y^2}+\frac{2xy}{4-z^2}\le 2$$

$$\frac{2yz}{4-x^2}+\frac{2xz}{4-y^2}+\frac{2xy}{4-z^2}\overset{A-G}{\lesseqgtr}\frac{y^2+z^2}{4-x^2}+\frac{x^2+z^2}{4-y^2}+\frac{x^2+y^2}{4-z^2}=$$

$$\frac{3-x^2}{4-x^2}+\frac{3-y^2}{4-y^2}+\frac{3-z^2}{4-z^2}=1-\frac{1}{4-x^2}+1-\frac{1}{4-y^2}+1-\frac{1}{4-z^2}=$$

$$3-\left(\frac{1}{4-x^2}+\frac{1}{4-y^2}+\frac{1}{4-z^2}\right)\overset{Bergstrom}{\lesseqgtr}3-\frac{(1+1+1)^2}{12-(x^2+y^2+z^2)}=$$

$$3-\frac{9}{12-3}=3-\frac{9}{9}=2$$

$$Equality\ \ holds\ \ for\ \ x=y=z=1$$

1636. **If $a,b,c>0$, then:**

$$\frac{6\sqrt{5}}{5}\le\sqrt{\frac{2a+b+c}{a+2b+2c}}+\sqrt{\frac{2b+c+a}{b+2c+2a}}+\sqrt{\frac{2c+a+b}{c+2a+2b}}<2\sqrt{2}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $a+b+c=1$. The desired inequality becomes

$$\frac{6\sqrt{5}}{5}\le\sqrt{\frac{a+1}{2-a}}+\sqrt{\frac{b+1}{2-b}}+\sqrt{\frac{c+1}{2-c}}<2\sqrt{2}$$

We will prove that

$$\frac{27a+31}{20\sqrt{5}}\le\sqrt{\frac{a+1}{2-a}}<\frac{a+1}{\sqrt{2}}. \qquad (1)$$

$$\sqrt{\frac{a+1}{2-a}}=\frac{a+1}{\sqrt{(a+1)(2-a)}}=\frac{a+1}{\sqrt{2+a(1-a)}}\underset{<}{\overset{0<a<1}{\smile}}\frac{a+1}{\sqrt{2}}$$

$$\frac{27a+31}{20\sqrt{5}}\le\sqrt{\frac{a+1}{2-a}}\Leftrightarrow(27a+31)^2(2-a)\le 2000(a+1)$$

$$\Leftrightarrow 729a^3+216a^2-387a+78\ge 0\ \Leftrightarrow\ (3a-1)^2(81a+78)\ge 0,$$

which is true and the proof of (1) is complete. Similarly, we have

$$\frac{27b+31}{20\sqrt{5}}\le\sqrt{\frac{b+1}{2-b}}<\frac{b+1}{\sqrt{2}}\ \text{ and }\ \frac{27c+31}{20\sqrt{5}}\le\sqrt{\frac{c+1}{2-c}}<\frac{c+1}{\sqrt{2}}$$

$$\frac{6\sqrt{5}}{5}=\frac{27(a+b+c)+3.31}{20\sqrt{5}}\le\sqrt{\frac{a+1}{2-a}}+\sqrt{\frac{b+1}{2-b}}+\sqrt{\frac{c+1}{2-c}}<\frac{a+b+c+3}{\sqrt{2}}=2\sqrt{2}$$

1637. **If $a,b,c>0$ and $\forall\, n\in\mathbb{N}$, then prove that:**

$$\frac{a^{2n+1}+a^nc^{n+1}}{b^{2n}a^{n+1}c+c^{2n+1}b^na}+\frac{b^{2n+1}+b^na^{n+1}}{c^{2n}b^{n+1}a+a^{2n+1}c^nb}+\frac{c^{2n+1}+c^nb^{n+1}}{a^{2n}c^{n+1}b+b^{2n+1}a^nc}\ge\frac{3}{(abc)^{\frac{n+1}{3}}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\, A', B', C' > 0, (A'+B'), (B'+C'), (C'+A')$ form sides of a triangle
$(\because (A'+B')+(B'+C')>(C'+A')$ and analogs$)$
$\Rightarrow \sqrt{A'+B'}, \sqrt{B'+C'}, \sqrt{C'+A'}$ form sides of a triangle with area F (say) and $16F^2$

$$=2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$

$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$

$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\rightarrow(1)$$

$$\text{Now, }\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{a^{2n+1} + a^n c^{n+1}}{b^{2n} a^{n+1} c + c^{2n+1} b^n a} + \frac{b^{2n+1} + b^n a^{n+1}}{c^{2n} b^{n+1} a + a^{2n+1} c^n b} + \frac{c^{2n+1} + c^n b^{n+1}}{a^{2n} c^{n+1} b + b^{2n+1} a^n c}$$

$$= \frac{a^n(a^{n+1} + c^{n+1})}{ab^n c(a^n b^n + c^{2n})} + \frac{b^n(b^{n+1} + a^{n+1})}{bc^n a(c^n b^n + a^{2n})} + \frac{c^n(c^{n+1} + b^{n+1})}{ca^n b(a^n c^n + b^{2n})}$$

$$= \frac{a^n\left(\frac{a^{n+1} + c^{n+1}}{a^n c^n}\right)}{ab^n c\left(\frac{b^n}{c^n} + \frac{c^n}{a^n}\right)} + \frac{b^n\left(\frac{b^{n+1} + a^{n+1}}{a^n b^n}\right)}{bc^n a\left(\frac{c^n}{a^n} + \frac{a^n}{b^n}\right)} + \frac{c^n\left(\frac{c^{n+1} + b^{n+1}}{b^n c^n}\right)}{ca^n b\left(\frac{a^n}{b^n} + \frac{b^n}{c^n}\right)}$$

$$= \frac{\frac{a^n}{b^n}}{\frac{b^n}{c^n} + \frac{c^n}{a^n}} \cdot \left(\frac{1}{c^{n+1}} + \frac{1}{a^{n+1}}\right) + \frac{\frac{b^n}{c^n}}{\frac{c^n}{a^n} + \frac{a^n}{b^n}} \cdot \left(\frac{1}{a^{n+1}} + \frac{1}{b^{n+1}}\right) + \frac{\frac{c^n}{a^n}}{\frac{a^n}{b^n} + \frac{b^n}{c^n}} \cdot \left(\frac{1}{b^{n+1}} + \frac{1}{c^{n+1}}\right)$$

$$= \frac{x}{y+z}(B' + C') + \frac{y}{z+x}(C' + A') + \frac{z}{x+y}(A' + B')$$

$$\left(x = \frac{a^n}{b^n}, y = \frac{b^n}{c^n}, z = \frac{c^n}{a^n}, A' = \frac{1}{b^{n+1}}, B' = \frac{1}{c^{n+1}}, C' = \frac{1}{a^{n+1}}\right)$$

$$= \frac{x}{y+z} \cdot \sqrt{B' + C'}^2 + \frac{y}{z+x} \cdot \sqrt{C' + A'}^2 + \frac{z}{x+y} \cdot \sqrt{A' + B'}^2$$

$$\overset{\text{Oppenheim}}{\geq} 4F. \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} A'B'} \cdot \frac{\sqrt{3}}{2}$$

$$= \sqrt{3} \cdot \sqrt{\sum_{cyc} \left(\frac{1}{b^{n+1}} \cdot \frac{1}{c^{n+1}}\right)} \overset{A-G}{\geq} 3 \cdot \sqrt[6]{\frac{1}{(a^2 b^2 c^2)^{n+1}}} = \frac{3}{(abc)^{\frac{n+1}{3}}} \therefore$$

$$\frac{a^{2n+1} + a^n c^{n+1}}{b^{2n} a^{n+1} c + c^{2n+1} b^n a} + \frac{b^{2n+1} + b^n a^{n+1}}{c^{2n} b^{n+1} a + a^{2n+1} c^n b} + \frac{c^{2n+1} + c^n b^{n+1}}{a^{2n} c^{n+1} b + b^{2n+1} a^n c} \geq \frac{3}{(abc)^{\frac{n+1}{3}}}$$

$\forall a, b, c > 0$ and $\forall n \in \mathbb{N}$, " = " iff $a = b = c$ (QED)

1638. If $a, b \geq 1$ and $\lambda \geq 2$, then :

$$\lambda\left(\frac{a^4}{b^2+1} + \frac{b^4}{a^2+1}\right) + (\lambda+1)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 3\lambda + 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}=\frac{a^4-b^4+b^4}{b^2+1}+\frac{b^4-a^4+a^4}{a^2+1}$$

$$=(a^4-b^4)\left(\frac{1}{b^2+1}-\frac{1}{a^2+1}\right)+\frac{b^4-1+1}{b^2+1}+\frac{a^4-1+1}{a^2+1}$$

$$=\frac{(a^4-b^4)(a^2-b^2)}{(a^2+1)(b^2+1)}+b^2-1+\frac{1}{b^2+1}+a^2-1+\frac{1}{a^2+1}$$

$$=\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+a^2-1+b^2-1+\frac{1}{a^2+1}+\frac{1}{b^2+1}$$

$$\Rightarrow\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}-1=$$

$$\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+a^2-1+b^2-1-\left(\frac{1}{2}-\frac{1}{a^2+1}\right)-\left(\frac{1}{2}-\frac{1}{b^2+1}\right)$$

$$=\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+a^2-1+b^2-1-\frac{a^2-1}{2(a^2+1)}-\frac{b^2-1}{2(b^2+1)}$$

$$=\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}$$

$$\therefore \boxed{\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)-\lambda}=\lambda.\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+$$

$$\lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)$$

$$\boxed{\overset{\lambda\geq 2>0}{\geq}\lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)}$$

$$\Rightarrow\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)+(\lambda+1)\left(\frac{1}{a}+\frac{1}{b}\right)-(3\lambda+2)$$

$$=\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)-\lambda-(\lambda+1)\left(2-\frac{1}{a}-\frac{1}{b}\right)\geq$$

$$\lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)-(\lambda+1)\left(\frac{a-1}{a}+\frac{b-1}{b}\right)$$

$$\geq\lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)-\left(\lambda+\frac{\lambda}{2}\right)\left(\frac{a-1}{a}+\frac{b-1}{b}\right)$$

$$\left(\because 1\leq\frac{\lambda}{2}\text{ and }a,b\geq 1\Rightarrow\frac{a-1}{a}+\frac{b-1}{b}\geq 0\right)$$

$$=\frac{\lambda(a-1)}{2}\left(\frac{(2a^2+1)(a+1)}{a^2+1}-\frac{3}{a}\right)+\frac{\lambda(b-1)}{2}\left(\frac{(2b^2+1)(b+1)}{b^2+1}-\frac{3}{b}\right)$$

$$=\frac{\lambda(a-1)(2a^4+2a^3-2a^2+a-3)}{2a(a^2+1)}+\frac{\lambda(b-1)(2b^4+2b^3-2b^2+b-3)}{2b(b^2+1)}$$

$$=\frac{\lambda(a-1)^2(2a^3+4a^2+2a+3)}{2a(a^2+1)}+\frac{\lambda(b-1)^2(2b^3+4b^2+2b+3)}{2b(b^2+1)}\geq 0$$

$$\therefore \lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)-\lambda-(\lambda+1)\left(2-\frac{1}{a}-\frac{1}{b}\right)\geq 0$$

$$\Rightarrow \lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)+(\lambda+1)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 3\lambda+2$$

$$\forall\, a,b\geq 1 \text{ and } \lambda\geq 2,''=''\ \text{ iff } a=b=1\ (\text{QED})$$

1639. If $a,b,c>0$ and, $a+b+c=3,\lambda\geq 0$ then:

$$\sum\frac{a^2}{\sqrt{a+\lambda b}}\geq\frac{3}{\sqrt{\lambda+1}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{a^2}{\sqrt{a+\lambda b}}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{\sum\sqrt{a+\lambda b}}\overset{CBS}{\geq}$$

$$\frac{(a+b+c)^2}{\sqrt{3(a+b+c)(\lambda+1)}}\overset{a+b+c=3}{=}\frac{9}{3}\frac{1}{\sqrt{\lambda+1}}=\frac{3}{\sqrt{\lambda+1}}$$

Equality holds for $a=b=c=1$

1640. If $a,b>0$ then:

$$2\left(\frac{a^3}{(b+1)^2}+\frac{b^3}{(a+1)^2}\right)+\frac{1}{a}+\frac{1}{b}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$2\left(\frac{a^3}{(b+1)^2}+\frac{b^3}{(a+1)^2}\right)+\frac{1}{a}+\frac{1}{b}\overset{Radon\ \&Bergstrom}{\geq}$$

$$\geq\frac{2(a+b)^3}{(a+b+2)^2}+\frac{(1+1)^2}{a+b}\overset{a+b=t>0}{=}\frac{2t^3}{(t+2)^2}+\frac{4}{t}$$

We need to show:

$$\frac{2t^3}{(t+2)^2}+\frac{4}{t}\geq 3\ or\,,$$

$$\frac{2t^4+4t^2+16t+16}{3t^3+12t^2+12t}\geq 3\ or,$$

$$2t^4 - 3t^3 - 8t^2 + 4t + 16 \geq 0 \ or,$$

$$(t-2)^2(2t^2 + 5t + 4) \geq 0 \ true$$

$$Equality\ holds\ for\ t = a + b = 2or\ a = b = 1$$

1641. Let $a, b, c \geq 0, ab + bc + ca > 0$. Prove that

$$\frac{a^2 + bc}{b + c + 2a} + \frac{b^2 + ca}{c + a + 2b} + \frac{c^2 + ab}{a + b + 2c} \geq \frac{a + b + c}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$2(a+b+c)\sum_{cyc}\frac{a^2 + bc}{b + c + 2a} = \sum_{cyc}(a^2 + bc) + \sum_{cyc}\frac{(a^2 + bc)(b + c)}{b + c + 2a}.$$

By CBS inequality, we have

$$\sum_{cyc}\frac{bc(b+c)}{b + c + 2a} \geq \frac{\left(\sum_{cyc} bc\right)^2}{\sum_{cyc} bc\left(1 + \frac{2a}{b+c}\right)} \overset{HM-AM}{\geq} \frac{\left(\sum_{cyc} bc\right)^2}{\sum_{cyc}\left(bc + \frac{a(b+c)}{2}\right)} = \frac{1}{2}\sum_{cyc} bc$$

$$\sum_{cyc}\frac{a^2(b+c)}{b + c + 2a} \geq \frac{(a+b+c)^2}{\sum_{cyc}\left(1 + \frac{2a}{b+c}\right)} \overset{?}{\geq} \frac{1}{2}\sum_{cyc} bc \Leftrightarrow \sum_{cyc} bc \geq \sum_{cyc}\frac{2abc}{b+c} \Leftrightarrow \sum_{cyc}\frac{a(b-c)^2}{b+c} \geq 0$$

Using the results, we get

$$2(a+b+c)\sum_{cyc}\frac{a^2 + bc}{b + c + 2a} \geq \sum_{cyc} a^2 + 2\sum_{cyc} bc = (a+b+c)^2$$

as desired. Equality holds iff $(a = b = c > 0)$ and $(a = b > 0, c = 0)$ and its permutations.

1642. If $a, b > 0, \left(\sqrt{a} + 1\right)\left(\sqrt{b} + 1\right) = 4$ then:

$$\frac{a^3}{(a+1)^2} + \frac{b^3}{(b+1)^2} \geq \frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(\sqrt{a}+1\right)\left(\sqrt{b}+1\right)=4 \text{ or } \sqrt{ab}+\left(\sqrt{a}+\sqrt{b}\right)+1\geq 4 \text{ or}$$

$$\frac{a+b}{2}+\sqrt{2(a+b)}\geq 3\ (Am-Gm\ \&\ CBS)$$

$$u^2+2\sqrt{2}u-6\overset{u=\sqrt{a+b}>0}{\geq} 0 \text{ or } \left(u+3\sqrt{2}\right)\left(u-\sqrt{2}\right)\geq 0 \text{ or}$$

$$u-\sqrt{2}\geq 0\ (as\ \ u>0) \text{ or } u\geq\sqrt{2} \text{ or } u^2\geq 2 \text{ or}, a+b\geq 2\ (1)$$

$$\frac{a^3}{(a+1)^2}+\frac{b^3}{(b+1)^2}\overset{CBS}{\geq}\left(\frac{a^3}{2(a^2+1)}+\frac{b^3}{2(b^2+1)}\right)=$$

$$=\frac{1}{2}\left(a-\frac{a}{a^2+1}+b-\frac{b}{b^2+1}\right)\overset{AM-GM}{\geq}$$

$$\geq\frac{1}{2}\left((a+b)-\left(\frac{a}{2a}+\frac{b}{2b}\right)\right)\overset{(1)}{\geq}\frac{1}{2}\left(2-\left(\frac{1}{2}+\frac{1}{2}\right)\right)=\frac{1}{2}$$

$$Equality\ holds\ \ for\ a=b=1$$

1643. **If $x,y,z>0\ ,x+y+z=3$ then:**

$$\frac{x^5}{x^2+1}+\frac{y^5}{y^2+1}+\frac{z^5}{z^2+1}\geq\frac{3}{2}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$Lemma:\frac{t^5}{t^2+1}\geq 2t-\frac{3}{2}, t\in(0,3)$$

$$Proof:\frac{t^5}{t^2+1}\geq 2t-\frac{3}{2} \text{ or } 2t^5-4t^3+3t^2-4t+3\geq 0$$

$$(t-1)^2(2t^3+4t^2+2t+3)\geq 0\ true\ as\ t<3$$

$$\frac{x^5}{x^2+1}+\frac{y^5}{y^2+1}+\frac{z^5}{z^2+1}=\sum\frac{x^5}{x^2+1}\overset{Lemma}{\geq}$$

$$\geq \sum\left(2x-\frac{3}{2}\right)=2(x+y+z)-\frac{9}{2}=2\times 3-\frac{9}{2}=\frac{3}{2}$$

Equality holds for $x=y=z=1$

1644. If $x,y,z>0$, $xy+yz+zx=3xyz$ then:

$$\frac{1}{x^3+y+z}+\frac{1}{y^3+z+x}+\frac{1}{z^3+x+y}\leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$xy+yz+zx=3xyz$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$$

$$3=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{x+y+z}$$

$$3(x+y+z)\geq 9$$

$$x+y+z\geq 3\ (1)$$

$$x+y+z\geq 3$$

$$y+z\geq 3-x$$

$$x^3+y+z\overset{(1)}{\geq} x^3+3-x=(x^3+1+1)+1-x\overset{AM-GM}{\geq} 3x+1-x=2x+1\ \ (2)$$

$$\frac{1}{x^3+y+z}+\frac{1}{y^3+z+x}+\frac{1}{z^3+x+y}=\sum\frac{1}{x^3+y+z}\overset{(2)}{\leq}$$

$$\leq\sum\frac{1}{2x+1}=\sum\frac{1}{x+x+1}\overset{AM-HM}{\leq}\frac{1}{9}\sum\left(\frac{1}{x}+\frac{1}{x}+1\right)=$$

$$=\frac{1}{9}\left(\sum\frac{1}{x}+\sum\frac{1}{x}+\sum 1\right)\overset{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3}{=}\frac{1}{9}(3+3+3)=1$$

