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3001. In any Δ ABC, the following relationship holds :

$$\sqrt{r_a}\sin\frac{A}{2}+\sqrt{r_b}\sin\frac{B}{2}+\sqrt{r_c}\sin\frac{C}{2}\ge\frac{s^2}{\sqrt{R(4R+r+s)(4R+r-s)}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x)=\sqrt{\tan\frac{x}{2}}.\sin\frac{x}{2}\ \forall x\in(0,\pi) \text{ and then :}$$

$$f''(x)=\frac{\left(4\sec^2\frac{x}{2}-4\right)\left(\sin\frac{x}{2}\right)\left(\tan^2\frac{x}{2}\right)+4\left(\sec^2\frac{x}{2}\right)\left(\sin\frac{x}{2}\right)-\left(\sin\frac{x}{2}\right)\left(\sec^4\frac{x}{2}\right)}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}$$

$$=\frac{\sin\frac{x}{2}}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}.\left(4\left(\sec^4\frac{x}{2}-2\sec^2\frac{x}{2}+1\right)+4\sec^2\frac{x}{2}-\sec^4\frac{x}{2}\right)$$

$$=\frac{\sin\frac{x}{2}}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}.\left(3\sec^4\frac{x}{2}-4\sec^2\frac{x}{2}+4\right)$$

$$=\frac{\sin\frac{x}{2}}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}.\left(2\sec^4\frac{x}{2}+\left(\sec^2\frac{x}{2}-2\right)^2\right)>0\Rightarrow f(x)=\sqrt{\tan\frac{x}{2}}.\sin\frac{x}{2}$$

$$\forall x\in(0,\pi) \text{ is convex}$$

$$\therefore\sqrt{r_a}\sin\frac{A}{2}+\sqrt{r_b}\sin\frac{B}{2}+\sqrt{r_c}\sin\frac{C}{2}=$$

$$=\sqrt{s}.\sum_{cyc}\left(\sqrt{\tan\frac{A}{2}}.\sin\frac{A}{2}\right)\overset{\text{Jensen}}{\ge}3\sqrt{s}.\sqrt{\tan\frac{\pi}{6}}.\sin\frac{\pi}{6}=\frac{3\sqrt{s}}{2}.\sqrt{\frac{1}{\sqrt{3}}}\overset{?}{\ge}$$

$$\frac{s^2}{\sqrt{R(4R+r+s)(4R+r-s)}}\Leftrightarrow\frac{9R}{4\sqrt{3}}\underset{(1)}{\overset{?}{\ge}}\frac{s^3}{((4R+r)^2-s^2)}$$

$$\text{Now, LHS of (1)}\overset{\text{Mitrinovic}}{\ge}\frac{9.2s}{(4\sqrt{3})(3\sqrt{3})}=\frac{s}{2}\overset{?}{\ge}\frac{s^3}{((4R+r)^2-s^2)}$$

$$\Leftrightarrow(4R+r)^2-s^2\overset{?}{\ge}2s^2\Leftrightarrow(4R+r)^2\overset{?}{\ge}3s^2\to\text{true via Doucet}\Rightarrow(1)\text{ is true}$$

$$\therefore\sqrt{r_a}\sin\frac{A}{2}+\sqrt{r_b}\sin\frac{B}{2}+\sqrt{r_c}\sin\frac{C}{2}\ge\frac{s^2}{\sqrt{R(4R+r+s)(4R+r-s)}}\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3002. In ΔABC the following relationship holds:

$$\sqrt{\frac{g_a^2-h_a^2}{w_a^2-h_a^2}}+\sqrt{\frac{g_b^2-h_b^2}{w_b^2-h_b^2}}+\sqrt{\frac{g_c^2-h_c^2}{w_c^2-h_c^2}}=2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$g_a^2=s(s-a)-\frac{(s-a)(b-c)^2}{a},h_a^2=s(s-a)-\frac{s(s-a)(b-c)^2}{a^2},w_a^2=s(s-a)-\frac{s(s-a)}{(b+c)^2}(b-c)^2$$