Equality holds for $x=y=z=1$

1645. Let $a,b,c\ge 0, ab+bc+ca>0$ and $a+b+c=3$. Prove that :

$$\frac{13a-4bc}{b+c}+\frac{13b-4ca}{c+a}+\frac{13c-4ab}{a+b}\le\frac{27}{2}\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } p:=a+b+c=3, q:=ab+bc+ca\le\frac{p^2}{3}=3, r:=abc.$$

$$\sum_{cyc}\frac{13a-4bc}{b+c}=\sum_{cyc}\left(\frac{13p-4q}{b+c}-13+4a\right)=(13.3-4q).\frac{p^2+q}{pq-r}-39+12=$$

$$=\frac{(39-4q)(9+q)}{3q-r}-27\overset{?}{\le}\frac{27}{2}.\frac{a^2+b^2+c^2}{ab+bc+ca}=\frac{27(9-2q)}{2q}\Leftrightarrow r\le\frac{q(27-6q+8q^2)}{243}.$$

From the known identity

$$0\le(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$

It follows

$$r\le\frac{-2p^3+9pq+2\sqrt{(p^2-3q)^3}}{27}=\frac{-18+9q+2\sqrt{3(3-q)^3}}{9}.$$

So it suffices to prove that

$$\frac{-18+9q+2\sqrt{3(3-q)^3}}{9}\le\frac{q(27-6q+8q^2)}{243}\Leftrightarrow 27\sqrt{3(3-q)^3}\le(3-q)(81-9q-4q^2)$$

$$\overset{3-q\ge0}{\Leftrightarrow}\ 27^2.3(3-q)\le(81-9q-4q^2)^2\ \Leftrightarrow 0\le q(q+9)(4q-9)^2,$$

which is true and the proof is complete. Equality holds iff $(q=3\Leftrightarrow a=b=c=1)$ and

$$\left(q=\frac{9}{4}\Leftrightarrow a=b=\frac{1}{2},c=2\right)\text{ and its permutations.}$$

1646. If $x,y,z>0, xyz=1$ then:

$$\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}}\ge\frac{3\sqrt{2}}{2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Hai Duong-Vietnam

Let be $f,g:\mathbb{R}_+^* X\mathbb{R}_+^* X\mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$

$$g(x,y,z)=3\sqrt[3]{\frac{xyz}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}},\ f(x,y,z)=\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}}$$

$$f(x,y,z)=\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}}\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{xyz}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}}=g(x,y,z)$$

$$f(x,y,z)\geq g(x,y,z)\ (1)$$

$$\frac{3\sqrt{2}}{2}=\frac{3}{\sqrt{1+\sqrt[3]{xyz}}}=3\sqrt[3]{\frac{1}{\sqrt{\left(1+\sqrt[3]{xyz}\right)^3}}}\overset{HOLDER}{\geq}3\sqrt[3]{\frac{1}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}}=$$

$$=3\sqrt[3]{\frac{xyz}{\sqrt{(y^2+z)(z^2+x)(x^2+y)}}}=g(x,y,z)\Rightarrow Maxg(x,y,z)=\frac{3\sqrt{2}}{2}(2)$$

By $(1),(2)$:

$$\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}}\geq\frac{3\sqrt{2}}{2}$$

Equality holds for $x=y=z=1$

1647. If $x,y,z>0,xyz=1$ then:

$$\frac{x}{(x+2)(y+2)}+\frac{y}{(y+2)(z+2)}+\frac{z}{(z+2)(x+2)}\geq\frac{1}{3}$$

Proposed by Shirvan Tahirov, Gulkhanim Piriyeva-Azerbaijan

Solution by Ertan Yildirim-Turkiye

$$\frac{x}{(x+2)(y+2)}+\frac{y}{(y+2)(z+2)}+\frac{z}{(z+2)(x+2)}\geq\frac{1}{3}\Leftrightarrow$$

$$\frac{x(z+2)+y(x+2)+z(y+2)}{(x+2)(y+2)(z+2)}\geq\frac{1}{3}\Leftrightarrow$$

$$3[xz+xy+yz+2(x+y+z)]\geq(x+2)(y+2)(z+2)\Leftrightarrow$$

$$3(xy+xz+yz)+6(x+y+z)\geq\underbrace{xyz}_{1}+2(xy+xz+yz)+4(x+y+z)+8\Leftrightarrow$$

$$xy+xz+yz+2(x+y+z)\overset{?}{\geq}9$$
$$\frac{1}{z}+\frac{1}{y}+\frac{1}{x}+2(x+y+z)\overset{?}{\geq}9$$
$$\underbrace{\left(x+\frac{1}{x}\right)}_{\geq 2}+\underbrace{\left(y+\frac{1}{y}\right)}_{\geq 2}+\underbrace{\left(x+\frac{1}{x}\right)}_{\geq 2}+x+y+z\geq 6+x+y+z\overset{A-G}{\geq}$$
$$\geq 6+3\cdot\sqrt[3]{xyz}=6+3\cdot 1=9$$

$$Equality\ \ holds\ \ for\ \ x=y=z=1$$

1648. If $x,y,z>0$ then:

$$\frac{x}{\sqrt{x^2+3yz}}+\frac{y}{\sqrt{y^2+3xz}}+\frac{z}{\sqrt{z^2+3xy}}\geq\frac{3}{2}$$

Proposed by Shirvan Tahirov, Gulkhanim Piriyeva-Azerbaijan

Solution by Hai Duong-Vietnam

$$Lemma:\forall\, t>0\ :\ f(t)=\frac{1}{\sqrt{1+3t}}+\frac{3}{16}lnt\geq\frac{1}{2}$$
$$f'(t)=-\frac{3}{2(1+3t)\sqrt{1+3t}}+\frac{3}{16t}=3\cdot\frac{(1+3t)\sqrt{1+3t}-8t}{8(1+3t)\sqrt{1+3t}}$$
$$f'(t)=0\to(1+3t)\sqrt{1+3t}=8t\to t=1$$
$$and\ \ f'(t)>0\to\ t>1\ \ and\ \ f'(t)<0\to t<1$$
$$\forall\, t>0\ :\ f(t)\geq Minf(t)=f(1)=\frac{1}{2}$$
$$\frac{1}{\sqrt{1+3t}}+\frac{3}{16}lnt\geq\frac{1}{2}$$
$$t=\frac{xy}{z^2}\to\frac{1}{\sqrt{1+3\cdot\frac{xy}{z^2}}}+\frac{3}{16}ln\frac{xy}{z^2}\geq\frac{1}{2}\ (1)$$
$$t=\frac{yz}{x^2}\to\frac{1}{\sqrt{1+3\cdot\frac{yz}{x^2}}}+\frac{3}{16}ln\frac{yz}{x^2}\geq\frac{1}{2}\ (2)$$
$$t=\frac{zx}{y^2}\to\frac{1}{\sqrt{1+3\cdot\frac{zx}{y^2}}}+\frac{3}{16}ln\frac{zx}{y^2}\geq\frac{1}{2}\ (3)$$

By adding $(1),(2),(3)$:

$$\frac{1}{\sqrt{1+3\cdot\frac{xy}{z^2}}}+\frac{1}{\sqrt{1+3\cdot\frac{yz}{x^2}}}+\frac{1}{\sqrt{1+3\cdot\frac{zx}{y^2}}}+\frac{3}{16}ln\left(\frac{xy}{z^2}\cdot\frac{yz}{x^2}\cdot\frac{zx}{y^2}\right)\geq\frac{3}{2}$$

$$\frac{x}{\sqrt{x^2+3yz}}+\frac{y}{\sqrt{y^2+3xz}}+\frac{z}{\sqrt{z^2+3xy}}+ln1\geq\frac{3}{2}$$
$$\frac{x}{\sqrt{x^2+3yz}}+\frac{y}{\sqrt{y^2+3xz}}+\frac{z}{\sqrt{z^2+3xy}}\geq\frac{3}{2}$$

1649. If $x,y,z>0\ ,x+y+z=3$ then:

$$\frac{x^5}{x^2+1}+\frac{y^5}{y^2+1}+\frac{z^5}{z^2+1}\geq\frac{3}{2}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$Lemma:\frac{t^5}{t^2+1}\geq 2t-\frac{3}{2},t\in(0,3)$$
$$Proof:\frac{t^5}{t^2+1}\geq 2t-\frac{3}{2}\ or\ 2t^5-4t^3+3t^2-4t+3\geq 0$$

$$(t-1)^2(2t^3+4t^2+2t+3)\geq 0\ true\ as\ t<3$$

$$\frac{x^5}{x^2+1}+\frac{y^5}{y^2+1}+\frac{z^5}{z^2+1}=\sum\frac{x^5}{x^2+1}\overset{Lemma}{\geq}$$

$$\geq\sum\left(2x-\frac{3}{2}\right)=2(x+y+z)-\frac{9}{2}=2\times 3-\frac{9}{2}=\frac{3}{2}$$

$$Equality\ holds\ for\ \ x=y=z=1$$

1650. If $a,b,c>0$ and $a^2+b^2+c^2+2abc=1$, then prove that :

$$\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+1\right)^2\geq 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2+b^2+c^2+2abc=1\Rightarrow a^2+2bc.a+b^2+c^2-1=0$$

$$\Rightarrow a=\frac{-2bc\pm 2\sqrt{b^2c^2-b^2-c^2+1}}{2}=-bc\pm\sqrt{(1-b^2)(1-c^2)}$$

$$(\because 1-b^2 = c^2 + a^2 + 2abc > 0 \text{ and analogously}, 1-c^2 > 0)$$

$$= -bc + \sqrt{(1-b^2)(1-c^2)}\ (\because a > 0)$$

$$= -bc + \frac{1}{3}.\sqrt{9(1-b)(1-c)*(1+b)(1+c)}$$

$$\overset{A-G}{\leq} -bc + \frac{9(1-b)(1-c)+(1+b)(1+c)}{6} = -bc + \frac{10-8(b+c)+10bc}{6}$$

$$= \frac{5-4(b+c)+2bc}{3} \Rightarrow 3a \leq 5-4(b+c)+2bc \text{ and analogs}$$

$$\therefore 3\sum_{cyc} a \leq 15 - 8\sum_{cyc} a + 2\sum_{cyc} ab \Rightarrow 11\sum_{cyc} a \leq 15 + 2\sum_{cyc} ab \leq 15 + \frac{2}{3}\left(\sum_{cyc} a\right)^2$$

$$\Rightarrow 2t^2 - 33t + 45 \geq 0 \left(t = \sum_{cyc} a\right) \Rightarrow (2t-3)(t-15) \geq 0 \Rightarrow t \leq \frac{3}{2}$$

$$\left(\because a,b,c < 1 \Rightarrow \sum_{cyc} a < 3 \Rightarrow t \ngeq 15\right) \therefore \frac{3}{4} \geq \frac{1}{2}\sum_{cyc} a \rightarrow (1)$$

$$\text{Again}, 1 = a^2 + b^2 + c^2 + 2abc \overset{A-G}{\geq} 3p^2 + 2p^3\ (p = \sqrt[3]{abc})$$

$$\Rightarrow (2p-1)(p+1)^2 \leq 0 \Rightarrow p = \sqrt[3]{abc} \leq \frac{1}{2} \Rightarrow 1 - 2abc \geq \frac{3}{4}$$

$$\Rightarrow \sum_{cyc} a^2 \geq \frac{3}{4} \overset{\text{via } (1)}{\geq} \frac{1}{2}\sum_{cyc} a \rightarrow (2)$$

$$\text{We have}: \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + 1\right)^2 \geq 4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1$$

$$\Leftrightarrow \left(\sum_{cyc} a^2 + abc\right)^2 \geq 4\sum_{cyc} a^2b^2 + a^2b^2c^2 \overset{a^2+b^2+c^2+2abc=1}{\Leftrightarrow}$$

$$(1-abc)^2 \geq 4\sum_{cyc} a^2b^2 + a^2b^2c^2 \Leftrightarrow 1 - 2abc \geq 4\sum_{cyc} a^2b^2 \overset{a^2+b^2+c^2+2abc=1}{\Leftrightarrow}$$

$$\sum_{cyc} a^2 \geq 4\sum_{cyc} a^2b^2 \overset{a^2+b^2+c^2+2abc=1}{\Leftrightarrow} \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2 + 2abc\right) \geq 4\sum_{cyc} a^2b^2$$

$$\Leftrightarrow \sum_{cyc} a^4 + 2\sum_{cyc} a^2b^2 + 2abc\left(\sum_{cyc} a^2\right) \geq 4\sum_{cyc} a^2b^2$$

$$\Leftrightarrow \sum_{cyc} a^4 + 2abc\left(\sum_{cyc} a^2\right) \overset{(*)}{\geq} 2\sum_{cyc} a^2b^2$$

$$\text{Now, via ②, } \sum_{cyc} a^4 + 2abc\left(\sum_{cyc} a^2\right) \geq \sum_{cyc} a^4 + abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{Schur}}{\geq} \sum_{cyc} a^3b + \sum_{cyc} ab^3 \overset{A-G}{\geq} 2\sum_{cyc} a^2b^2 \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + 1\right)^2 \geq 4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1$$

$$\forall\, a, b, c > 0 \mid a^2 + b^2 + c^2 + 2abc = 1, \text{"="} \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$

1651. If $a, b, c > 0$ and $(a+b)(b+c)(c+a) = 8$, then prove that :

$$\frac{\sqrt{a^2+ab+b^2}}{\sqrt{ab}+2} + \frac{\sqrt{b^2+bc+c^2}}{\sqrt{bc}+2} + \frac{\sqrt{c^2+ca+a^2}}{\sqrt{ca}} \geq \sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \sqrt{a} = x, \sqrt{b} = y, \sqrt{c} = z \text{ and then : } \sum_{cyc} \frac{\sqrt{b^2+bc+c^2}}{\sqrt{bc}+2} \geq \sum_{cyc} \frac{\frac{\sqrt{3}}{2}(y^2+z^2)}{yz+2}$$

$$= \sqrt{3}.\sum_{cyc} \frac{(y^2+z^2)^2}{2yz(y^2+z^2)+4(y^2+z^2)} \overset{\text{Bergstrom}}{\geq} \sqrt{3}.\frac{\left(\sum_{cyc}(y^2+z^2)\right)^2}{\sum_{cyc}\left(2yz\left(\sum_{cyc} x^2 - x^2\right)\right) + 8\sum_{cyc} x^2}$$

$$\overset{\prod_{cyc}(x^2+y^2)=8}{=} \sqrt{3}.\frac{2\left(\sum_{cyc} x^2\right)^2}{\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) - xyz\sum_{cyc} x + 2\left(\sum_{cyc} x^2\right).\sqrt[3]{\left(\prod_{cyc}(x^2+y^2)\right)}} \overset{?}{\geq} \sqrt{3}$$

$$\Leftrightarrow 2\left(\sum_{cyc} x^2\right)^2 - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) + xyz\sum_{cyc} x \overset{?}{\geq} 2\left(\sum_{cyc} x^2\right).\sqrt[3]{\left(\prod_{cyc}(x^2+y^2)\right)}$$

$$\Leftrightarrow \left(2\left(\sum_{cyc} x^2\right)^2 - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) + xyz\sum_{cyc} x\right)^3$$

$$\overset{?}{\geq} 8\left(\prod_{cyc}(x^2+y^2)\right)\left(\sum_{cyc} x^2\right)^3$$

$$\Leftrightarrow \left(2\left(\sum_{cyc} x^2\right)^2 - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) + xyz\sum_{cyc} x\right)^3$$

$$\overset{?}{\underset{(*)}{\geq}} 8\left(\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x^2y^2\right) - x^2y^2z^2\right)\left(\sum_{cyc} x^2\right)^3$$

Assigning $y+z=X, z+x=Y, x+y=Z \Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ **and** $Z+X-Y=2y>0 \Rightarrow X+Y>Z, Y+Z>X, Z+X>Y \Rightarrow X,Y,Z$ *form* **sides of** *a* **triangle with semiperimeter, circumradius and inradius** $= s, R, r$ (*say*)

yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s-X, y = s-Y, z = s-Z$

$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s$ **and such substitutions** $\Rightarrow \sum_{cyc} xy = \sum_{cyc}(s-X)(s-Y)$

$$\Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2 \text{ and } \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2 \text{ and also,}$$

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=} (4Rr + r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} r^2((4R+r)^2 - 2s^2) \therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet\bullet) \Rightarrow (*) \Leftrightarrow$$

$$(2(s^2-8Rr-2r^2)^2 - (s^2-8Rr-2r^2)(4Rr+r^2) + r^2s^2)^3 \overset{?}{\geq}$$
$$8(r^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2) - r^4s^2)(s^2-8Rr-2r^2)^3$$
$$\Leftrightarrow s^{12} - (54Rr + 10r^2)s^{10} + r^2(1196R^2 + 480Rr + 47r^2)s^8$$
$$-r^3(13960R^3 + 8976R^2r + 1908Rr^2 + 134r^3)s^6$$
$$+r^4(90816R^4 + 82400R^3r + 27996R^2r^2 + 4224Rr^3 + 239r^4)s^4$$
$$-r^5(312832R^5 + 373248R^4r + 178240R^3r^2 + 42592R^2r^3 + 5094Rr^4 + 244r^5)s^2$$
$$+r^6\left(\begin{matrix}446464R^6 + 669696R^5r + 418560R^4r^2 + 139520R^3r^3 + 26160R^2r^4\\ +2616Rr^5 + 109r^6\end{matrix}\right) \overset{?}{\underset{(**)}{\geq}} 0$$

$$\text{Now, } 3132t^3 - 9088t^2 + 9000t - 3037 \left(t = \frac{R}{r}\right)$$

$$= (t-2)(3132t^2 - 2824t + 3352) + 3667 \overset{\text{Euler}}{\geq} 3667 > 0,$$
$$14496t^4 - 57364t^3 + 86922t^2 - 59604t + 15587$$
$$= (t-2)(14496t^3 - 28372t^2 + 30178t + 752) + 17091 \overset{\text{Euler}}{\geq} 17091 > 0 \text{ and}$$
$$8000t^5 - 41552t^4 + 87477t^3 - 92947t^2 + 49798t - 10773$$
$$= (t-2)(t^2(8000t^2 - 25552t + 26272) + 10101t(t-2) + t + 9396) + 8019$$
$$\overset{\text{Euler}}{\geq} 8019 \left(\begin{array}{c}\because \text{discriminant of } (8000t^2 - 25552t + 26272) = \\ -187799296 < 0 \Rightarrow t^2(8000t^2 - 25552t + 26272) > 0\end{array}\right) > 0$$
$$\therefore P = (s^2 - 16Rr + 5r^2)^6 + 16r(21R - 20r)(s^2 - 16Rr + 5r^2)^5$$
$$+16r^2(358R^2 - 685Rr + 336r^2)(s^2 - 16Rr + 5r^2)^4$$
$$+16r^3(3132R^3 - 9088R^2r + 9000Rr^2 - 3037r^3)(s^2 - 16Rr + 5r^2)^3$$
$$+16r^4\left(\begin{array}{c}14496R^4 - 57364R^3r + 86922R^2r^2 \\ -59604Rr^3 + 15587r^4\end{array}\right)(s^2 - 16Rr + 5r^2)^2$$
$$+64r^5\left(\begin{array}{c}8000R^5 - 41552R^4r + 87477R^3r^2 - 92947R^2r^3 \\ +49798Rr^4 - 10773r^5\end{array}\right)(s^2 - 16Rr + 5r^2)$$
$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove : } \boxed{\text{LHS of } (**) \overset{?}{\geq} P}$$
$$\Leftrightarrow 5632t^6 - 40256t^5 + 117000t^4 - 177919t^3$$
$$+150265t^2 - 67251t + 12538 \overset{?}{\geq} 0 \Leftrightarrow$$
$$(t-2)\Big((t-2)\big((t-2)(5632t^3 - 6464t^2 + 10632t + 8497) + 21951\big) + 3645\Big)$$
$$\overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{\sqrt{a^2 + ab + b^2}}{\sqrt{ab} + 2} + \frac{\sqrt{b^2 + bc + c^2}}{\sqrt{bc} + 2} + \frac{\sqrt{c^2 + ca + a^2}}{\sqrt{ca}} \geq \sqrt{3}$$
$$\forall a, b, c > 0 \,\big|\, (a+b)(b+c)(c+a) = 8, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$

1652. If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{a+b}{b^2c^2} \cdot \frac{c^2\sqrt{b+c} + b^2\sqrt{c+a}}{a+b+2c} \geq 3 \cdot \sqrt{\frac{2}{abc}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A', B', C' > 0, (A' + B'), (B' + C'), (C' + A') \text{ form sides of } a \text{ triangle}$$
$$(\because (A' + B') + (B' + C') > (C' + A') \text{ and analogs})$$

$$\Rightarrow \sqrt{A' + B'}, \sqrt{B' + C'}, \sqrt{C' + A'} \text{ form sides of } a \text{ triangle with area F (say) and}$$

$$16F^2=2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$

$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$

$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\rightarrow(1)$$

Now, $\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$

Via Bergstrom, LHS of $(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

We have :

$$\frac{a+b}{b^2c^2}\cdot\frac{c^2.\sqrt{b+c}+b^2.\sqrt{c+a}}{a+b+2c}+\frac{b+c}{c^2a^2}\cdot\frac{c^2.\sqrt{a+b}+a^2.\sqrt{c+a}}{b+c+2a}$$
$$+\frac{c+a}{a^2b^2}\cdot\frac{b^2.\sqrt{a+b}+a^2.\sqrt{b+c}}{c+a+2b}=$$
$$=\frac{a+b}{(b+c)+(c+a)}\cdot\left(\frac{\sqrt{b+c}}{b^2}+\frac{\sqrt{c+a}}{c^2}\right)+\frac{b+c}{(c+a)+(a+b)}\cdot\left(\frac{\sqrt{c+a}}{c^2}+\frac{\sqrt{a+b}}{a^2}\right)$$
$$+\frac{c+a}{(a+b)+(b+c)}\cdot\left(\frac{\sqrt{a+b}}{a^2}+\frac{\sqrt{b+c}}{b^2}\right)$$
$$=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$
$$\left(x=a+b,y=b+c,z=c+a,A'=\frac{\sqrt{a+b}}{a^2},B'=\frac{\sqrt{b+c}}{b^2},C'=\frac{\sqrt{c+a}}{c^2}\right)$$
$$\overset{\text{Oppenheim}}{\geq}4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}$$
$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{\sqrt{a+b}}{a^2}.\frac{\sqrt{b+c}}{b^2}\right)}\overset{A-G}{\geq}3.\sqrt[6]{\frac{(a+b)(b+c)(c+a)}{a^4b^4c^4}}\overset{\text{Cesaro}}{\geq}3.\sqrt[6]{\frac{8abc}{a^4b^4c^4}}$$
$$=3.\sqrt{\frac{2}{abc}}\ \forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$

1653. If $a,b,c>0$ and $a^4+b^4+c^4=3$ then prove that:

$$\frac{1}{4-a}+\frac{1}{4-b}+\frac{1}{4-c}\le 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\frac{1}{4-a}\overset{?}{\le}\frac{a^4+11}{36}\Leftrightarrow 8-11a+4a^4-a^5\overset{?}{\ge}0$$

$$\Leftrightarrow (1-2a+a^2)(8+5a+2a^2-a^3)\overset{?}{\ge}0,$$

which is true for all $a\le 2$. Therefore :

$$\frac{1}{4-a}+\frac{1}{4-b}+\frac{1}{4-c}\le\frac{a^4+b^4+c^4+3.11}{36}=1.$$

Equality holds iff $a=b=c=1$.

1654. If $a,b,c>0$ and $a^2+b^2+c^2+2abc=1$, then prove that :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Moroco

Let $x:=\frac{bc}{a},y:=\frac{ca}{b},z:=\frac{ab}{c}$. The given condition becomes $xy+yz+zx+2xyz=1$ or

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=2,$$ and we need to prove that $x+y+z\ge 2(xy+yz+zx)$.

By CBS inequality, we have :

$$1=\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}\ge\frac{(x+y+z)^2}{x(1+x)+y(1+y)+z(1+z)}$$

then $x+y+z\ge 2(xy+yz+zx)$. Equality holds iff $a=b=c=\frac{1}{2}$.

1655. If $a,b,c>0$ and $a^2+b^2+c^2+2abc=1$, then prove that :

$$\left(\frac{a}{bc}\right)^3+\left(\frac{b}{ca}\right)^3+\left(\frac{c}{ab}\right)^3\ge 2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2+b^2+c^2+2abc=1\Rightarrow a^2+2bc.a+b^2+c^2-1=0$$

$$\Rightarrow a=\frac{-2bc\pm 2\sqrt{b^2c^2-b^2-c^2+1}}{2}=-bc\pm\sqrt{(1-b^2)(1-c^2)}$$

$$(\because 1-b^2=c^2+a^2+2abc>0 \text{ and analogously}, 1-c^2>0)$$

$$=-bc+\sqrt{(1-b^2)(1-c^2)}\ (\because a>0)$$

$$=-bc+\frac{1}{3}.\sqrt{9(1-b)(1-c)*(1+b)(1+c)}$$

$$\overset{A-G}{\le}-bc+\frac{9(1-b)(1-c)+(1+b)(1+c)}{6}=-bc+\frac{10-8(b+c)+10bc}{6}$$

$$=\frac{5-4(b+c)+2bc}{3}\Rightarrow 3a\le 5-4(b+c)+2bc \text{ and analogs}$$

$$\therefore 3\sum_{cyc}a\le 15-8\sum_{cyc}a+2\sum_{cyc}ab\Rightarrow 11\sum_{cyc}a\le 15+2\sum_{cyc}ab\le 15+\frac{2}{3}\left(\sum_{cyc}a\right)^2$$

$$\Rightarrow 2t^2-33t+45\ge 0\ \left(t=\sum_{cyc}a\right)\Rightarrow(2t-3)(t-15)\ge 0\Rightarrow t=\sum_{cyc}a\le\frac{3}{2}$$

$$\left(\because a,b,c<1\Rightarrow\sum_{cyc}a<3\Rightarrow t\ngeq 15\right)\therefore 1\ge\frac{2}{3}\sum_{cyc}a\to(\blacksquare)$$

$$\text{Now, }\left(\frac{a}{bc}\right)^3+\left(\frac{b}{ca}\right)^3+\left(\frac{c}{ab}\right)^3\overset{\text{via }(\blacksquare)}{\ge}\frac{2}{3a^3b^3c^3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^6\right)\overset{?}{\ge}2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

$$=\frac{2}{a^2b^2c^2}\left(\sum_{cyc}a^2b^2\right)\Leftrightarrow\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^6\right)\underset{(*)}{\overset{?}{\ge}}3abc\left(\sum_{cyc}a^2b^2\right)$$

$$\text{But, }\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^6\right)\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^4\right)\ge$$

$$\frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right) \overset{A-G}{\geq} \frac{1}{3}.9abc.\left(\sum_{cyc} a^2b^2\right) \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\frac{a}{bc}\right)^3 + \left(\frac{b}{ca}\right)^3 + \left(\frac{c}{ab}\right)^3 \geq 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$\forall\, a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 + 2abc = 1, '' ='' \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$

1656. **Let $a, b, c, d \geq 0$, prove that :**

$$3(a+b+c+d)^3 + 18(abc+bcd+cda+dab) \geq 11(a+b+c+d)(ab+bc+cd+da+ac+bd)$$

Proposed by Nguyen Van Hoa-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $d = min(a, b, c, d)$.

Let $p := a+b+c, q := ab+bc+ca, r := abc$. The desired inequality becomes :

$$3(p+d)^3 + 18(r+qd) \geq 11(p+d)(q+pd).$$

By Schur's inequality, we have $9r \geq 4pq - p^3$, so it suffices to prove that :

$$p^3 - 2p^2d - 2pd^2 + 3d^3 \geq q(3p - 7d).$$

Since $q \leq \frac{p^2}{3}$ and $p \geq 3d$, so it suffices to prove that :

$$p^3 - 2p^2d - 2pd^2 + 3d^3 \geq \frac{p^2}{3}(3p - 7d) \text{ or } \frac{d}{3}(p - 3d)^2 \geq 0,$$

which is true and the proof is complete.

Equality holds iff $(a = b = c = d)$ and $(a = b = c, d = 0)$ and its permutations.

1657. **Let $a, b, c \geq 0, ab + bc + ca > 0$ and $a + b + c = 3$. Prove that :**

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} + \frac{27}{2} \geq 3(a^2+b^2+c^2+2abc)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequlity, we have :

$$\frac{a^2}{b^2+c^2}+2.\frac{a^2\sqrt{b^2+c^2}}{2\sqrt{2}} \geq 3\sqrt[3]{\frac{a^2}{b^2+c^2}.\left(\frac{a^2\sqrt{b^2+c^2}}{2\sqrt{2}}\right)^2}=\frac{3a^2}{2}.$$

By CBS inequality, we have

$$\sqrt{\frac{b^2+c^2}{2}}+\sqrt{bc} \leq \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)(b^2+c^2+2bc)}=b+c.$$

Then

$$\frac{a^2}{b^2+c^2} \geq \frac{3a^2}{2}-a^2\sqrt{\frac{b^2+c^2}{2}} \geq \frac{3a^2}{2}-a^2\left(b+c-\sqrt{bc}\right)\ \text{(and analogs)}$$

Therefore

$$\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{27}{2} \geq \sum_{cyc}\left(\frac{3a^2}{2}+a^2\sqrt{bc}-a^2(b+c)\right)+\frac{3}{2}(a+b+c)^2=$$

$$=3(a^2+b^2+c^2)+\sum_{cyc}a^2\sqrt{bc}-(a+b+c)(ab+bc+ca)+3abc+3(ab+bc+ca) \geq$$

$$\overset{AM-GM}{\geq} 3(a^2+b^2+c^2)+3abc+3abc=3(a^2+b^2+c^2+2abc).$$

Equality holds iff $a=b=c=1$.

1658. Let $a,b,c \geq 0, a+b+c+abc=4$. Prove that :

$$(ab+bc+ca-5)^2 \geq 3abc+1$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc$. We have 4

$$=p+r \overset{AM-GM}{\leq} p+\frac{p^3}{27}, \text{then } 3 \leq p \leq 4$$

$$\text{and } q \overset{Schur}{\leq} \frac{p^3+9r}{4p} = \frac{p^3+9(4-p)}{4p} = 4 - \frac{(4-p)(p^2+4p-9)}{4p} \leq 4.$$

The desired inequality can be rewritten as $q \leq 5-\sqrt{13-3p} = 5-x$,

where $x = \sqrt{13-3p}$.

Suppose, for the sake of contradiction, that $q > 5-x$.

By the fourth degree Schur's inequality, we have

$$p^4 + 6pr \geq 5p^2q - 4q^2 = f(q).$$

We have $f'(q) = 5p^2 - 8q > 0$, so f is strictly increasing, then

$$p^4 + 6pr \geq f(q) > f(5-x) = 5p^2(5-x) - 4(5-x)^2$$

$$\overset{p=\frac{13-x^2}{3}}{\Leftrightarrow} \left(\frac{13-x^2}{3}\right)^4 + 2(13-x^2)\left(4-\frac{13-x^2}{3}\right) > 5\left(\frac{13-x^2}{3}\right)^2(5-x) - 4(5-x)^2$$

$$\Leftrightarrow -(x-1)^2(2-x)(1033+400x-137x^2-41x^3+4x^4+x^5) > 0,$$

which is not true because $x = \sqrt{13-3p} \in [1,2]$.

Then $q \leq 5-\sqrt{13-3p}$, and the proof is done.

Equality holds iff $(a=b=c=1)$ and $(a=b=2, c=0)$ and its permutations.

1659. Let $a,b,c \geq 0, ab+bc+ca+abc=4$. Prove that :

$$\frac{1}{2a+2b+c} + \frac{1}{2b+2c+a} + \frac{1}{2c+2a+b} \leq \frac{21}{10a+10b+10c+5}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := a+b+c, q := ab+bc+ca, r := abc$. We have $q+r=4$, and

$$\sum_{cyc} \frac{1}{2b+2c+a} = \sum_{cyc} \frac{1}{2p-a} = \frac{\sum_{cyc}(2p-b)(2p-c)}{(2p-a)(2p-b)(2p-c)} = \frac{8p^2+q}{4p^3+2pq-r} =$$

$$\overset{q=4-r}{\Leftrightarrow}\ \frac{8p^2+4-r}{4p^3+8p-(2p+1)r}\overset{?}{\leq}\frac{21}{10p+5}\Leftrightarrow\ r\overset{?}{\leq}\frac{p^3-10p^2+32p-5}{4(2p+1)}=r_0.\ (1)$$

By AM − GM inequality, we have

$$4=q+r\geq 3\sqrt[3]{r^2}+r\ \Rightarrow\ r\leq 1\ \ \&\ \ p\geq\sqrt{3q}=\sqrt{3(4-r)}\geq 3.$$

If $p\geq 7$, **we have :** $\dfrac{p^3-10p^2+32p-5}{4(2p+1)}=1+\dfrac{(p-3)(p^2-7p+3)}{4(2p+1)}\geq 1\geq r.$

If $3\leq p\leq 7$ **: Suppose that** $r>r_0$.

Using the following known identity, we have

$$0\leq(a-b)^2(b-c)^2(c-a)^2=p^2q^2-4q^3+18pqr-4p^3r-27r^2=$$

$$=p^2(4-r)^2-4(4-r)^3+18p(4-r)r-4p^3r-27r^2=$$

$$=16p^2-256-(4p^3+8p^2-72p-192)r-(75+18p-p^2)r^2+4r^3=f(r).$$

We have

$$f''(r)=-2(75+18p-p^2)+24r=-126-24(1-r)-2p(18-p)<0\Rightarrow f'\downarrow$$

$$\Rightarrow f'(r)<f'(r_0)=-4p^3-8p^2+72p+192-2(75+18p-p^2)r_0+12{r_0}^2=$$

$$=-\frac{(p-3)[(p-3)(2343+802p+423p^2+192p^3-7p^4)+7560]}{4(2p+1)^3}\ \overset{3\leq p\leq 7}{\leq}\ 0\ \Rightarrow f\downarrow$$

$$\Rightarrow f(r)<f(r_0)=16p^2-256-(4p^3+8p^2-72p-192)r_0-(75+18p-p^2){r_0}^2+4{r_0}^3=$$

$$=-(p-3)^2(p-4)^2\left(69+43p+137p^2+293p^3+127p^4-3p^5\right)\ \overset{3\leq p\leq 7}{\leq}\ 0,$$

which contradicts $f(r)\geq 0$, **so** (1) **is true, and the proof is complete.**

Equality holds iff $(a=b=c=1)$ **and** $\left(a=b=\dfrac{1}{2},c=3\right)$ **and its permutations.**

1660. If $x,y,z>0\ ,xyz=1$ **then:**

$$\frac{x}{\sqrt{y^2+z}}+\frac{y}{\sqrt{z^2+x}}+\frac{z}{\sqrt{x^2+y}}\geq\frac{3\sqrt{2}}{2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Le Thu-Vietnam

$$\therefore \textit{Lemma} \left(\sum x\right)^3 \ge \frac{27(xyz+\sum x^2y)}{4} \quad \forall\, x,y,z \ge 0$$

$$\textit{Proof}\ : \textit{Let}\ y>0 \overset{WLOG}{=} mid(x,y,z)\ \textit{then by Rearangement inequality}:$$

$$xyz+\sum x^2y = xyz + x.xy + y.yz + z.zx \le$$

$$\le xyz + x.xy + y.xz + z.yz = y(x^2+2xz+z^2) =$$

$$= y(x+z)^2 \overset{AM-GM}{\le} 4.\frac{\left(y+\frac{x+z}{2}+\frac{x+z}{2}\right)^3}{27} = \frac{4\,(\sum x)^3}{27}$$

$$\therefore \textit{We have} \left(\sum \frac{x}{\sqrt{y^2+z}}\right)^2 \overset{Holder}{\ge} \frac{(\sum x)^3}{\sum x(y^2+z)} \overset{?}{\ge} \left(\frac{3}{\sqrt{2}}\right)^2$$

$$\leftrightarrow 2\left(\sum x\right)^3 \overset{?}{\ge} 9\left(\sum xy^2 + \sum xy\right).\ \textit{Using above lemma}$$

$$\textit{it suffices to prove that}\ : 2\left(\sum x\right)^3 \overset{?}{\ge} 27\left(\sum xy - xyz\right)$$

$$\textit{By AM-GM}\ ,\quad 3\sum xy \le \left(\sum x\right)^2\ \textit{since}\ xyz=1$$

$$\textit{we are thus left with}:\ 2\left(\sum x\right)^3 - 9\left(\sum x\right)^2 + 27 \overset{?}{\ge} 0$$

$$\overset{u=\sum x}{\Leftrightarrow} (u-3)^2(2u+3) \ge 0\ \textit{which completes the proof}$$

$$\textit{Equality holds iff}\ x=y=z=1.$$

1661. If $a, b, c > 0$, *then prove that* :

$$\sum_{cyc} \frac{ab^2c^2 + 2b^2c^3 + b^3c^2}{a^2c^2\sqrt{\frac{b+c}{a+b}} + a^2b^2\sqrt{\frac{c+a}{a+b}}} \ge 6\sqrt[3]{abc}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A', B', C' > 0, (A'+B'), (B'+C'), (C'+A')\ \text{form sides of}\ a\ \text{triangle}$$

$$(\because (A'+B')+(B'+C') > (C'+A')\ \textit{and analogs}) \Rightarrow \sqrt{A'+B'}, \sqrt{B'+C'}, \sqrt{C'+A'}$$

$$\text{form sides of}\ a\ \Delta\ \text{with}\ area\ F\ (say)\ \text{and}\ 16F^2 =$$

$$2\sum_{cyc}(A'+B')(B'+C') - \sum_{cyc}(A'+B')^2$$

$$= 2\sum_{cyc}\left(\sum_{cyc} A'B' + B'^2\right) - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B'$$

$$= 6\sum_{cyc} A'B' + 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc} A'B'} \to (1)$$

$$\text{Now}, \forall\, x, y, z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} =$$

$$\frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true}$$

$$\therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{ab^2c^2 + 2b^2c^3 + b^3c^2}{a^2c^2.\sqrt{\frac{b+c}{a+b}} + a^2b^2.\sqrt{\frac{c+a}{a+b}}} + \frac{bc^2a^2 + 2c^2a^3 + c^3a^2}{b^2c^2.\sqrt{\frac{a+b}{b+c}} + a^2b^2.\sqrt{\frac{c+a}{b+c}}}$$

$$+ \frac{cb^2a^2 + 2a^2b^3 + a^3b^2}{b^2c^2.\sqrt{\frac{a+b}{a+c}} + a^2c^2.\sqrt{\frac{b+c}{a+c}}}$$

$$= \frac{a+2c+b}{\frac{a^2}{b^2}\cdot\sqrt{\frac{b+c}{a+b}} + \frac{a^2}{c^2}\cdot\sqrt{\frac{c+a}{a+b}}} + \frac{b+2a+c}{\frac{b^2}{a^2}\cdot\sqrt{\frac{a+b}{b+c}} + \frac{b^2}{c^2}\cdot\sqrt{\frac{c+a}{b+c}}} + \frac{c+2b+a}{\frac{c^2}{a^2}\cdot\sqrt{\frac{a+b}{a+c}} + \frac{c^2}{b^2}\cdot\sqrt{\frac{b+c}{a+c}}}$$