(Reference: Bogdan Fustei & Mohamed Amine Ben Ajiba- ABOUT NAGEL AND GERGONNE'S CEVIANS-www.ssmrmh.ro)

$$g_a^2-h_a^2=s(s-a)-\frac{(s-a)(b-c)^2}{a}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}=\frac{(s-a)(b-c)^2}{a^2}(s-a)=\frac{(s-a)^2(b-c)^2}{a^2}$$

$$w_a^2-h_a^2=s(s-a)-\frac{s(s-a)}{(b+c)^2}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}=$$

$$=\frac{s(s-a)(b-c)^2}{a^2}\left(1-\frac{a^2}{(b+c)^2}\right)=$$

$$=\frac{s(s-a)(b-c)^2}{a^2}\frac{(a+b+c)(b+c-a)}{(b+c)^2}=\frac{2s^2(s-a)(b-c)^2}{a^2}\frac{2(s-a)}{(b+c)^2}$$

Using above result we get:

$$\frac{g_a^2-h_a^2}{w_a^2-h_a^2}=\frac{(b+c)^2}{4s^2},\sqrt{\frac{g_a^2-h_a^2}{w_a^2-h_a^2}}=\frac{b+c}{2s}$$

$$\sqrt{\frac{g_a^2-h_a^2}{w_a^2-h_a^2}}+\sqrt{\frac{g_b^2-h_b^2}{w_b^2-h_b^2}}+\sqrt{\frac{g_c^2-h_c^2}{w_c^2-h_c^2}}=\sum\sqrt{\frac{g_a^2-h_a^2}{w_a^2-h_a^2}}=$$

$$=\sum\frac{b+c}{2s}=\frac{2(a+b+c)}{2s}=\frac{4s}{2s}=2$$

3003. If $m > 0$ then in ΔABC the following relationship holds:

$$\frac{a^{m+2}}{h_a^m}+\frac{b^{m+2}}{h_b^m}+\frac{c^{m+2}}{h_c^m}\ge 2^{m+2}\left(\sqrt{3}\right)^{1-m}F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

Solution by Tapas Das-India

$$\frac{a^{m+2}}{h_a^m}+\frac{b^{m+2}}{h_b^m}+\frac{c^{m+2}}{h_c^m}=\sum\frac{a^{m+2}}{h_a^m}=\frac{1}{2^mF^m}\sum a^{m+2}.a^m=$$

$$=\frac{1}{2^mF^m}\sum(a^2)^{m+1}\overset{CBS}{\ge}\frac{1}{2^mF^m}\frac{(a^2+b^2+c^2)^{m+1}}{3^m}\overset{Ionescu-Weitzenbock}{\ge}$$

$$\ge\frac{1}{2^mF^m}\frac{\left(4\sqrt{3}F\right)^{m+1}}{3^m}=\frac{1}{2^mF^m}\frac{2^{2m+2}F^{m+1}\left(\sqrt{3}\right)^{m+1}}{\left(\sqrt{3}\right)^{2m}}=2^{m+2}\left(\sqrt{3}\right)^{1-m}F$$

Equality holds for an equilateral triangle.

3004. In acute ΔABC the following relationship holds:

$$\frac{1}{\cos A\cos B}+\frac{1}{\cos B\cos C}+\frac{1}{\cos C\cos A}\ge\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{1}{\sin\frac{C}{2}\sin\frac{A}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\cos A\cos B=\frac{s^2+r^2-4R^2}{4R^2}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2-4R^2}{4R^2}=\frac{Rr+r^2}{R^2}\ (1)$$

$$\frac{1}{\cos A\cos B}+\frac{1}{\cos B\cos C}+\frac{1}{\cos C\cos A}\overset{CBS}{\ge}\frac{(1+1+1)^2}{\sum\cos A\cos B}\overset{(1)}{\ge}\frac{9R^2}{Rr+r^2}$$

$$\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{1}{\sin\frac{C}{2}\sin\frac{A}{2}}=\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\sum\sin\frac{A}{2}\overset{Jensen}{\le}$$

$$\le\frac{4R}{r}.3\sin\left(\frac{A+B+C}{6}\right)=\frac{4R}{r}.\frac{3}{2}$$

We need to show:

$$\frac{9R^2}{Rr+r^2}\geq\frac{4R}{r}\cdot\frac{3}{2}\ or,\frac{3R}{R+r}\geq 2\ \ or,R\geq 2r\ \ \ EULER$$

Equality holds for an equilateral triangle.