$$= \frac{\frac{\sqrt{a+b}}{a^2}.(b+c+c+a)}{\frac{\sqrt{b+c}}{b^2} + \frac{\sqrt{c+a}}{c^2}} + \frac{\frac{\sqrt{b+c}}{b^2}.(c+a+a+b)}{\frac{\sqrt{c+a}}{c^2} + \frac{\sqrt{a+b}}{a^2}} + \frac{\frac{\sqrt{c+a}}{c^2}.(a+b+b+c)}{\frac{\sqrt{a+b}}{a^2} + \frac{\sqrt{b+c}}{b^2}}$$

$$= \frac{x}{y+z}(B'+C') + \frac{y}{z+x}(C'+A') + \frac{z}{x+y}(A'+B')$$

$$\left(x = \frac{\sqrt{a+b}}{a^2}, y = \frac{\sqrt{b+c}}{b^2}, z = \frac{\sqrt{c+a}}{c^2}, A' = a+b, B' = b+c, C' = c+a\right)$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}((a+b)(b+c))} \overset{A-G}{\geq} 3.\sqrt[6]{(a+b)^2(b+c)^2(c+a)^2} \overset{\text{Cesaro}}{\geq} 3.\sqrt[6]{64a^2b^2c^2}$$

$$=6.\sqrt[3]{abc}\ \forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1662. **If $a,b,c>0$ and $abc=1$, then prove that :**

$$\frac{\sqrt{b}.\sqrt[3]{\frac{a}{b}}+\sqrt{c}.\sqrt[3]{\frac{c}{b}}}{\sqrt[3]{c}.\sqrt[5]{\frac{c}{a}}+\sqrt[3]{a}.\sqrt[5]{\frac{b}{a}}}+\frac{\sqrt{c}.\sqrt[3]{\frac{b}{c}}+\sqrt{a}.\sqrt[3]{\frac{a}{c}}}{\sqrt[3]{a}.\sqrt[5]{\frac{a}{b}}+\sqrt[3]{b}.\sqrt[5]{\frac{c}{b}}}+\frac{\sqrt{a}.\sqrt[3]{\frac{c}{a}}+\sqrt{b}.\sqrt[3]{\frac{b}{a}}}{\sqrt[3]{b}.\sqrt[5]{\frac{b}{c}}+\sqrt[3]{c}.\sqrt[5]{\frac{a}{c}}}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\, A',B',C'>0,(A'+B'),(B'+C'),(C'+A')$ form sides of a triangle
$(\because (A'+B')+(B'+C')>(C'+A')$ and analogs$)\Rightarrow\sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}$
form sides of a Δ with area F (say) and

$$16F^2=2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$

$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$

$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\to(1)$$

$$\text{Now, }\forall\, X,Y,Z>0,\ \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{X^2Y^2}{XY(Y+Z)(Z+X)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}XY\right)^2}{\sum_{cyc}\left(XY\left(\sum_{cyc}XY+Z^2\right)\right)}=\frac{\left(\sum_{cyc}XY\right)^2}{\left(\sum_{cyc}XY\right)^2+XYZ\sum_{cyc}X}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}XY\right)^2\overset{?}{\geq}3XYZ\sum_{cyc}X\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\geq\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}: \frac{\sqrt{b}.\sqrt[3]{\frac{a}{b}}+\sqrt{c}.\sqrt[3]{\frac{c}{b}}}{\sqrt[3]{c}.\sqrt[5]{\frac{c}{a}}+\sqrt[3]{a}.\sqrt[5]{\frac{b}{a}}}+\frac{\sqrt{c}.\sqrt[3]{\frac{b}{c}}+\sqrt{a}.\sqrt[3]{\frac{a}{c}}}{\sqrt[3]{a}.\sqrt[5]{\frac{a}{b}}+\sqrt[3]{b}.\sqrt[5]{\frac{c}{b}}}+\frac{\sqrt{a}.\sqrt[3]{\frac{c}{a}}+\sqrt{b}.\sqrt[3]{\frac{b}{a}}}{\sqrt[3]{b}.\sqrt[5]{\frac{b}{c}}+\sqrt[3]{c}.\sqrt[5]{\frac{a}{c}}}$$

$$=\frac{y^{15}\left(\frac{x}{y}\right)^{10}+z^{15}\left(\frac{z}{y}\right)^{10}}{z^{10}\left(\frac{z}{x}\right)^{6}+x^{10}\left(\frac{y}{x}\right)^{6}}+\frac{z^{15}\left(\frac{y}{z}\right)^{10}+x^{15}\left(\frac{x}{z}\right)^{10}}{x^{10}\left(\frac{x}{y}\right)^{6}+y^{10}\left(\frac{z}{y}\right)^{6}}+\frac{x^{15}\left(\frac{z}{x}\right)^{10}+y^{15}\left(\frac{y}{x}\right)^{10}}{y^{10}\left(\frac{y}{z}\right)^{6}+z^{10}\left(\frac{x}{z}\right)^{6}}$$

$$\left(x=\sqrt[30]{a}, y=\sqrt[30]{b}, z=\sqrt[30]{c}\right)$$

$$=\frac{x^{16}y^{15}+x^{6}z^{25}}{z^{16}y^{10}+x^{10}y^{16}}+\frac{y^{16}z^{15}+y^{6}x^{25}}{x^{16}z^{10}+y^{10}z^{16}}+\frac{z^{16}x^{15}+z^{6}y^{25}}{y^{16}x^{10}+z^{10}x^{16}}$$

$$=\frac{x^{16}y^{15}+x^{6}z^{25}}{z^{6}(yz)^{10}+y^{6}(xy)^{10}}+\frac{y^{16}z^{15}+y^{6}x^{25}}{x^{6}(zx)^{10}+z^{6}(yz)^{10}}+\frac{z^{16}x^{15}+z^{6}y^{25}}{y^{6}(xy)^{10}+x^{6}(zx)^{10}}$$

$$\overset{xyz=1}{=}\frac{x^{16}y^{15}+x^{6}z^{25}}{\frac{z^6}{x^{10}}+\frac{y^6}{z^{10}}}+\frac{y^{16}z^{15}+y^{6}x^{25}}{\frac{x^6}{y^{10}}+\frac{z^6}{x^{10}}}+\frac{z^{16}x^{15}+z^{6}y^{25}}{\frac{y^6}{z^{10}}+\frac{x^6}{y^{10}}}$$

$$=\frac{\frac{x^6}{y^{10}}\left(x^{10}y^{25}+y^{10}z^{25}\right)}{\frac{y^6}{z^{10}}+\frac{z^6}{x^{10}}}+\frac{\frac{y^6}{z^{10}}\left(y^{10}z^{25}+z^{10}x^{25}\right)}{\frac{z^6}{x^{10}}+\frac{x^6}{y^{10}}}+\frac{\frac{z^6}{x^{10}}\left(z^{10}x^{25}+x^{10}y^{25}\right)}{\frac{x^6}{y^{10}}+\frac{y^6}{z^{10}}}$$

$$=\frac{X}{Y+Z}(B'+C')+\frac{Y}{Z+X}(C'+A')+\frac{Z}{X+Y}(A'+B')$$

$$\left(X=\frac{x^6}{y^{10}}, Y=\frac{y^6}{z^{10}}, Z=\frac{z^6}{x^{10}}, A'=z^{10}x^{25}, B'=x^{10}y^{25}, C'=y^{10}z^{25}\right)\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}A'B'.\frac{\sqrt{3}}{2}}=\sqrt{3}.\sqrt{\sum_{cyc}(z^{10}x^{25}.x^{10}y^{25})}$$

$$\overset{xyz=1}{=}\sqrt{3}.\sqrt{\sum_{cyc}x^{25}y^{15}}\overset{A-G}{\geq}3.\sqrt[6]{(xyz)^{25}.(xyz)^{15}}\overset{xyz=1}{=}3$$

$$\therefore\frac{\sqrt{b}.\sqrt[3]{\frac{a}{b}}+\sqrt{c}.\sqrt[3]{\frac{c}{b}}}{\sqrt[3]{c}.\sqrt[5]{\frac{c}{a}}+\sqrt[3]{a}.\sqrt[5]{\frac{b}{a}}}+\frac{\sqrt{c}.\sqrt[3]{\frac{b}{c}}+\sqrt{a}.\sqrt[3]{\frac{a}{c}}}{\sqrt[3]{a}.\sqrt[5]{\frac{a}{b}}+\sqrt[3]{b}.\sqrt[5]{\frac{c}{b}}}+\frac{\sqrt{a}.\sqrt[3]{\frac{c}{a}}+\sqrt{b}.\sqrt[3]{\frac{b}{a}}}{\sqrt[3]{b}.\sqrt[5]{\frac{b}{c}}+\sqrt[3]{c}.\sqrt[5]{\frac{a}{c}}}\geq 3$$

$$\forall a,b,c>0 \mid abc=1, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1663. Let $a,b,c\geq 0, a^2+b^2+c^2=3$. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+abc\geq 4$$

Proposed by Nguyen Van Hoa-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that b is between a and c. By AM − GM inequality, we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+abc$$
$$=\left(\frac{a^2}{b}+a^2b\right)+\left(\frac{b^2}{c}+b^2c\right)+\left(\frac{c^2}{a}+c^2a\right)+abc-(a^2b+b^2c+c^2a)\geq$$
$$\geq 2a^2+2b^2+2c^2+abc-(a^2b+b^2c+c^2a)=6+c(a-b)(b-c)-b(a^2+c^2)\geq$$
$$\geq 6-b(3-b^2)=4+(b+2)(b-1)^2\geq 4,$$

as desired. Equality holds iff $a=b=c=1$.

1664.

For $a,b,c\in\mathbb{R}$ such that $ab+bc+ca\geq 0, a+b+c=3$. Prove that:

$$(2a+b-3c)(2b+c-3a)(2c+a-3b)\leq 42\sqrt{21}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $ab+bc+ca\geq 0$, then we have $18=2(a+b+c)^2$
$$\geq (a-b)^2+(b-c)^2+(c-a)^2$$
so it suffices to prove that:
$$(2a+b-3c)(2b+c-3a)(2c+a-3b)$$
$$\leq 2\sqrt{7.\frac{(a-b)^2+(b-c)^2+(c-a)^2}{6}}^{\,3}\quad (1)$$
WLOG, we assume that $c=min\{a,b,c\}$. Let $x:=a-c\geq 0, y:=b-c\geq 0$.
The inequality (1) becomes:
$$(2x+y)(2y-3x)(x-3y)\leq 2\sqrt{7\cdot\frac{x^2-xy+y^2}{3}}^{\,3}$$
The inequality is true for $y=0$. Now, we assume that $y>0$,
let $t:=\frac{x}{y}$, the inequality becomes
$$(2t+1)(2-3t)(t-3)\leq 2\sqrt{7\cdot\frac{t^2-t+1}{3}}^{\,3}\quad (2)$$
This inequality is true if $LHS_{(2)}<0$. We assume now that $LHS_{(2)}\geq 0$.

After squaring, the inequality is equivalent to

$$(t+4)^2(5t-1)^2(4t-5)^2 \geq 0,$$

which is true and the proof is complete. Equality holds iff

$$(a,b,c) \in \left\{\left(1+\frac{2\sqrt{21}}{7}, 1+\frac{\sqrt{21}}{7}, 1-\frac{3\sqrt{21}}{7}\right), \left(1-\frac{3\sqrt{21}}{7}, 1+\frac{2\sqrt{21}}{7}, 1+\frac{\sqrt{21}}{7}\right), \left(1+\frac{\sqrt{21}}{7}, 1-\frac{3\sqrt{21}}{7}, 1+\frac{2\sqrt{21}}{7}\right)\right\}.$$

1665. Let $a,b,c \geq 0, ab+bc+ca > 0$, then prove that :

$$\frac{\sqrt{b+c}}{a+\sqrt{bc}} + \frac{\sqrt{c+a}}{b+\sqrt{ca}} + \frac{\sqrt{a+b}}{c+\sqrt{ab}} \geq 3\sqrt{\frac{a+b+c}{2(ab+bc+ca)}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\left(\sum_{cyc} \frac{\sqrt{b+c}}{a+\sqrt{bc}}\right)^2 \sum_{cyc} (b+c)^2\left(a+\sqrt{bc}\right)^2 \geq 8(a+b+c)^3,$$

so it suffices to prove that

$$9\sum_{cyc} (b+c)^2\left(a+\sqrt{bc}\right)^2 \leq 16(a+b+c)^2(ab+bc+ca).$$

By AM − GM inequality, we have

$$18\sum_{cyc} (b+c)^2\left(a+\sqrt{bc}\right)^2 = 9\sum_{cyc} (b+c)^2\left(2a^2+2bc+4a\sqrt{bc}\right)$$

$$\leq 9\sum_{cyc} (b+c)^2\left[2a^2+2bc+a\left(b+c+\frac{4bc}{b+c}\right)\right]$$

$$= 9\sum_{cyc} (b+c)[2(a^2+bc)(b+c)+a(b^2+c^2+6bc)] =$$

$$= 27\sum_{cyc} a^3(b+c) + 72\sum_{cyc} b^2c^2 + 162abc\sum_{cyc} a \overset{?}{\leq} 32\left(\sum_{cyc} a\right)^2 \sum_{cyc} bc$$

$$\Leftrightarrow 8\sum_{cyc} b^2c^2 + 2abc\sum_{cyc} a \leq 5\sum_{cyc} a^3(b+c) \Leftrightarrow 5\sum_{cyc} bc(b-c)^2 + \sum_{cyc} a^2(b-c)^2 \geq 0,$$

which is true and the proof is complete. Equality holds if $a=b=c$.

1666. Let $a,b,c\geq 0, ab+bc+ca>0.$ Prove that:

$$\frac{a(b+c)}{ab+ac+4bc}+\frac{b(c+a)}{bc+ba+4ca}+\frac{c(a+b)}{ca+cb+4ab}\leq\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(ab+bc+ca)\sum_{cyc}\frac{a(b+c)}{ab+ac+4bc}=\sum_{cyc}\left(a(b+c)-\frac{3abc(b+c)}{ab+ac+4bc}\right)\leq$$

$$\overset{AM-GM}{\leq}\ 2(ab+bc+ca)-3abc\sum_{cyc}\frac{b+c}{ab+ac+(b+c)^2}=$$

$$=2(ab+bc+ca)-\frac{9abc}{a+b+c}\overset{Schur}{\leq}a^2+b^2+c^2$$

which completes the proof.

Equality holds iff $a=b=c=1$ & $a=b, c=0$ and permutations.

1667. Let $a,b,c\geq 0, a+b+c=4.$ Prove that:

$$\frac{a}{a+4b+2}+\frac{b}{b+4c+2}+\frac{c}{c+4a+2}\leq\frac{2}{3}\qquad(*)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*)\Leftrightarrow\frac{3a}{a+4b+2}+\frac{3b}{b+4c+2}+\frac{3c}{c+4a+2}\leq 2=\frac{a}{6-a}+\frac{3(b+c)}{6-a}$$

$$\Leftrightarrow a\left(\frac{1}{6-a}-\frac{3}{a+4b+2}\right)+3b\left(\frac{1}{6-a}-\frac{1}{b+4c+2}\right)+3c\left(\frac{1}{6-a}-\frac{1}{c+4a+2}\right)\geq 0$$

$$\Leftrightarrow -\frac{4ac}{(6-a)(a+4b+2)}+\frac{9bc}{(6-a)(b+4c+2)}+\frac{3c(4a-b)}{(6-a)(c+4a+2)}\geq 0$$

$$\Leftrightarrow 4ca\left(\frac{3}{c+4a+2}-\frac{1}{a+4b+2}\right)+3bc\left(\frac{3}{b+4c+2}-\frac{1}{c+4a+2}\right)\geq 0$$

$$\Leftrightarrow \frac{52abc}{(c+4a+2)(a+4b+2)}+\frac{52abc}{(b+4c+2)(c+4a+2)}\geq 0,$$

which is true and the proof is complete.

Equality holds if $a=b=2, c=0$ and its permutations.

1668. If $a,b,c>0, a+b+c=3$ and $\lambda\geq\frac{3}{5}$, then :

$$\sum_{cyc}\frac{a^3}{a^2+\lambda b}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3}{a^2+\lambda b}=\sum_{cyc}\frac{a^4}{a^3+\lambda ab}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^3+\lambda\sum_{cyc}ab}\overset{a+b+c=3}{=}$$

$$\frac{\left(\sum_{cyc}a^2\right)^2}{\frac{\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)}{3}+\frac{\lambda\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2}{9}}=\frac{9\left(\sum_{cyc}a^2\right)^2}{3\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)+\lambda\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2}$$

$$\overset{?}{\geq}\frac{3}{\lambda+1}\Leftrightarrow 3\lambda\left(\sum_{cyc}a^2\right)^2+3\left(\sum_{cyc}a^2\right)^2\overset{?}{\geq}3\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

$$+\lambda\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\Leftrightarrow$$

$$\lambda\left(3\left(\sum_{cyc}a^2\right)^2-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\right)+3\left(\sum_{cyc}a^2\right)^2\ \overset{?}{\underset{(*)}{\geq}}\ 3\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

$$\text{Now, }3\left(\sum_{cyc}a^2\right)^2-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\geq$$

$$3\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2=\left(\sum_{cyc}ab\right)\left(3\sum_{cyc}a^2-\left(\sum_{cyc}a\right)^2\right)$$

$$\geq 0 \text{ and } \because \lambda \geq \frac{3}{5} \therefore \text{LHS of } ① \geq$$

$$\frac{3}{5}\cdot\left(3\left(\sum_{cyc}a^2\right)^2-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\right)+3\left(\sum_{cyc}a^2\right)^2\overset{?}{\geq}3\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow 8\left(\sum_{cyc}a^2\right)^2\ \boxed{\overset{?}{\geq}_{(**)}}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2+5\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ **and** $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ **form sides of** a **triangle with semiperimeter, circumradius and inradius** $= s, R, r$ **(say);**

so $2\sum_{cyc}a=\sum_{cyc}x=2s \Rightarrow \sum_{cyc}a=s \to (1) \Rightarrow a=s-x, b=s-y, c=s-z$ **and**

such substitutions $\Rightarrow \sum_{cyc}ab=\sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc}ab=4Rr+r^2 \to (2)$ **and**

$$\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (2)}}{=}s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc}a^2=s^2-8Rr-2r^2 \to (3) \text{ and finally, } \sum_{cyc}a^3=$$

$$\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)\overset{\text{via (1)}}{=}s^3-12Rrs \Rightarrow$$

$$\sum_{cyc}a^3=s(s^2-12Rr) \to (4) \therefore \text{via (1), (2), (3) and (4), } (**) \Leftrightarrow$$

$$8(s^2-8Rr-2r^2)^2\overset{?}{\geq}(4Rr+r^2)s^2+5s^2(s^2-12Rr)$$

$$\Leftrightarrow 3s^4-(72Rr+33r^2)s^2+32r^2(4R+r)^2 \underset{(***)}{\overset{?}{\geq}} 0 \text{ and } \because 3(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***), \text{it suffices to prove :}$$

$$\text{LHS of } (***) \overset{?}{\geq} 3(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (24R-63r)s^2 \underset{(****)}{\overset{?}{\geq}} r(256R^2-736Rr+43r^2)$$

$$\boxed{Case\ 1}\ 24R-63r \geq 0 \text{ and then : LHS of } (****) \overset{\text{Gerretsen}}{\geq}$$

$$(24R-63r)(16Rr-5r^2) \overset{?}{\geq} r(256R^2-736Rr+43r^2)$$

$$\Leftrightarrow 8r^2(16R^2-49Rr+34r^2) \overset{?}{\geq} 0 \Leftrightarrow 8r^2(R-2r)(16R-17r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (****) \text{ is true}$$

$$\boxed{Case\ 2}\ 24R-63r < 0 \text{ and then : LHS of } (****) \overset{\text{Gerretsen}}{\geq}$$

$$(24R-63r)(4R^2+4Rr+3r^2) \overset{?}{\geq} r(256R^2-736Rr+43r^2)$$

$$\Leftrightarrow 24t^3-103t^2+139t-58 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)\big((t-2)(24t-7)+15\big) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \therefore t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \text{ is true and combining both cases, } (****) \Rightarrow$$

$$(***) \text{ is true } \forall \text{ triangle}_{s,R,r} \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{a^3}{a^2+\lambda b} \geq \frac{3}{\lambda+1}$$

$$\forall a,b,c>0 \,\middle|\, a+b+c=3 \wedge \lambda \geq \frac{3}{5}, \text{"="} \text{ iff } a=b=c=1 \text{ (QED)}$$

1669.