3005. In any Δ ABC, the following relationship holds :

$$\frac{1}{4}s^2.\sqrt{s^2-16Rr+21r^2}\leq m_a m_b m_c\leq\frac{1}{4}s^2.\sqrt{s^2-11Rr+11r^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\to(1)$$

$$\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\to(2)\ and,$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\to(3)$$

$$\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2=$$

$$\frac{1}{64}\left(\begin{array}{c}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{array}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2}{64}$$

$$\Rightarrow m_a^2m_b^2m_c^2=\frac{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3}{16}\to(m)$$

$$\therefore (m)\Rightarrow m_am_bm_c\ge\frac{1}{4}s^2.\sqrt{s^2-16Rr+21r^2}$$

$$\Leftrightarrow s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\ge s^4(s^2-16Rr+21r^2)$$

$$\Leftrightarrow (4R+12r)s^4-r(60R^2+120Rr+33r^2)s^2-r^2(4R+r)^3\overset{(*)}{\ge}0$$

Now, LHS of $(*)\overset{\text{Gerretsen}}{\ge}(4R+12r)(16Rr-5r^2)s^2-r(60R^2+120Rr+33r^2)s^2$

$$-r^2(4R+r)^3\overset{?}{\ge}0\Leftrightarrow(4R^2+52Rr-93r^2)s^2-r(4R+r)^3\underset{(**)}{\overset{?}{\ge}}0$$

Again, LHS of $(**)\overset{\text{Gerretsen}}{\ge}(4R^2+52Rr-93r^2)(16Rr-5r^2)-r(4R+r)^3\overset{?}{\ge}0$

$$\Leftrightarrow 4r(191R^2-440Rr+116r^2)\overset{?}{\ge}0\Leftrightarrow(R-2r)(191R-58r)\overset{?}{\ge}0\to\text{true}$$

$\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(**)\Rightarrow(*)$ is true $\therefore m_am_bm_c\ge\frac{1}{4}s^2.\sqrt{s^2-16Rr+21r^2}$

and also, via (m), $m_am_bm_c\le\frac{1}{4}s^2.\sqrt{s^2-11Rr+11r^2}$

$$\Leftrightarrow -(11Rr-11r^2)s^4\ge -s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3$$

$$\Leftrightarrow (R-22r)s^4+r(60R^2+120Rr+33r^2)s^2+r^2(4R+r)^3\overset{(\bullet)}{\ge}0$$

and $(\bullet)$ is trivially true if $R-22r\ge0$ and so, we now focus on the case when: $R-22r<0$ and then: LHS of $(\bullet)\overset{\text{Gerretsen}}{\ge}(R-22r)(4R^2+4Rr+3r^2)s^2$

$$+r(60R^2+120Rr+33r^2)s^2+r^2(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow(4R^3-24R^2r+35Rr^2-33r^3)s^2+r^2(4R+r)^3\underset{(\bullet\bullet)}{\overset{?}{\ge}}0$$

and $(\bullet\bullet)$ is trivially true if $4R^3-24R^2r+35Rr^2-33r^3\ge0$ and so,

we now focus on the case when: $4R^3-24R^2r+35Rr^2-33r^3<0$ and then:

LHS of $(\bullet\bullet)\overset{\text{Gerretsen}}{\ge}$

$$(4R^3 - 24R^2r + 35Rr^2 - 33r^3)(4R^2 + 4Rr + 3r^2) + r^2(4R + r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow 16t^5 - 80t^4 + 120t^3 - 16t^2 - 15t - 98 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)(6t^3 + 2t(t^2-4) + 8t^2(t-2) + 16) + 81\Big) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \therefore m_a m_b m_c \leq \frac{1}{4}s^2.\sqrt{s^2 - 11Rr + 11r^2} \text{ and so,}$$