If $a,b,c>0$, then prove that :

$$\sum_{cyc} \frac{(b^3+c^3)^3 . \sqrt[3]{\frac{c^3+a^3}{a^3+b^3}} + (c^3+a^3)^3 . \sqrt[3]{\frac{b^3+c^3}{a^3+b^3}}}{\sqrt[3]{b^3+c^3}+\sqrt[3]{c^3+a^3}} \geq \frac{12.\sqrt[6]{48}.a^3b^3c^3}{\sqrt[6]{a^6+b^6+c^6}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A', B', C' > 0, (A'+B'), (B'+C'), (C'+A')$ **form sides of *a* triangle**
$(\because (A'+B')+(B'+C') > (C'+A')$ *and analogs*$) \Rightarrow \sqrt{A'+B'}, \sqrt{B'+C'}, \sqrt{C'+A'}$
form sides of *a* Δ with *area* F (*say*) *and* $16F^2 =$

$$2\sum_{cyc}(A'+B')(B'+C') - \sum_{cyc}(A'+B')^2$$

$$= 2\sum_{cyc}\left(\sum_{cyc}A'B' + B'^2\right) - 2\sum_{cyc}A'^2 - 2\sum_{cyc}A'B'$$

$$= 6\sum_{cyc}A'B' + 2\sum_{cyc}A'^2 - 2\sum_{cyc}A'^2 - 2\sum_{cyc}A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc}A'B'} \to (1)$$

$$\text{Now}, \forall X, Y, Z > 0, \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{X^2Y^2}{XY(Y+Z)(Z+X)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}XY\right)^2}{\sum_{cyc}\left(XY\left(\sum_{cyc}XY + Z^2\right)\right)} = \frac{\left(\sum_{cyc}XY\right)^2}{\left(\sum_{cyc}XY\right)^2 + XYZ\sum_{cyc}X}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc}XY\right)^2 \overset{?}{\geq} 3XYZ\sum_{cyc}X \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \geq \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{(b^3+c^3)^3 . \sqrt[3]{\frac{c^3+a^3}{a^3+b^3}} + (c^3+a^3)^3 . \sqrt[3]{\frac{b^3+c^3}{a^3+b^3}}}{\sqrt[3]{b^3+c^3} + \sqrt[3]{c^3+a^3}} +$$

$$\frac{(a^3+b^3)^3 . \sqrt[3]{\frac{c^3+a^3}{b^3+c^3}} + (c^3+a^3)^3 . \sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}}{\sqrt[3]{a^3+b^3} + \sqrt[3]{c^3+a^3}}$$

$$+ \frac{(a^3+b^3)^3 . \sqrt[3]{\frac{b^3+c^3}{c^3+a^3}} + (b^3+c^3)^3 . \sqrt[3]{\frac{a^3+b^3}{c^3+a^3}}}{\sqrt[3]{a^3+b^3} + \sqrt[3]{b^3+c^3}}$$

$$= \frac{y^9\left(\frac{z}{x}\right) + z^9\left(\frac{y}{x}\right)}{y+z} + \frac{x^9\left(\frac{z}{y}\right) + z^9\left(\frac{x}{y}\right)}{z+x} + \frac{x^9\left(\frac{y}{z}\right) + y^9\left(\frac{x}{z}\right)}{x+y}$$

$$\left(x = \sqrt[3]{a^3+b^3}, y = \sqrt[3]{b^3+c^3}, z = \sqrt[3]{c^3+a^3}\right)$$

$$= \frac{y^9\left(\frac{1}{xy}\right) + z^9\left(\frac{1}{zx}\right)}{\frac{1}{z}+\frac{1}{y}} + \frac{x^9\left(\frac{1}{xy}\right) + z^9\left(\frac{1}{yz}\right)}{\frac{1}{x}+\frac{1}{z}} + \frac{x^9\left(\frac{1}{zx}\right) + y^9\left(\frac{1}{yz}\right)}{\frac{1}{y}+\frac{1}{x}}$$

$$=\frac{\frac{1}{x}\cdot(y^8+z^8)}{\frac{1}{y}+\frac{1}{z}}+\frac{\frac{1}{y}\cdot(z^8+x^8)}{\frac{1}{z}+\frac{1}{x}}+\frac{\frac{1}{z}\cdot(x^8+y^8)}{\frac{1}{x}+\frac{1}{y}}$$

$$=\frac{X}{Y+Z}(B'+C')+\frac{Y}{Z+X}(C'+A')+\frac{Z}{X+Y}(A'+B')$$

$$\left(X=\frac{1}{x},Y=\frac{1}{y},Z=\frac{1}{z},A'=x^8,B'=y^8,C'=z^8\right)\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}(a^3+b^3)^{\frac{8}{3}}(b^3+c^3)^{\frac{8}{3}}}\overset{AM-GM}{\geq}3.\sqrt[6]{(a^3+b^3)^{\frac{16}{3}}(b^3+c^3)^{\frac{16}{3}}(c^3+a^3)^{\frac{16}{3}}}\overset{\text{Cesaro}}{\geq}$$

$$3.\sqrt[6]{(8a^3b^3c^3)^{\frac{16}{3}}}\overset{?}{\geq}\frac{12.\sqrt[6]{48}.a^3b^3c^3}{\sqrt[6]{a^6+b^6+c^6}}\Leftrightarrow 2^{16}(abc)^{16}\left(\sum_{cyc}a^6\right)\overset{?}{\geq}2^{12}.2^4.3(abc)^{18}$$

$$\Leftrightarrow\sum_{cyc}a^6\overset{?}{\geq}3a^2b^2c^2\to\text{true via AM}-\text{GM}\therefore$$

$$\frac{(b^3+c^3)^3.\sqrt[3]{\frac{c^3+a^3}{a^3+b^3}}+(c^3+a^3)^3.\sqrt[3]{\frac{b^3+c^3}{a^3+b^3}}}{\sqrt[3]{b^3+c^3}+\sqrt[3]{c^3+a^3}}$$

$$+\frac{(a^3+b^3)^3.\sqrt[3]{\frac{c^3+a^3}{b^3+c^3}}+(c^3+a^3)^3.\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}}{\sqrt[3]{a^3+b^3}+\sqrt[3]{c^3+a^3}}$$

$$+\frac{(a^3+b^3)^3.\sqrt[3]{\frac{b^3+c^3}{c^3+a^3}}+(b^3+c^3)^3.\sqrt[3]{\frac{a^3+b^3}{c^3+a^3}}}{\sqrt[3]{a^3+b^3}+\sqrt[3]{b^3+c^3}}\geq\frac{12.\sqrt[6]{48}.a^3b^3c^3}{\sqrt[6]{a^6+b^6+c^6}}\forall a,b,c>0,$$

$$''=''\text{ iff } a=b=c\text{ (QED)}$$

1670. If $a,b,c>0\,,a+b+c=3\,,\lambda\geq 0$ then:

$$\sum\frac{ab}{\sqrt{(\lambda+1)a^2-\lambda ab+(\lambda+1)b^2}}\leq\frac{3}{\sqrt{\lambda+2}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$(\lambda+1)a^2-\lambda ab+(\lambda+1)b^2\overset{AM-GM}{\geq}2(\lambda+1)ab-\lambda ab=(\lambda+2)ab\ (1)$$

$$\sum\sqrt{ab}\overset{CBS}{\leq}\sqrt{3(ab+bc+ca)}\leq\sqrt{\frac{3(a+b+c)^2}{3}}=a+b+c=3\ (2)$$

$$\sum\frac{ab}{\sqrt{(\lambda+1)a^2-\lambda ab+(\lambda+1)b^2}}\overset{(1)}{\leq}\sum\frac{ab}{\sqrt{ab(\lambda+2)}}=\frac{1}{\sqrt{\lambda+2}}\sum\sqrt{ab}\overset{(2)}{\leq}\frac{3}{\sqrt{\lambda+2}}$$

Equality holds for a=b=c=1.

1671. If $x,y,z>0$ then:

$$4\sum x+\sum\frac{9}{1+x}\geq 24$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$4\sum x+\sum\frac{9}{1+x}=4\sum(x+1)+\sum\frac{9}{1+x}-12\geq$$

$$\overset{CBS}{\geq}4\sum(x+1)+9\frac{(1+1+1)^2}{\sum(x+1)}-12\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{4\sum(x+1).9\frac{(3)^2}{\sum(x+1)}}-12=2\cdot 18-12=24$$

$$\textit{Equality holds for } x=y=z=\frac{1}{2}.$$

1672. If $a,b,c>0, ab+bc+ca+2abc=1,$ then :

$$\sum_{cyc}\frac{1}{a^5}\geq 2\sum_{cyc}\frac{1}{b^2c^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall first prove the following : }\sum_{cyc}\frac{1}{a^{2n+1}}\geq 2\sum_{cyc}\frac{1}{b^nc^n}$$

$$\forall a,b,c>0\,|\,ab+bc+ca+2abc=1 \text{ and } n\in\mathbb{N}$$

$$\text{We have : }\sum_{cyc}\big((1+b)(1+c)\big)=2(1+a)(1+b)(1+c)\Rightarrow\sum_{cyc}\frac{1}{1+a}=2$$

$$\Rightarrow\sum_{cyc}\frac{1}{1+\frac{1}{x}}=2\left(x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\right)\Rightarrow\sum_{cyc}\frac{x+1-1}{x+1}=2$$

$$\Rightarrow 3-2=1=\sum_{cyc}\frac{1}{1+x}\overset{\text{Bergstrom}}{\geq}\frac{9}{\sum_{cyc}x+3}\Rightarrow\sum_{cyc}x\geq 6\rightarrow(m)\text{ and}$$

$$\sum_{cyc}\frac{1}{a^{2n+1}}\geq 2\sum_{cyc}\frac{1}{b^nc^n}\text{ becomes : }\sum_{cyc}x^{2n+1}\geq 2\sum_{cyc}y^nz^n\rightarrow(*)$$

Case 1 $n = 0$ *and then* : $(*)$ is equivalent to : $\sum_{cyc} x \geq 6$, which is true via (m)

$\Rightarrow (*)$ is true

Case 2 $n \in \mathbb{N}^*$ *and then* : $\sum_{cyc} x^{2n+1} \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^{2n}\right) \overset{\text{via (m)}}{\geq}$

$2\left(\sum_{cyc}(x^n)^2\right) \geq 2\sum_{cyc} y^n z^n \Rightarrow (*)$ is true and hence, combining both cases,

$(*)$ is true $\forall x, y, z > 0 \mid \sum_{cyc} \frac{1}{1+x} = 1$ and $n \in \mathbb{N}$, " = " iff $x = y = z = 2$

$$\therefore \sum_{cyc} \frac{1}{a^{2n+1}} \geq 2\sum_{cyc} \frac{1}{b^n c^n} \ \forall a, b, c > 0 \mid ab + bc + ca + 2abc = 1 \text{ and } n \in \mathbb{N},$$

" = " iff $a = b = c = \frac{1}{2}$ and putting $n = 2$, we get :

$$\sum_{cyc} \frac{1}{a^5} \geq 2\sum_{cyc} \frac{1}{b^2 c^2} \ \forall a, b, c > 0 \mid ab + bc + ca + 2abc = 1,$$

" = " iff $a = b = c = \frac{1}{2}$ (QED)

1673. If $a, b, c > 0, ab + bc + ca + 2abc = 1$ and $n \in \mathbb{N}$ then :

$$\sum_{cyc} \frac{1}{a^{2n+1}} \geq 2\sum_{cyc} \frac{1}{b^n c^n}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

We have : $\sum_{cyc}\left((1+b)(1+c)\right) = 2(1+a)(1+b)(1+c) \Rightarrow \sum_{cyc} \frac{1}{1+a} = 2$

$$\Rightarrow \sum_{cyc} \frac{1}{1+\frac{1}{x}} = 2 \left(x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}\right) \Rightarrow \sum_{cyc} \frac{x+1-1}{x+1} = 2$$

$$\Rightarrow 3 - 2 = 1 = \sum_{cyc} \frac{1}{1+x} \overset{\text{Bergstrom}}{\geq} \frac{9}{\sum_{cyc} x + 3} \Rightarrow \sum_{cyc} x \geq 6 \to (\text{m}) \text{ and}$$

$$\sum_{cyc} \frac{1}{a^{2n+1}} \geq 2\sum_{cyc} \frac{1}{b^n c^n} \text{ becomes : } \sum_{cyc} x^{2n+1} \geq 2\sum_{cyc} y^n z^n \to (*)$$

$\boxed{\text{Case 1}}$ $n = 0$ *and* then : $(*)$ is equivalent to : $\sum_{cyc} x \geq 6$, which is true via (m)

$\Rightarrow (*)$ is true

$\boxed{\text{Case 2}}$ $n \in \mathbb{N}^*$ *and* then : $\sum_{cyc} x^{2n+1} \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^{2n}\right) \overset{\text{via }(m)}{\geq}$

$2\left(\sum_{cyc} (x^n)^2\right) \geq 2\sum_{cyc} y^n z^n \Rightarrow (*)$ is true *and* hence, combining both cases,

$(*)$ is true $\forall\, x, y, z > 0 \left|\, \sum_{cyc} \frac{1}{1+x} = 1\right.$ *and* $n \in \mathbb{N}$, $'' = ''$ iff $x = y = z = 2$

$\therefore \sum_{cyc} \frac{1}{a^{2n+1}} \geq 2\sum_{cyc} \frac{1}{b^n c^n}\ \forall\, a, b, c > 0 \,|\, ab + bc + ca + 2abc = 1$ *and* $n \in \mathbb{N}$,

$'' = ''$ iff $a = b = c = \frac{1}{2}$ (QED)

1674. If $x, y, z > 0, x^3 + y^3 + z^3 + 3xyz = 6$ *and* $\lambda \geq 0$, then :

$$(xy + yz + zx)^3 + \lambda xyz \leq 27 + \lambda$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$6 = \sum_{cyc} x^3 + 3xyz \overset{A-G}{\geq} 6xyz \Rightarrow 1 - xyz \geq 0 \text{ and } \because \lambda \geq 0 \therefore \lambda(1 - xyz) \geq 0$$

$$\Rightarrow \text{it suffices to prove : } 27 \geq \left(\sum_{cyc} xy\right)^3 \Leftrightarrow \frac{27}{36}\left(\sum_{cyc} x^3 + 3xyz\right)^2 \geq \left(\sum_{cyc} xy\right)^3$$

$$\left(\because 6 = \sum_{cyc} x^3 + 3xyz\right) \Leftrightarrow 3\left(\sum_{cyc} x^3 + 3xyz\right)^2 \boxed{\overset{(*)}{\geq}} 4\left(\sum_{cyc} xy\right)^3$$

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of *a* triangle with semiperimeter, circumradius *and* inradius $= s, R, r$ (say)

yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c$

$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2s \text{ and}, \sum_{cyc} xy = \sum_{cyc}(s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2$$

$$\text{and also}, \sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3(x+y)(y+z)(z+x) \overset{\text{via }(\bullet)}{=} s^3 - 3.4Rrs$$

$$\Rightarrow \sum_{cyc} x^3 \overset{(\bullet\bullet\bullet\bullet)}{=} s^3 - 12Rrs$$

Via $(\bullet\bullet),(\bullet\bullet\bullet)$ and $(\bullet\bullet\bullet\bullet),(*)$ becomes : $3(s^3-12Rrs+3r^2s)^2 \geq 4(4Rr+r^2)^3$

$$\Leftrightarrow 3s^2(s^2-12Rr+3r^2)^2 \overset{(**)}{\geq} 4r^3(4R+r)^3$$

Now, $s^2 \overset{\text{Gerretsen}}{\geq} 12Rr + 3r^2 + 4r(R-2r) \overset{\text{Euler}}{\geq} 3r(4R+r)$ and so, in order to prove $(**)$, it suffices to prove : $9(s^2-12Rr+3r^2)^2 \geq 4r^2(4R+r)^2$

$$\Leftrightarrow 3(s^2-12Rr+3r^2) \geq 2r(4R+r) \Leftrightarrow 3s^2 \geq 44Rr - 7r^2 \rightarrow \text{true}$$

$\because 3s^2 \overset{\text{Gerretsen}}{\geq} 44Rr - 7r^2 + 4r(R-2r) \overset{\text{Euler}}{\geq} 44Rr - 7r^2 \Rightarrow (**) \Rightarrow (*)$ is true

$\therefore (xy+yz+zx)^3 + \lambda xyz \leq 27 + \lambda \; \forall \, x,y,z > 0 \,\big|\, x^3+y^3+z^3+3xyz = 6$ and $\lambda \geq 0$, " = " iff $x = y = z = 1$ (QED)

1675. If $a,b,c > 0, a+b+c = 3$ and $\lambda \geq \frac{1}{2}$, then :

$$\lambda \sum_{cyc} a^3 \geq \sum_{cyc} a^2 + 3(\lambda - 1)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda \sum_{cyc} a^3 \geq \sum_{cyc} a^2 + 3(\lambda-1) \Leftrightarrow \lambda\left(\sum_{cyc} a^3 - \frac{3}{27}\left(\sum_{cyc} a\right)^3\right) \geq$$

$$\frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - \frac{3}{27}\left(\sum_{cyc} a\right)^3 \quad \left(\because \sum_{cyc} a = 3\right)$$

$$\Leftrightarrow \lambda\left(9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3\right) \geq 3\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - \left(\sum_{cyc} a\right)^3 \text{ and } \because \lambda \geq \frac{1}{2}$$

$$\therefore \text{it suffices to prove : } 9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3 \geq 6\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - 2\left(\sum_{cyc} a\right)^3$$

$$\Leftrightarrow 4\sum_{cyc} a^3 + 6abc \overset{(*)}{\geq} 3\sum_{cyc} a^2b + 3\sum_{cyc} ab^2$$

$$\text{Now, } 4\sum_{cyc} a^3 + 12abc \underbrace{\overset{Schur}{\geq}}_{①} 4\sum_{cyc} a^2b + 4\sum_{cyc} ab^2 \text{ and}$$

$$\sum_{cyc} a^2b + \sum_{cyc} ab^2 \underbrace{\overset{A-G}{\geq}}_{②} 6abc \text{ and via } ① + ②, (*) \text{ is true}$$

$$\therefore \lambda\sum_{cyc} a^3 \geq \sum_{cyc} a^2 + 3(\lambda-1)\ \forall\, a,b,c>0 \,\middle|\, a+b+c=3 \text{ and } \lambda \geq \frac{1}{2},$$

$$'' =''\ \text{ iff } a=b=c=1\ (QED)$$

1676. If $x,y,z>0, x+y+z=1$ and $\lambda \geq \frac{27}{2}$, then :

$$\sum_{cyc}\frac{x^2}{1-z} + \lambda xyz \leq \frac{1}{2} + \frac{\lambda}{27}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{x^2}{1-z} + \lambda xyz \leq \frac{1}{2} + \frac{\lambda}{27} \overset{x+y+z=1}{\Leftrightarrow} \frac{1}{\sum_{cyc} x}\cdot\sum_{cyc}\frac{x^2}{x+y} - \frac{1}{2} \leq \frac{\lambda}{27} - \frac{\lambda xyz}{\left(\sum_{cyc} x\right)^3}$$

$$\Leftrightarrow \frac{1}{\left(\sum_{cyc} x\right).\prod_{cyc}(x+y)}\cdot\left(\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) + \sum_{cyc} x^2y^2\right) - \frac{1}{2}$$

$$\leq \frac{\lambda}{27\left(\sum_{cyc} x\right)^3}\cdot\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right)$$

$$\Leftrightarrow \frac{\lambda}{27\left(\sum_{cyc} x\right)^2}\cdot\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) \overset{(*)}{\geq}$$

$$\frac{1}{2\prod_{cyc}(x+y)}\cdot\left(2\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} xy\right) + 2\sum_{cyc} x^2y^2 - \left(\sum_{cyc} x\right).\prod_{cyc}(x+y)\right)$$