$$\frac{1}{4}s^2.\sqrt{s^2 - 16Rr + 21r^2} \leq m_a m_b m_c \leq \frac{1}{4}s^2.\sqrt{s^2 - 11Rr + 11r^2}\ \forall \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3006. In any Δ ABC, the following relationship holds :

$$\frac{n_a}{w_a} + \frac{n_b}{w_b} + \frac{n_c}{w_c} \leq \frac{13R}{8r} - \frac{1}{4}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - a + \frac{(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}.(b-c)^2 \text{ and analogs}$$

$$\therefore \frac{n_a}{w_a} + \frac{n_b}{w_b} + \frac{n_c}{w_c} = \sum_{cyc}\left(\sqrt{\frac{as(s-a) + s(b-c)^2}{a}}.\frac{(b+c).\sqrt{(s-b)(s-c)}}{2\sqrt{bc}.\sqrt{s(s-a)(s-b)(s-c)}}\right)$$
$$= \frac{1}{2\sqrt{4Rrs}.rs}.\sum_{cyc}\left(\sqrt{(as(s-a)+s(b-c)^2)(b+c)}.\sqrt{(b+c)(s-b)(s-c)}\right) \overset{CBS}{\leq}$$
$$\frac{1}{2\sqrt{4Rrs}.rs}.\sqrt{\sum_{cyc}\big((as(s-a)+s(b-c)^2)(b+c)\big)}.\sqrt{\sum_{cyc}\big((b+c)(s-b)(s-c)\big)} \to (a)$$

$$\text{Now,} \sum_{cyc}\left((as(s-a)+s(b-c)^2)(b+c)\right)$$
$$= s\sum_{cyc}\left(a(s-a)(2s-a)\right) + s\sum_{cyc}\left((2s-a)(b-c)^2\right)$$
$$= 2s^2\left(s(2s)-\sum_{cyc}a^2\right) - s\left(s\sum_{cyc}a^2 - \sum_{cyc}a^3\right) + 4s^2\left(\sum_{cyc}a^2 - \sum_{cyc}ab\right)$$
$$-s\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) - 9abc\right)$$
$$= 2s^2\left(2s^2-2(s^2-4Rr-r^2)\right) - 2s^2\left((s^2-4Rr-r^2)-(s^2-6Rr-3r^2)\right)$$
$$+4s^2(s^2-12Rr-3r^2) - 2s^2(s^2-14Rr+r^2)$$
$$\Rightarrow \sum_{cyc}\left((as(s-a)+s(b-c)^2)(b+c)\right) = 2s^2(s^2-4Rr-7r^2) \to (m) \text{ and}$$
$$\sum_{cyc}\left((b+c)(s-b)(s-c)\right) = \sum_{cyc}\left((s-b)(s-c)(2s-a)\right)$$
$$= 2s(4Rr+r^2) - \sum_{cyc}\left(a(-s^2+sa+bc)\right)$$
$$= 2s(4Rr+r^2) + s^2(2s) - 2s(s^2-4Rr-r^2) - 12Rrs$$
$$\Rightarrow \sum_{cyc}\left((b+c)(s-b)(s-c)\right) = 4rs(R+r) \to (n) \therefore (a),(m),(n) \Rightarrow$$
$$\left(\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c}\right)^2 \le \frac{2s^2(s^2-4Rr-7r^2).4rs(R+r)}{16Rrs.r^2s^2} \overset{?}{\le} \left(\frac{13R}{8r}-\frac{1}{4}\right)^2$$
$$= \frac{(13R-2r)^2}{64r^2} \Leftrightarrow 32(R+r)(s^2-4Rr-7r^2) \overset{?}{\underset{①}{\le}} R(13R-2r)^2$$
$$\text{Now, LHS of ① } \overset{\text{Rouche}}{\le} 32(R+r)\left(2R^2+6Rr-8r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$
$$\overset{?}{\le} R(13R-2r)^2 \Leftrightarrow 105R^3-308R^2r+68Rr^2+256r^3 \overset{?}{\ge}$$
$$64(R+r)(R-2r).\sqrt{R^2-2Rr}$$
$$\Leftrightarrow (R-2r)(105R^2-98Rr-128r^2) \overset{?}{\underset{②}{\ge}} 64(R+r)(R-2r).\sqrt{R^2-2Rr}$$
$$\text{Now, } 105R^2-98Rr-128r^2 = (R-2r)(105R+112r)+96r^2 \overset{\text{Euler}}{\ge} 96r^2 > 0 \text{ and}$$
$$\because R-2r \overset{\text{Euler}}{\ge} 0 \therefore \text{in order to prove ②, it suffices to prove :}$$
$$(105R^2-98Rr-128r^2)^2 > 4096(R^2-2Rr)(R+r)^2$$
$$\Leftrightarrow 6929t^4-20580t^3-4988t^2+33280t+16384 > 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \frac{1}{27}\left((20787t^2+35266t+36437)(3t-7)^2+700880(t-2)+58715\right)>0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ② \Rightarrow ① \text{ is true} \therefore \left(\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c}\right)^2 \leq \left(\frac{13R}{8r}-\frac{1}{4}\right)^2 \text{ and so,}$$