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0$, $b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a$, $c+a>b \Rightarrow a,b,c$ form sides of a triangle with semiperimeter,

$$\text{circumradius and inradius} = s, R, r\ (say)\ \text{yielding}\ 2\sum_{cyc} x = \sum_{cyc} a = 2s$$

$$\Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c \therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and},$$

$$\sum_{cyc} xy = \sum_{cyc}(s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2 \text{ and},$$

$$\sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=} s^2 - (4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2 \text{ and}, \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x$$

$$\overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=} (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} r^2((4R + r)^2 - 2s^2)$$

$$\because \lambda \geq \frac{27}{2} \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\frac{s^3 - 27r^2s}{2s^2} \geq \frac{2(4Rr + r^2)(s^2 - 8Rr - 2r^2) + 2r^2((4R + r)^2 - 2s^2) - 4Rrs^2}{8Rrs}$$

$$\big(\text{via } (\bullet), (\bullet\bullet), (\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet\bullet)\big) \Leftrightarrow s^2 + 16R^2 - 46Rr + r^2 \geq 0 \to \text{true}$$

$$\because s^2 + 16R^2 - 46Rr + r^2 \overset{\text{Gerretsen}}{\geq} 16R^2 - 30Rr - 4r^2 = 2(8R + r)(R - 2r) \overset{\text{Euler}}{\geq} 0$$

$$\Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{x^2}{1 - z} + \lambda xyz \leq \frac{1}{2} + \frac{\lambda}{27}$$

$$\forall\, x, y, z > 0 \,\big|\, x + y + z = 1 \text{ and } \lambda \geq \frac{27}{2}, ''='' \text{ iff } x = y = z = \frac{1}{3}\ (QED)$$

1677. If $a, b, c > 0$ and $a^2 + b^2 + c^2 = 12$, then prove that:

$$(a^3 + 4a + 8)(b^3 + 4b + 8)(c^3 + 4c + 8) \leq 24^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $a^3 + 4a + 8 = \dfrac{(a^2 + 8)^2 - (a - 2)^2(a^2 - 2a + 4)}{6} \leq \dfrac{(a^2 + 8)^2}{6}, \quad \forall a > 0$, then

$$(a^3 + 4a + 8)(b^3 + 4b + 8)(c^3 + 4c + 8) \leq \frac{1}{6^3}[(a^2 + 8)(b^2 + 8)(c^2 + 8)]^2$$

$$\overset{AM-GM}{\leq}\frac{1}{6^3}\left[\left(\frac{a^2+8+b^2+8+c^2+8}{3}\right)^3\right]^2=24^3$$

Equality holds iff $a=b=c=2$.

1678. If $a,b,c>0, ab+bc+ca+2abc=1$ then:

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq 2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ a=\frac{x}{y+z}, b=\frac{y}{z+x}, c=\frac{z}{x+y}\ then\ ab+bc+ca+2abc=1$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\left(\frac{1}{a}\right)^3+\left(\frac{1}{b}\right)^3+\left(\frac{1}{c}\right)^3\overset{Chebycv}{\geq}\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2\right)\geq$$

$$\geq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

We need to show:

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq 2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$$or\ \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\geq 2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$$or\ \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 2\ or\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 6\ or\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\geq 6\ true$$

$$since,\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}=\left(\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\right)\overset{AM-GM}{\geq}$$

$$\geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}}=6$$

$$Equality\ holds\ for\ x=y=z\ or\ a=b=c=\frac{1}{2}.$$

1679. If $a,b,c>0, ab+bc+ca+2abc=1$ then:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 4(a+b+c)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$ab+bc+ca+2abc=1. Let\ a=\frac{x}{y+z}, b=\frac{y}{z+x}, c=\frac{z}{x+y}$$

We need to show:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 4(a+b+c)\ or\ \frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\geq 4\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)$$

$$4\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)\overset{AM-HM}{\leq} 4\left(\frac{1}{4}\left(\frac{x}{y}+\frac{x}{z}\right)+\frac{1}{4}\left(\frac{y}{z}+\frac{y}{x}\right)+\frac{1}{4}\left(\frac{z}{x}+\frac{z}{y}\right)\right)=$$

$$=\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}$$

$$Equality\ holds\ for\ x=y=z\ or,a=b=c=\frac{1}{2}.$$

1680. If $a,b,c>0$ and $a^2+b^2+c^2+abc=4$, then prove that :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\Leftrightarrow\sum_{cyc}a^2b^2\geq abc\sum_{cyc}a^2$$

$$\overset{a^2+b^2+c^2+abc=4}{\Leftrightarrow} a^2(b^2+c^2)+b^2c^2\geq abc(4-abc)$$

$$\overset{a^2+b^2+c^2+abc=4}{\Leftrightarrow} a^2(4-a^2-abc)+b^2c^2\geq 4abc-a^2b^2c^2$$

$$\Leftrightarrow(1+a^2)b^2c^2-(a^3+4a)bc+4a^2-a^4\overset{(*)}{\geq}0$$

Now, LHS of $(*)$ is a quadratic polynomial in bc with discriminant, $\delta=$

$(a^3+4a)^2-4(1+a^2)(4a^2-a^4)=a^4(5a^2-4)$ and if $a^2\leq\frac{4}{5}$, $(*)$ is

trivially true and so, we now focus on the scenario when : $a^2>\frac{4}{5}$

and in order to prove $(*)$, it suffices to prove : $bc\overset{(**)}{\leq}\frac{a^3+4a-a^2.\sqrt{5a^2-4}}{2(1+a^2)}$

Now, $4-a^2=b^2+c^2+abc\overset{A-G}{\geq}bc(2+a)\Rightarrow bc\leq 2-a\overset{?}{\leq}$

$$\frac{a^3+4a-a^2.\sqrt{5a^2-4}}{2(1+a^2)}\Leftrightarrow 3a^3-4a^2+6a-4\underset{(***)}{\overset{?}{\geq}}a^2.\sqrt{5a^2-4}$$

$$We\ have:3a^3-4a^2+6a-4=\frac{1}{81}\left(\frac{(9a-4)^3}{3}+342\left(a-\frac{454}{513}\right)\right)>0$$

$$\because a > \frac{2}{\sqrt{5}} > \frac{454}{513} > \frac{4}{9} \therefore (***) \Leftrightarrow (3a^3 - 4a^2 + 6a - 4)^2 \overset{?}{\geq} a^4(5a^2 - 4)$$

$$\Leftrightarrow a^6 - 6a^5 + 14a^4 - 18a^3 + 17a^2 - 12a + 4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (a-1)^2(a-2)^2(a^2+1) \overset{?}{\geq} 0 \to \text{true} \therefore (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \; \forall\, a,b,c > 0 \,\Big|\, a^2+b^2+c^2+abc = 4,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1681. If $a,b,c > 0$ and $a^2+b^2+c^2+abc = 4$, then prove that :

$$\left(\frac{2a}{bc} + \frac{2b}{ca} + \frac{2c}{ab} + 1\right)^2 \geq 16\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{2a}{bc} + \frac{2b}{ca} + \frac{2c}{ab} + 1\right)^2 \geq 16\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1 \overset{a^2+b^2+c^2+abc=4}{\Longleftrightarrow}$$

$$\frac{1}{a^2b^2c^2}\left(2\sum_{cyc} a^2 + 4 - \sum_{cyc} a^2\right)^2 \geq \frac{1}{a^2b^2c^2}\left(16\sum_{cyc} a^2b^2 + \left(4 - \sum_{cyc} a^2\right)^2\right)$$

$$\Leftrightarrow 16 + \left(\sum_{cyc} a^2\right)^2 + 8\sum_{cyc} a^2 \geq 16\sum_{cyc} a^2b^2 + 16 + \left(\sum_{cyc} a^2\right)^2 - 8\sum_{cyc} a^2$$

$$\Leftrightarrow a^2+b^2+c^2 \geq a^2(b^2+c^2) + b^2c^2 \overset{a^2+b^2+c^2+abc=4}{\Longleftrightarrow}$$

$$a^2 - b^2c^2 + (1-a^2)(4-a^2-abc) \geq 0$$

$$\Leftrightarrow a^2 + (1-a^2)(4-a^2) \overset{(*)}{\geq} b^2c^2 + a(1-a^2)bc$$

$$\text{Now}, 4 - a^2 = b^2+c^2+abc \overset{A-G}{\geq} bc(2+a) \Rightarrow bc \leq 2 - a \Rightarrow \text{LHS of } (*) \leq$$

$$(2-a)^2 + a(1-a^2)(2-a) \overset{?}{\leq} a^2 + (1-a^2)(4-a^2) \Leftrightarrow 2a(a^2-2a+1) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2a(a-1)^2 \overset{?}{\geq} 0 \to \text{true} \therefore (*) \text{ is true} \Rightarrow$$

$$\left(\frac{2a}{bc} + \frac{2b}{ca} + \frac{2c}{ab} + 1\right)^2 \geq 16\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1$$

$$\forall\, a,b,c > 0 \,\Big|\, a^2+b^2+c^2+abc = 4, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1682. If $a, b, c > 0$ and $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ then prove that :

$$a + b + c \geq \frac{3}{a + b + c} + \frac{2}{abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a + b + c \geq \frac{3}{a + b + c} + \frac{2}{abc} \Leftrightarrow \frac{(\sum_{cyc} a)(\sum_{cyc} ab)}{abc \sum_{cyc} a} \geq$$

$$\frac{3}{\sum_{cyc} a} + \frac{2}{abc} \cdot \frac{abc \sum_{cyc} a}{\sum_{cyc} ab} \left(\because 1 = \frac{\sum_{cyc} ab}{abc \sum_{cyc} a}\right)$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)^2 \overset{(*)}{\geq} 3abc \sum_{cyc} ab + 2abc\left(\sum_{cyc} a\right)^2$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s - x)(s - y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3) \text{ and via } (1), (2) \text{ and } (3), (*) \Leftrightarrow$$

$$s(4Rr + r^2)^2 \geq 3r^2 s(4Rr + r^2) + 2r^2 s^3 \Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2 \to \text{true}$$

$$\because s^2 \overset{\text{Gerretsen}}{\leq} 4R^2 + 4Rr + 3r^2 = 8R^2 - 2Rr - r^2 - 2(R - 2r)(2R + r) \overset{\text{Euler}}{\leq}$$

$$8R^2 - 2Rr - r^2 \Rightarrow (*) \text{ is true } \therefore a + b + c \geq \frac{3}{a + b + c} + \frac{2}{abc}$$

$$\forall\, a, b, c > 0 \,\middle|\, a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1683. If $a, b, c > 0$ and $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ then prove that :

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \le 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \le 1 \Leftrightarrow \sum_{cyc}\left((b^2+2)(c^2+2)\right)$$

$$\le (a^2+2)(b^2+2)(c^2+2) \Leftrightarrow a^2b^2c^2 + \sum_{cyc} a^2b^2 \ge 4$$

$$\Leftrightarrow \frac{\sum_{cyc} ab}{abc\sum_{cyc} a}.a^2b^2c^2 + \sum_{cyc} a^2b^2 \ge 4\left(\frac{abc\sum_{cyc} a}{\sum_{cyc} ab}\right)^2 \left(\because 1 = \frac{\sum_{cyc} ab}{abc\sum_{cyc} a}\right)$$

$$\Leftrightarrow \left(abc\sum_{cyc} ab + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right)\right)\left(\sum_{cyc} ab\right)^2 \overset{(*)}{\ge} 4a^2b^2c^2\left(\sum_{cyc} a\right)^3$$

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow$ x, y, z form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$\therefore abc = r^2s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3) \text{ and } \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr+r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \rightarrow (4)$$

and via $(1), (2), (3)$ and $(4), (*) \Leftrightarrow$

$$\left(r^2s(4Rr+r^2) + sr^2((4R+r)^2 - 2s^2)\right)\left((4Rr+r^2)\right)^2 \ge 4r^4s^5$$

$$\Leftrightarrow ((4R+r)^2 - 2s^2)(4R+r)^2 + r(4R+r)^3 \overset{(**)}{\ge} 4s^4$$

Now, via Doucet (or Trucht) and via Gerretsen, LHS of $(**)$ $-$ RHS of $(**) \ge$ $s^2(16R^2+8Rr+r^2) + r(4R+r)^3 - 4s^2(4R^2+4Rr+3r^2)$

$$= r(4R+r)^3 - s^2(8Rr+11r^2) \overset{\text{Gerretsen}}{\ge}$$

$$r(4R+r)^3-(4R^2+4Rr+3r^2)(8Rr+11r^2)=4r(R-2r)(8R^2+9Rr+4r^2)$$
$$\overset{\text{Euler}}{\geq} 0 \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \leq 1$$
$$\forall a,b,c>0 \,\Big|\, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \text{"}=\text{" iff } a=b=c=1 \text{ (QED)}$$

1684. If $x,y>0, x+y=8$ then:

$$\frac{\sqrt{x}+\sqrt{y}}{8+\sqrt{xy}} \geq \frac{1}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x+y=8 \text{ or }, 2\sqrt{xy} \overset{AM-GM}{\leq} 8 \text{ or}, \sqrt{xy} \leq 4 \ (1)$$
We need to show:
$$\frac{\sqrt{x}+\sqrt{y}}{8+\sqrt{xy}} \geq \frac{1}{3} \text{ or } 3(\sqrt{x}+\sqrt{y}) \geq 8+\sqrt{xy} \text{ or},$$
$$9(x+y+2\sqrt{xy}) \overset{Squaring}{\geq} 64+xy+16\sqrt{xy}$$
$$\text{or } 72+18\sqrt{xy} \overset{x+y=8}{\geq} 64+xy+16\sqrt{xy}$$
$$xy-2\sqrt{xy}-8 \leq 0 \text{ or } (\sqrt{xy}-4)(\sqrt{xy}+2) \leq 0$$
$$\text{True as } \sqrt{xy} \leq 4 \ (\text{from } (1))$$
Equality holds for x=y=4.

1685. If $a,b,c>0$ and $ab+bc+ca+2abc=1$, then prove that :

$$a+b+c \geq 2(ab+bc+ca)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$ab+bc+ca+2abc=1 \Rightarrow a(b+c+2bc)=1-bc \Rightarrow b+c$$
$$=\frac{1-bc}{a}-2bc \text{ and so}, a+b+c \geq 2(ab+bc+ca) \text{ becomes :}$$
$$a+\frac{1-bc}{a}-2bc \geq 2(1-2abc) \left(\because \sum_{cyc} ab = 1-2abc\right)$$
$$\Leftrightarrow a^2+1-bc-2abc \geq 2a-4a^2bc \Leftrightarrow (1+4bc)a^2-2(1+bc)a+1-bc \overset{(*)}{\geq} 0$$
Now, LHS of $(*)$ is a quadratic polynomial in a with discriminant, $\delta =$

$$4(1+bc)^2-4(1+4bc)(1-bc)=4(5b^2c^2-bc)\ \text{and if}\ bc\le\frac{1}{5},$$

$(*)$ is trivially true and so, we now focus on the scenario when : $bc>\frac{1}{5}$

and in order to prove $(*)$, it suffices to prove : $a\overset{(**)}{\le}\frac{1+bc-\sqrt{5b^2c^2-bc}}{1+4bc}$

$$\text{Now, }1-bc=a(b+c)+2abc\overset{A-G}{\ge}2a.\sqrt{bc}+2abc=2a.\sqrt{bc}\left(1+\sqrt{bc}\right)$$

$$\Rightarrow\left(1+\sqrt{bc}\right)\left(1-\sqrt{bc}\right)\ge 2a.\sqrt{bc}\left(1+\sqrt{bc}\right)\Rightarrow 1-\sqrt{bc}\ge 2a.\sqrt{bc}$$

$$\Rightarrow a\le\frac{1-\sqrt{bc}}{2\sqrt{bc}}\overset{?}{\le}\frac{1+bc-\sqrt{5b^2c^2-bc}}{1+4bc}\Leftrightarrow\frac{1-m}{2m}\overset{?}{\le}\frac{1+m^2-\sqrt{5m^4-m^2}}{1+4m^2}$$

$$\left(m=\sqrt{bc}\right)\Leftrightarrow 6m^3-4m^2+3m-1\underset{(***)}{\overset{?}{\ge}}2m.\sqrt{5m^4-m^2}$$

$$\text{We have : }6m^3-4m^2+3m-1=\frac{1}{81}\left(\frac{2(9m-2)^3}{3}+171\left(m-\frac{227}{513}\right)\right)>0$$

$$\because bc>\frac{1}{5}\Rightarrow m>\frac{1}{\sqrt{5}}>\frac{227}{513}>\frac{2}{9}\therefore(***)\Leftrightarrow$$

$$(6m^3-4m^2+3m-1)^2\overset{?}{\ge}4m^2(5m^4-m^2)$$

$$\Leftrightarrow 16m^6-48m^5+56m^4-36m^3+17m^2-6m+1\overset{?}{\ge}0$$

$$\Leftrightarrow(m-1)^2(2m-1)^2(4m^2+1)\overset{?}{\ge}0\to\text{true}\therefore(***)\Rightarrow(**)\Rightarrow(*)\ \text{is true}$$

$$\therefore a+b+c\ge 2(ab+bc+ca)\ \forall\, a,b,c>0\,\big|\,ab+bc+ca+2abc=1,$$

$$''=''\ \text{iff}\ a=b=c=\frac{1}{2}\ \text{(QED)}$$

1686. Let $a,b,c\ge 0, ab+bc+ca>0$. Prove that

$$\frac{15}{2}\cdot\frac{a^2+b^2+c^2}{(a+b+c)^2}+\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\ge 4$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality can be rewritten as follows

$$\frac{15(a^2+b^2+c^2)}{(a+b+c)^2}+\frac{(a+b)^2}{a^2+b^2}+\frac{(b+c)^2}{b^2+c^2}+\frac{(c+a)^2}{c^2+a^2}\ge 11.$$

WLOG, assume that $a+b+c=1$. Let $p:=a+b+c=1, q:=ab+bc+ca\le\frac{p^2}{3}$

$$=\frac{1}{3}, r:=abc.$$

By CBS inequality, we have

$$\sum_{cyc}\frac{(b+c)^2}{b^2+c^2}\ge\frac{\left(\sum_{cyc}(b+c)(5a+4)\right)^2}{\sum_{cyc}(b^2+c^2)(5a+4)^2}$$

$$=\frac{(10q+8p)^2}{50(q^2-2pr)+40(pq-3r)+32(p^2-2q)}$$

$$=\frac{2(5q+4)^2}{16-12q+25q^2-110r}.$$

So it suffices to prove that

$$15(1-2q)+\frac{2(5q+4)^2}{16-12q+25q^2-110r}\ge 11\ \Leftrightarrow\ \frac{(5q+4)^2}{16-12q+25q^2-110r}\ge 15q-2\ \ (1)$$

If $15q-2\le 0$, the inequality (1) is true. Assume now that $15q-2>0$.

$$(1)\Leftrightarrow r\ge\frac{375q^3-255q^2+224q-48}{110(15q-2)}.$$

From the known identity

$$0\le(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$

It follows

$$r\ge\frac{-2p^3+9pq-2\sqrt{(p^2-3q)^3}}{27}=\frac{-2+9q-2\sqrt{(1-3q)^3}}{27}.$$

So it suffices to prove that

$$\frac{-2+9q-2\sqrt{(1-3q)^3}}{27}\ge\frac{375q^3-255q^2+224q-48}{110(15q-2)}$$

$$\Leftrightarrow(1-3q)(1736-6120q+3375q^2)\ge 220(15q-2)\sqrt{(1-3q)^3}$$

$$\overset{squaring}{\Leftrightarrow}\quad 27(1-3q)^2(25q-8)^2(675q^2-80q+1632)\ge 0,$$

which is true and the proof is complete.