$$\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c} \leq \frac{13R}{8r}-\frac{1}{4}\ \forall\,\Delta\,ABC, \text{with equality iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3007. If $t>0$ then in ΔABC the following relationship holds:

$$\frac{m_a^{t+2}}{(Rm_b+rm_c)^t}+\frac{m_b^{t+2}}{(Rm_c+rm_a)^t}+\frac{m_c^{t+2}}{(Rm_a+rm_b)^t} \geq \frac{3\sqrt{3}}{(R+r)^t}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$\frac{m_a^{t+2}}{(Rm_b+rm_c)^t}+\frac{m_b^{t+2}}{(Rm_c+rm_a)^t}+\frac{m_c^{t+2}}{(Rm_a+rm_b)^t}=$$

$$=\sum\frac{m_a^{t+2}}{(Rm_b+rm_c)^t}=\sum\frac{m_a^{t+2}m_a^t}{(Rm_bm_a+rm_am_c)^t}=$$

$$=\sum\frac{(m_a^2)^{t+1}}{(Rm_bm_a+rm_am_c)^t} \overset{Radon}{\geq} \frac{\left(m_a^2+m_b^2+m_c^2\right)^{t+1}}{\left((R+r)(m_am_b+m_bm_c+m_cm_a)\right)^t} \geq$$

$$\geq \frac{\left(m_a^2+m_b^2+m_c^2\right)^{t+1}}{\left((R+r)\left(m_a^2+m_b^2+m_c^2\right)\right)^t}=\frac{\sum m_a^2}{(R+r)^t}=\frac{3}{4}\frac{\sum a^2}{(R+r)^t} \overset{Ionescu-Weitzenbock}{\geq}$$

$$\geq \frac{3}{4}\cdot\frac{4\sqrt{3}F}{(R+r)^t}=\frac{3\sqrt{3}}{(R+r)^t}\cdot F$$

Equality holds for an equilateral triangle.

3008.

In any Δ ABC, the following relationship holds :

$$r_a+\frac{R}{r}h_a \geq \sqrt{3}s$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_a+\frac{R}{r}h_a=\frac{rs(s-b)(s-c)}{(s-a)(s-b)(s-c)}+\frac{R}{r}.\frac{2rs}{4Rrs}.bc=\frac{4(s-b)(s-c)}{4r}+\frac{2bc}{4r}$$
$$=\frac{a^2-(b-c)^2+2bc}{4r}\overset{?}{\geq}\sqrt{3}s\Leftrightarrow(a^2-(b-c)^2+2bc)^2\overset{?}{\geq}3.16F^2$$
$$\Leftrightarrow(a^2-(b-c)^2+2bc)^2\overset{?}{\geq}3\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$
$$\Leftrightarrow a^4-2a^2b^2-2a^2c^2+b^4+c^4+3b^2c^2-2b^3c-2bc^3+2a^2bc\overset{?}{\geq}0$$
$$\Leftrightarrow\left((b^2+c^2)^2+a^4-2a^2(b^2+c^2)\right)+b^2c^2-2bc(b^2+c^2-a^2)\overset{?}{\geq}0$$
$$\Leftrightarrow(b^2+c^2-a^2)^2+b^2c^2