Equality holds iff $a=b=c$ and $a=b=2c$ and its permutation.

1687. Let a,b,c be non − negative real numbers such that

$$a+b+c=ab+bc+ac.\text{ Prove that :}$$

$$\frac{a}{\sqrt{7a+2}}+\frac{b}{\sqrt{7b+2}}+\frac{c}{\sqrt{7c+2}}\ge 1$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Yusuf Wasef-Egypt

Let $a=\frac{1}{x}, b=\frac{1}{y}, c=\frac{1}{z}$, WLOG we can assume that x & z are on the same side of unity.

$\Rightarrow x+y+z=xy+xz+yz.$ **Thus it's sufficient to show that :**

$$\frac{1}{\sqrt{2x^2+7x}}+\frac{1}{\sqrt{2y^2+7y}}+\frac{1}{\sqrt{2z^2+7z}}\geq 1.$$

Since we have that $f(u)=\frac{1}{\sqrt{2u^2+7u}}$ **is a convex function thus we can conclude that**

$$\frac{1}{\sqrt{2x^2+7x}}+\frac{1}{\sqrt{2z^2+7z}}\overset{\text{Jensen's}}{\geq}\frac{2}{\sqrt{2\left(\frac{x+z}{2}\right)^2+7\left(\frac{x+z}{2}\right)}}.$$

Let assume that $x+z=2t$ **and It's given that**

$$\begin{cases} x+y+z=xy+xz+yz \\ (x-1)(z-1)\geq 0 \Rightarrow xz\geq 2t-1 \end{cases} \Rightarrow \begin{cases} y=\frac{2t-xz}{2t-1}\leq\frac{1}{2t-1} & \text{for } \frac{1}{2}<t. \\ y=\frac{xz-2t}{1-2t}\overset{AM-GM}{\leq}\frac{t^2-2t}{1-2t} & \text{for } 0\leq t<\frac{1}{2}. \end{cases}$$

And since that the function

$f(u)=\frac{1}{\sqrt{2u^2+7u}}$ **is a decreasing function we conclude that**

Case (I):

$$\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2y^2+7y}}\geq\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2\left(\frac{1}{2t-1}\right)^2+7\left(\frac{1}{2t-1}\right)}}\geq 1.$$

Case (II):

$$\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2y^2+7y}}\geq\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2\left(\frac{t^2-2t}{1-2t}\right)^2+7\left(\frac{t^2-2t}{1-2t}\right)}}\geq 1.$$

With equality at $t=1$ *or* $t\to 0.5 \Rightarrow (a,b,c)=(1,1,1)$ *or* (a,b,c)

$=(2,2,0)$ **and their permutations.**

1688. If $a,\mathbf{b},\mathbf{c}>0$, *then prove that* **:**

$$\sum_{cyc}\frac{\sqrt{\frac{c^2(b^3+c^3)}{b}}+\sqrt{\frac{b^2(c^3+a^3)}{c}}}{ac\cdot\frac{b+c}{a+b}+ab\cdot\frac{c+a}{a+b}}\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A',B',C'>0,(A'+B'),(B'+C'),(C'+A')\text{ form sides of a triangle}$$
$$(\because\ (A'+B')+(B'+C')>(C'+A')\text{ and analogs})\Rightarrow\sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}$$
$$\text{form sides of a }\Delta\text{ with area F (say) and }16F^2=$$
$$2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$
$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\to(1)$$
$$\text{Now},\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$
$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=$$
$$\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}$$
$$\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$
$$\text{We have}:\frac{\sqrt{\frac{c^2(b^3+c^3)}{b}}+\sqrt{\frac{b^2(c^3+a^3)}{c}}}{ac.\frac{b+c}{a+b}+ab.\frac{c+a}{a+b}}+\frac{\sqrt{\frac{c^2(a^3+b^3)}{a}}+\sqrt{\frac{a^2(c^3+a^3)}{c}}}{bc.\frac{a+b}{b+c}+ab.\frac{c+a}{b+c}}$$
$$+\frac{\sqrt{\frac{b^2(a^3+b^3)}{a}}+\sqrt{\frac{a^2(b^3+c^3)}{b}}}{bc.\frac{a+b}{a+c}+ac.\frac{b+c}{a+c}}$$
$$=\frac{\frac{a+b}{ab}.\sqrt{\frac{b^3+c^3}{b}}+\frac{a+b}{ca}.\sqrt{\frac{c^3+a^3}{c}}}{\frac{b+c}{b}+\frac{c+a}{c}}+\frac{\frac{b+c}{ab}.\sqrt{\frac{a^3+b^3}{a}}+\frac{b+c}{bc}.\sqrt{\frac{c^3+a^3}{c}}}{\frac{a+b}{a}+\frac{c+a}{c}}$$
$$+\frac{\frac{c+a}{ca}.\sqrt{\frac{a^3+b^3}{a}}+\frac{c+a}{bc}.\sqrt{\frac{b^3+c^3}{b}}}{\frac{a+b}{a}+\frac{b+c}{b}}$$

$$=\frac{\frac{a+b}{a}}{\frac{b+c}{b}+\frac{c+a}{c}}\cdot\left(\sqrt{\frac{b^3+c^3}{b^3}}+\sqrt{\frac{c^3+a^3}{c^3}}\right)+\frac{\frac{b+c}{b}}{\frac{c+a}{c}+\frac{a+b}{a}}\cdot\left(\sqrt{\frac{c^3+a^3}{c^3}}+\sqrt{\frac{a^3+b^3}{a^3}}\right)$$

$$+\frac{\frac{c+a}{c}}{\frac{a+b}{a}+\frac{b+c}{b}}\cdot\left(\sqrt{\frac{a^3+b^3}{a^3}}+\sqrt{\frac{b^3+c^3}{b^3}}\right)$$

$$=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\left(x=\frac{a+b}{a},y=\frac{b+c}{b},z=\frac{c+a}{c},A'=\sqrt{\frac{a^3+b^3}{a^3}},B'=\sqrt{\frac{b^3+c^3}{b^3}},C'=\sqrt{\frac{c^3+a^3}{c^3}}\right)$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\left(\sqrt{\frac{a^3+b^3}{a^3}}\right)\left(\sqrt{\frac{b^3+c^3}{b^3}}\right)\right)}\overset{A-G}{\geq} 3.\sqrt[6]{\frac{(a^3+b^3)(b^3+c^3)(c^3+a^3)}{a^3b^3c^3}}\overset{\text{Cesaro}}{\geq}$$

$$3.\sqrt[6]{8}=3\sqrt{2}\ \forall a,b,c>0,''=''\ \text{iff } a=b=c\ (\text{QED})$$

1689. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{(b+c)^3.\frac{c^3+a^3}{a^3+b^3}+(c+a)^3.\frac{b^3+c^3}{a^3+b^3}}{\left(\frac{b+c}{a+b}\right)^2.(c^3+a^3)+\left(\frac{c+a}{a+b}\right)^2.(b^3+c^3)}\geq\frac{36abc}{a^3+b^3+c^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A',B',C'>0,(A'+B'),(B'+C'),(C'+A')\ \text{form sides of a triangle}$$

$$(\because (A'+B')+(B'+C')>(C'+A')\ \text{and analogs})\Rightarrow\sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}$$

$$\text{form sides of a}\ \Delta\ \text{with area F (say) and } 16F^2=$$

$$2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$

$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$

$$= 6\sum_{cyc} A'B' + 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc} A'B'} \rightarrow (1)$$

Now, $\forall\, x, y, z > 0$, $\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} =$$

$$\frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \rightarrow \text{true}$$

$$\therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have}: \frac{(b+c)^3.\frac{c^3+a^3}{a^3+b^3} + (c+a)^3.\frac{b^3+c^3}{a^3+b^3}}{\left(\frac{b+c}{a+b}\right)^2.(c^3+a^3) + \left(\frac{c+a}{a+b}\right)^2.(b^3+c^3)} +$$

$$\frac{(a+b)^3.\frac{c^3+a^3}{b^3+c^3} + (c+a)^3.\frac{a^3+b^3}{b^3+c^3}}{\left(\frac{a+b}{b+c}\right)^2.(c^3+a^3) + \left(\frac{c+a}{b+c}\right)^2.(a^3+b^3)} + \frac{(a+b)^3.\frac{b^3+c^3}{c^3+a^3} + (b+c)^3.\frac{a^3+b^3}{c^3+a^3}}{\left(\frac{a+b}{c+a}\right)^2.(b^3+c^3) + \left(\frac{b+c}{c+a}\right)^2.(a^3+b^3)}$$

$$= \frac{\frac{(a+b)^2}{a^3+b^3}\cdot\left(\frac{(b+c)^3}{b^3+c^3} + \frac{(c+a)^3}{c^3+a^3}\right)}{\frac{(b+c)^2}{b^3+c^3} + \frac{(c+a)^2}{c^3+a^3}} + \frac{\frac{(b+c)^2}{b^3+c^3}\cdot\left(\frac{(a+b)^3}{a^3+b^3} + \frac{(c+a)^3}{c^3+a^3}\right)}{\frac{(a+b)^2}{a^3+b^3} + \frac{(c+a)^2}{c^3+a^3}}$$

$$+ \frac{\frac{(c+a)^2}{c^3+a^3}\cdot\left(\frac{(a+b)^3}{a^3+b^3} + \frac{(b+c)^3}{b^3+c^3}\right)}{\frac{(a+b)^2}{a^3+b^3} + \frac{(b+c)^2}{b^3+c^3}}$$

$$= \frac{x}{y+z}(B'+C') + \frac{y}{z+x}(C'+A') + \frac{z}{x+y}(A'+B')$$

$$\left(\begin{array}{l} x = \frac{(a+b)^2}{a^3+b^3}, y = \frac{(b+c)^2}{b^3+c^3}, z = \frac{(c+a)^2}{c^3+a^3}, \\ A' = \frac{(a+b)^3}{a^3+b^3}, B' = \frac{(b+c)^3}{b^3+c^3}, C' = \frac{(c+a)^3}{c^3+a^3} \end{array}\right)$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} A'B'}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{(a+b)^3}{a^3+b^3},\frac{(b+c)^3}{b^3+c^3}\right)}\overset{A-G}{\geq}3.\sqrt[6]{\frac{(a+b)^6(b+c)^6(c+a)^6}{(a^3+b^3)^2(b^3+c^3)^2(c^3+a^3)^2}}\overset{Cesaro}{\geq}$$

$$3.8abc.\frac{1}{\sqrt[3]{(a^3+b^3)(b^3+c^3)(c^3+a^3)}}\overset{A-G}{\geq}24abc.\frac{3}{2(a^3+b^3+c^3)}=\frac{36abc}{a^3+b^3+c^3}$$

$$\forall\, a,b,c>0,''=''\text{ iff } a=b=c\ (QED)$$

1690. If $a,b,c>0$, *then prove that* :

$$\sum_{cyc}\frac{a^{2024}(ab^{2025}+c^{2026})}{ab^{2025}+bc^{2025}}\geq 3(abc)^{\frac{2024}{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\, A',B',C'>0,(A'+B'),(B'+C'),(C'+A')$ form sides of a triangle
$(\because (A'+B')+(B'+C')>(C'+A')$ *and analogs*$)\Rightarrow\sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}$
form sides of $a\ \Delta$ with area F (say) and

$$16F^2=2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$

$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$

$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\to(1)$$

$$\text{Now, }\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=$$

$$\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}$$

$$\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{a^{2024}(ab^{2025}+c^{2026})}{ab^{2025}+bc^{2025}}+\frac{b^{2024}(bc^{2025}+a^{2026})}{bc^{2025}+ca^{2025}}+\frac{c^{2024}(ca^{2025}+b^{2026})}{ca^{2025}+ab^{2025}}$$

$$=\frac{\frac{a^{2024}}{b}\left(\frac{b^{2025}}{c}+\frac{c^{2025}}{a}\right)}{\frac{b^{2024}}{c}+\frac{c^{2024}}{a}}+\frac{\frac{b^{2024}}{c}\left(\frac{c^{2025}}{a}+\frac{a^{2025}}{b}\right)}{\frac{c^{2024}}{a}+\frac{a^{2024}}{b}}+\frac{\frac{c^{2024}}{a}\left(\frac{a^{2025}}{b}+\frac{b^{2025}}{c}\right)}{\frac{a^{2024}}{b}+\frac{b^{2024}}{c}}$$

$$=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\left(x=\frac{a^{2024}}{b},y=\frac{b^{2024}}{c},z=\frac{c^{2024}}{a},A'=\frac{a^{2025}}{b},B'=\frac{b^{2025}}{c},C'=\frac{c^{2025}}{a}\right)\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}A'B'.\frac{\sqrt{3}}{2}}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{a^{2025}}{b}.\frac{b^{2025}}{c}\right)}$$

$$\overset{A-G}{\geq}3.\sqrt[6]{\frac{(abc)^{4050}}{(abc)^2}}\therefore\frac{a^{2024}\left(ab^{2025}+c^{2026}\right)}{ab^{2025}+bc^{2025}}+\frac{b^{2024}\left(bc^{2025}+a^{2026}\right)}{bc^{2025}+ca^{2025}}+$$

$$\frac{c^{2024}\left(ca^{2025}+b^{2026}\right)}{ca^{2025}+ab^{2025}}\geq 3.(abc)^{\frac{2024}{3}}\ \forall a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1691. **If $a,b,c>0, ab+bc+ca=a+b+c$ and $n\in\mathbb{N},\lambda\geq 0$, then :**

$$n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}\geq 3(n+\lambda)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$n\in\mathbb{N}\Rightarrow n+1\geq 1\therefore n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}=$$

$$n(a+b+c)+\lambda\sum_{cyc}\left(1+\frac{1}{a}-1\right)^{n+1}\overset{\text{Bernoulli}}{\geq}n\sum_{cyc}a+\lambda\sum_{cyc}\left(1+(n+1)\left(\frac{1}{a}-1\right)\right)$$

$$=n\sum_{cyc}a+3\lambda+\lambda(n+1)\left(\sum_{cyc}\frac{1}{a}-3\right)\overset{?}{\geq}3(n+\lambda)$$

$$\Leftrightarrow n\left(\sum_{cyc}a-3\right)+\frac{\lambda(n+1)}{abc}\left(\sum_{cyc}ab-3abc\right)\overset{?}{\geq}0$$

$$\overset{ab+bc+ca=a+b+c}{\Leftrightarrow}n\left(\sum_{cyc}a-\frac{3\sum_{cyc}ab}{\sum_{cyc}a}\right)+\frac{\lambda(n+1)}{abc}\left(\frac{\sum_{cyc}ab}{\sum_{cyc}a}.\sum_{cyc}ab-3abc\right)\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{n}{\sum_{cyc} a}\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab\right) + \frac{\lambda(n+1)}{abc\sum_{cyc} a}\left(\left(\sum_{cyc} ab\right)^2 - 3abc\sum_{cyc} a\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because n, \lambda, \left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab, \left(\sum_{cyc} ab\right)^2 - 3abc\sum_{cyc} a \geq 0$$

$$\therefore n(a+b+c) + \lambda\sum_{cyc}\frac{1}{a^{n+1}} \geq 3(n+\lambda)\ \forall\, a,b,c > 0 \,\middle|\, ab+bc+ca = a+b+c$$

$$\text{and } n \in \mathbb{N}, \lambda \geq 0, '' ='' \text{ iff } n = 0 \wedge a = b = c = 1\ (\text{QED})$$

1692. In Δ ABC, if $a^2 + b^2 + c^2 = 3$, then prove that :

$$\frac{1}{3a+bc} + \frac{1}{3b+ca} + \frac{1}{3c+ab} + \frac{ab+bc+ca}{2} \leq \frac{9}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{3a+bc} \overset{a^2+b^2+c^2=3}{=} \frac{1}{\sqrt{3\sum_{cyc} a^2}\,.\,a + bc} \leq \frac{1}{a(a+b+c)+bc}$$

$$= \frac{1}{a(a+b)+c(a+b)} = \frac{1}{(a+b)(c+a)}$$

$$\Rightarrow \frac{1}{3a+bc} \leq \frac{b+c}{2s(s^2+2Rr+r^2)} \text{ and analogs}$$

$$\Rightarrow \frac{1}{3a+bc} + \frac{1}{3b+ca} + \frac{1}{3c+ab} + \frac{ab+bc+ca}{2} \overset{a^2+b^2+c^2=3}{\leq}$$

$$\left(\sum_{cyc} a^2\right).\sum_{cyc}\frac{b+c}{6s(s^2+2Rr+r^2)} + \frac{3\sum_{cyc} ab}{2\sum_{cyc} a^2}$$

$$= \frac{4s(s^2-4Rr-r^2)}{3s(s^2+2Rr+r^2)} + \frac{3(s^2+4Rr+r^2)}{4(s^2-4Rr-r^2)}$$

$$= \frac{16(s^2-4Rr-r^2)^2 + 9(s^2+4Rr+r^2)(s^2+2Rr+r^2)}{12(s^2+2Rr+r^2)(s^2-4Rr-r^2)} \overset{?}{\leq} \frac{9}{4}$$

$$\Leftrightarrow \underbrace{s^4 + (10Rr+7r^2)s^2 - r^2(272R^2+172Rr+26r^2) \overset{?}{\geq} 0}_{(*)}$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (26Rr+2r^2)s^2 - r^2(272R^2+172Rr+26r^2)$$

$$\overset{\text{Gerretsen}}{\geq} (26Rr+2r^2)(16Rr-5r^2) - r^2(272R^2+172Rr+26r^2)$$

$$= 18r^2(8R^2 - 15Rr - 2r^2) = 18r^2(8R + 5r)(R - 2r) \overset{\text{Euler}}{\geq} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{3a + bc} + \frac{1}{3b + ca} + \frac{1}{3c + ab} + \frac{ab + bc + ca}{2} \leq \frac{9}{4}$$

$$\forall \Delta ABC \,\big|\, a^2 + b^2 + c^2 = 3, ''='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1693. **If $a, b, c > 0$ *then prove that*:**

$$2 + \frac{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}{a + b + c} \leq 4\left(\frac{a + b}{3a + 3b + 2c} + \frac{b + c}{3b + 3c + 2a} + \frac{c + a}{3c + 3a + 2b}\right)$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kokata-India

$$2 + \frac{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}{a + b + c} - 4\left(\frac{a + b}{3a + 3b + 2c} + \frac{b + c}{3b + 3c + 2a} + \frac{c + a}{3c + 3a + 2b}\right)$$

$$\overset{\text{CBS}}{\leq} 2 + \frac{\sqrt{3\sum_{\text{cyc}} ab}}{\sum_{\text{cyc}} a} - 4\sum_{\text{cyc}} \frac{(b + c)^2}{3(b + c)^2 + 2a(b + c)} \overset{\text{Bergstrom}}{\leq}$$

$$2 + \frac{\sqrt{3\sum_{\text{cyc}} ab}}{\sum_{\text{cyc}} a} - \frac{16\left(\sum_{\text{cyc}} a\right)^2}{6\sum_{\text{cyc}} a^2 + 10\sum_{\text{cyc}} ab} \overset{?}{\leq} 0$$

$$\Leftrightarrow \frac{8(m + 2n) - 6m - 10n}{3m + 5n} \overset{?}{\geq} \sqrt{\frac{3n}{m + 2n}} \left(m = \sum_{\text{cyc}} a^2, n = \sum_{\text{cyc}} ab\right)$$

$$\Leftrightarrow (m + 2n)(2m + 6n)^2 \overset{?}{\geq} 3n(3m + 5n)^2 \Leftrightarrow 4m^3 + 5m^2n - 6mn^2 - 3n^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow (m - n)(4m^2 + 9mn + 3n^2) \overset{?}{\geq} 0 \to \text{true} \because m \geq n$$

$$\therefore 2 + \frac{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}{a + b + c} \leq 4\left(\frac{a + b}{3a + 3b + 2c} + \frac{b + c}{3b + 3c + 2a} + \frac{c + a}{3c + 3a + 2b}\right)$$

$$\forall a, b, c > 0, ''='' \text{ iff } a = b = c \text{ (QED)}$$

1694. If $a,b,c>0, a+b+c=3abc$ and $n\in\mathbb{N}, \lambda\ge 0$, then :

$$n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}\ge 3(n+\lambda)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$n\in\mathbb{N}\Rightarrow n+1\ge 1 \therefore n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}$$

$$=n(a+b+c)+\lambda\sum_{cyc}\left(1+\frac{1}{a}-1\right)^{n+1}\overset{\text{Bernoulli}}{\ge}$$

$$n\sum_{cyc}a+\lambda\sum_{cyc}\left(1+(n+1)\left(\frac{1}{a}-1\right)\right)$$

$$=n\sum_{cyc}a+3\lambda+\lambda(n+1)\left(\sum_{cyc}\frac{1}{a}-3\right)\overset{?}{\ge}3(n+\lambda)$$

$$\Leftrightarrow n\left(\sum_{cyc}a-3\right)+\frac{\lambda(n+1)}{abc}\left(\sum_{cyc}ab-3abc\right)\overset{?}{\ge}0$$

$$\overset{a+b+c=3abc}{\Leftrightarrow} n\left(\sum_{cyc}a-\frac{\sum_{cyc}a}{abc}\right)+\frac{\lambda(n+1)}{abc}\left(\sum_{cyc}ab-3abc\right)\overset{?}{\ge}0$$

$$\Leftrightarrow \frac{n}{abc}\left(\sum_{cyc}a\right)(abc-1)+\frac{\lambda(n+1)}{abc}\left(\sum_{cyc}ab-3abc\right)\boxed{\overset{?}{\underset{(*)}{\ge}}}0$$

$$\text{Now, } 3abc=\sum_{cyc}a\overset{A-G}{\ge}3\sqrt[3]{abc}\Rightarrow abc\ge 1\rightarrow(1) \text{ and } \sum_{cyc}ab\ge\sqrt{3abc\sum_{cyc}a}$$

$$\overset{a+b+c=3abc}{=}\sqrt{3abc.3abc}\Rightarrow\sum_{cyc}ab\ge 3abc\rightarrow(2)\therefore(1)\text{ and }(2)\Rightarrow(*)\text{ is true}$$

$$\therefore n(a+b+c)+\lambda\sum_{cyc}\frac{1}{a^{n+1}}\ge 3(n+\lambda)\ \forall a,b,c>0\,|\,a+b+c=3abc \text{ and}$$

$$n\in\mathbb{N},\lambda\ge 0,''='' \text{ iff } n=0\wedge a=b=c=1 \text{ (QED)}$$

1695. If $a, b, c > 0, \lambda \geq 0$ then:

$$\sum \frac{a^4 + \lambda b^4}{ab} + (\lambda + 1) \sum ab \geq 2(\lambda + 1) \sum a^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{a^4 + \lambda b^4}{ab} = \sum \frac{a^4}{ab} + \lambda \sum \frac{b^4}{ab} = \sum \frac{(a^2)^2}{ab} + \lambda \sum \frac{(b^2)^2}{ab}$$

$$\overset{CBS}{\geq} \frac{(\sum a^2)^2}{\sum ab} + \lambda \frac{(\sum a^2)^2}{\sum ab} = (\lambda + 1) \frac{(\sum a^2)^2}{\sum ab}$$

We need to show:

$$\sum \frac{a^4 + \lambda b^4}{ab} + (\lambda + 1) \sum ab \geq 2(\lambda + 1) \sum a^2$$

$$(\lambda + 1) \frac{(\sum a^2)^2}{\sum ab} + (\lambda + 1) \sum ab \geq 2(\lambda + 1) \sum a^2$$

$$\frac{(\sum a^2)^2}{\sum ab} + \sum ab \geq 2 \sum a^2$$

$$x + \frac{1}{x} \overset{\frac{\sum a^2}{\sum ab} = x \geq 1}{\geq} 2 \text{ or } (x - 1)^2 \geq 0 \text{ true}$$

Equality case for $a = b = c$.

1696. If $a, b, c > 0$ and $abc = ab + bc + ca$, then prove that :

$$2(a + b + c) - abc + \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} \geq -9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0,$
$y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius

$$= s, R, r\ (say);\ so\ 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1)$$

$$\Rightarrow a = s - x, b = s - y, c = s - z \therefore abc = r^2 s \to (2)\ and\ such\ substitutions \Rightarrow$$

$$\sum_{cyc} ab = \sum_{cyc} (s - x)(s - y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3)\ and$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{via\ (1)\ and\ (3)}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

$$\text{Now}, 2(a + b + c) - abc + \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} + 9 \overset{abc = ab+bc+ca}{=}$$

$$\frac{2abc(\sum_{cyc} a)}{\sum_{cyc} ab} - \sum_{cyc} ab + \frac{1}{2}\left(2\sum_{cyc} a^2 - 2\sum_{cyc} ab\right) + \frac{9a^2b^2c^2}{(\sum_{cyc} ab)^2}$$

$$\overset{via\ (1),(2),(3),(4)}{=} \frac{2r^2s^2}{4Rr + r^2} - (4Rr + r^2) + s^2 - 12Rr - 3r^2 + \frac{9r^4s^2}{(4Rr + r^2)^2}$$

$$= \frac{2r(4R + r)s^2 - r(4R + r)^3 + (s^2 - 12Rr - 3r^2)(4R + r)^2 + 9r^2s^2}{(4R + r)^2} \overset{?}{\geq} 0$$

$$\Leftrightarrow (4R^2 + 4Rr + 3r^2)s^2 \overset{?}{\geq} r(4R + r)^3 \quad (1)$$

$$\text{Now}, (4R^2 + 4Rr + 3r^2)s^2 \overset{\text{Rouche}}{\geq}$$

$$(4R^2 + 4Rr + 3r^2)\left(2R^2 + 10Rr - r^2 - 2(R - 2r).\sqrt{R^2 - 2Rr}\right) \overset{?}{\geq} r(4R + r)^3$$

$$\Leftrightarrow (4R^2 + 4Rr + 3r^2)(2R^2 + 10Rr - r^2) - r(4R + r)^3$$

$$\overset{?}{\geq} 2(R - 2r).\sqrt{R^2 - 2Rr}.(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 2(R - 2r)(4R^3 - 3R^2r + r^3) \overset{?}{\geq} 2(R - 2r).\sqrt{R^2 - 2Rr}.(4R^2 + 4Rr + 3r^2) \quad (2)$$

$$and \because R - 2r \overset{\text{Euler}}{\geq} 0 \therefore \text{in order to prove } (2), \text{it suffices to prove}:$$

$$(4R^3 - 3R^2r + r^3)^2 \overset{?}{>} (R^2 - 2Rr)(4R^2 + 4Rr + 3r^2)^2 \Leftrightarrow r^3(4R + r)^3 \overset{?}{>} 0 \to \text{true}$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true} \therefore 2(a + b + c) - abc + \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} \geq -9$$

$$\forall a, b, c > 0 \mid abc = ab + bc + ca, '' ='' \text{ iff } a = b = c = 1\ (\text{QED})$$

1697. In any Δ ABC, the following relationship holds :

$$\frac{a^3(a+b)}{a^2+b^2}+\frac{b^3(b+c)}{b^2+c^2}+\frac{c^3(c+a)}{c^2+a^2}\ge a^2+b^2+c^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3(a+b)}{a^2+b^2}+\frac{b^3(b+c)}{b^2+c^2}+\frac{c^3(c+a)}{c^2+a^2}-(a^2+b^2+c^2)=$$

$$=\sum_{cyc}\left(\frac{b^3(b+c)}{b^2+c^2}-\frac{b^2+c^2}{2}\right)=\sum_{cyc}\frac{2b^4+2b^3c-b^4-c^4-2b^2c^2}{2(b^2+c^2)}$$

$$=\sum_{cyc}\frac{(b^2+c^2)(b^2-c^2)}{2(b^2+c^2)}+\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\sum_{cyc}\left(b^2c(b-c)\left(\sum_{cyc}a^2b^2+a^4\right)\right)$$

$$=\frac{1}{2}\sum_{cyc}(b^2-c^2)+\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\left(\begin{array}{c}\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}b^3c-\sum_{cyc}b^2c^2\right)\\ +abc\left(\sum_{cyc}a^3b^2-abc\sum_{cyc}a^2\right)\end{array}\right)\overset{?}{\ge}0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}b^3c-\sum_{cyc}b^2c^2\right)+abc\left(\sum_{cyc}a^3b^2-abc\sum_{cyc}a^2\right)\boxed{\underset{(*)}{\overset{?}{\ge}}}0$$

$$\left(\because\sum_{cyc}(b^2-c^2)=0\right)$$

Let $s-a=x, s-b=y, s-c=z$ **& then :** $a=y+z, b=z+x, c=x+y\to(m)$

$$\text{Now},(m)\Rightarrow\sum_{cyc}b^3c-\sum_{cyc}b^2c^2=\sum_{cyc}(z+x)^3(x+y)-\sum_{cyc}(z+x)^2(x+y)^2$$

$$=\sum_{cyc}xy^3-xyz\sum_{cyc}x=xyz\sum_{cyc}\frac{y^2}{z}-xyz\sum_{cyc}x$$

$$\overset{\text{Bergstrom}}{\ge}xyz\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x}-xyz\sum_{cyc}x=0\therefore\sum_{cyc}b^3c-\sum_{cyc}b^2c^2\ge0\to①$$

$$\text{Again},(m)\Rightarrow\sum_{cyc}a^3b^2-abc\sum_{cyc}a^2=$$

$$\sum_{cyc}(y+z)^3(z+x)^2-(x+y)(y+z)(z+x)\sum_{cyc}(y+z)^2=\sum_{cyc}(x^5+xy^4-2x^3y^2)$$

$$=\sum_{cyc} x(x^2-y^2)^2 \ge 0 \therefore \sum_{cyc} a^3b^2 - abc\sum_{cyc} a^2 \ge 0 \to ② \therefore ① \text{ and } ② \Rightarrow$$

$$(*) \text{ is true} \therefore \frac{a^3(a+b)}{a^2+b^2}+\frac{b^3(b+c)}{b^2+c^2}+\frac{c^3(c+a)}{c^2+a^2} \ge a^2+b^2+c^2$$

$\forall$ ABC, with equality iff Δ ABC is equilateral (QED)

1698. If $a,b,c>0$ then:

$$\sum \frac{\frac{c}{a}\sqrt[2025]{b^{2025}+c^{2025}}+\frac{b}{a}\sqrt[2025]{c^{2025}+a^{2025}}}{c\sqrt[2024]{\frac{b^{2024}+c^{2024}}{a^{2024}+b^{2024}}}+b\sqrt[2024]{\frac{c^{2024}+a^{2024}}{a^{2024}+b^{2024}}}} \ge 3\sqrt[2025]{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

Walter Janous inequality: a',b',c' *and* x',y',z' *be positive real numbers then*:

$$\frac{x'}{y'+z'}(b'+c')+\frac{y'}{z'+x'}(c'+a')+\frac{z'}{x'+y'}(b'+a') \ge \sqrt{3(a'b'+b'c'+c'a')} \ (1)$$

$$\text{Let } x'=\frac{\sqrt[2024]{(a^{2024}+b^{2024})}}{a}, y'=\frac{\sqrt[2024]{(c^{2024}+b^{2024})}}{b}, z'=\frac{\sqrt[2024]{(a^{2024}+c^{2024})}}{c}$$

$$\text{and } a'=\frac{\sqrt[2025]{(b^{2025}+a^{2025})}}{a}, b'=\frac{\sqrt[2025]{(b^{2025}+c^{2025})}}{b}, c'=\frac{\sqrt[2025]{(c^{2025}+a^{2025})}}{c}$$

$$\sum a'b'=\sum\left(\frac{\sqrt[2025]{(b^{2025}+a^{2025})}}{a}\cdot\frac{\sqrt[2025]{(b^{2025}+c^{2025})}}{b}\right) \overset{AM-GM}{\ge}$$

$$\ge \sum \frac{\sqrt[2025]{2(ab)^{\frac{2025}{2}}.2(bc)^{\frac{2025}{2}}}}{bc} = 2^{\frac{2}{2025}}\sum\frac{\sqrt{abbc}}{bc} = 2^{\frac{2}{2025}}\sum\sqrt{\frac{a}{c}} \ (2)$$

$$\sum \frac{\frac{c}{a}\sqrt[2025]{b^{2025}+c^{2025}}+\frac{b}{a}\sqrt[2025]{c^{2025}+a^{2025}}}{c\sqrt[2024]{\frac{b^{2024}+c^{2024}}{a^{2024}+b^{2024}}}+b\sqrt[2024]{\frac{c^{2024}+a^{2024}}{a^{2024}+b^{2024}}}} =$$

$$=\sum \frac{\sqrt[2024]{(a^{2024}+b^{2024})}}{a}\cdot\left(\frac{\frac{\sqrt[2025]{(b^{2025}+c^{2025})}}{b}+\frac{\sqrt[2025]{(c^{2025}+a^{2025})}}{c}}{\frac{\sqrt[2024]{(c^{2024}+b^{2024})}}{b}+\frac{\sqrt[2024]{(a^{2024}+c^{2024})}}{c}}\right)=$$

$$= \sum \frac{x'(b'+c')}{y'+z'} \overset{(1)}{\geq} \sqrt{3(a'b'+b'c'+c'a')} \overset{(2)}{\geq} \sqrt{3 \cdot 2^{\frac{2}{2025}} \sum \sqrt{\frac{a}{c}}} \overset{AM-GM}{\geq}$$

$$\geq \sqrt{3 \cdot 2^{\frac{2}{2025}} \cdot 3} = 3\sqrt[2025]{2}$$

Equality holds for $a = b = c$.

1699. If $a, b, c > 0$ then:

$$\frac{a^{2025}}{b} + \frac{b^{2025}}{c} + \frac{c^{2025}}{a} \geq \frac{(ab^{2023}+bc^{2023}+ca^{2023})^{2025}}{(a^{2024}+b^{2024}+c^{2024})^{2024}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\left(\frac{a^{2025}}{b} + \frac{b^{2025}}{c} + \frac{c^{2025}}{a}\right)(b^{2024}+c^{2024}+a^{2024})^{2024} \geq$$

$$\overset{Holder}{\geq} \left(\sum \sqrt[2025]{\frac{a^{2025}}{b} \cdot b^{(2024\times 2024)}}\right)^{2025} = \left(\sum \sqrt[2025]{\frac{a^{2025}}{b} \cdot b^{(4096576)}}\right)^{2025} =$$

$$= \left(\sum \sqrt[2025]{\frac{a^{2025}}{b} \cdot b^{(4096575)} \cdot b}\right)^{2025} = \left(\sum \sqrt[2025]{a^{2025} \cdot b^{4096575}}\right)^{2025} =$$

$$= \left(\sum \sqrt[2025]{(ab^{2023})^{2025}}\right)^{2025} = (ab^{2023}+bc^{2023}+ca^{2023})^{2025}$$

$$\frac{a^{2025}}{b} + \frac{b^{2025}}{c} + \frac{c^{2025}}{a} \geq \frac{(ab^{2023}+bc^{2023}+ca^{2023})^{2025}}{(a^{2024}+b^{2024}+c^{2024})^{2024}}$$

Equality holds for $a = b = c$.

1700.

If $a, b, c \in \left[0; \frac{1}{2}\right]$ and $a + b + c = 1$, then prove that :

$$a^3 + b^3 + c^3 + 4abc \leq \frac{9}{32}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^3+b^3+c^3+4abc-\frac{9}{32}=$$

$$=3abc+\left(\sum_{cyc}a\right)\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)+4abc-\frac{9}{32}$$

$$\overset{a+b+c=1}{=}7abc+1-3\big(bc+a(1-a)\big)-\frac{9}{32}=3a^2-3a+1+bc(7a-3)-\frac{9}{32}$$

$$\therefore a^3+b^3+c^3+4abc-\frac{9}{32}=\frac{32(3a^2-3a)+32bc(7a-3)+23}{32}\rightarrow (m)$$

Case 1 $7a-3\ge 0$ and then : $a^3+b^3+c^3+4abc-\frac{9}{32}\overset{via\,(m)}{=}$

$$\frac{32(3a^2-3a)+32bc(7a-3)+23}{32}\overset{A-G}{\le}\frac{32(3a^2-3a)+8(b+c)^2(7a-3)+23}{32}$$

$$\overset{a+b+c=1}{=}\frac{32(3a^2-3a)+8(1-a)^2(7a-3)+23}{32}=\frac{56a^3-40a^2+8a-1}{32}$$

$$=\frac{(2a-1)(19a^2+(3a-1)^2)}{32}\le 0\ \left(\because a\in\left[0;\frac{1}{2}\right]\right)\therefore a^3+b^3+c^3+4abc\le\frac{9}{32}$$

Case 2 $7a-3<0$ and $\because\left(b-\frac{1}{2}\right),\left(c-\frac{1}{2}\right)\le 0\therefore\left(b-\frac{1}{2}\right)\left(c-\frac{1}{2}\right)\ge 0$

$$\Rightarrow bc-\frac{b+c}{2}+\frac{1}{4}\ge 0\overset{a+b+c=1}{\Rightarrow}bc-\frac{1-a}{2}+\frac{1}{4}\ge 0\Rightarrow bc-\frac{1-2a}{4}\ge 0$$

$$\Rightarrow (7a-3)\left(bc-\frac{1-2a}{4}\right)\le 0\Rightarrow bc(7a-3)\le\frac{(7a-3)(1-2a)}{4}\text{ and then :}$$

$$a^3+b^3+c^3+4abc-\frac{9}{32}\overset{via\,(m)}{=}\frac{32(3a^2-3a)+32bc(7a-3)+23}{32}$$

$$\le\frac{32(3a^2-3a)+8(7a-3)(1-2a)+23}{32}=\frac{-(4a-1)^2}{32}\le 0$$

$$\therefore a^3+b^3+c^3+4abc\le\frac{9}{32}\therefore\text{ combining both cases,}$$

$$a^3+b^3+c^3+4abc\le\frac{9}{32}\ \forall\, a,b,c\in\left[0;\frac{1}{2}\right]\wedge a+b+c=1,$$

$$''='' \text{ iff }\left(a=\frac{1}{2},b=c=\frac{1}{4}\right)\text{ or }\left(b=\frac{1}{2},c=a=\frac{1}{4}\right)\left(c=\frac{1}{2},a=b=\frac{1}{4}\right)\text{ (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru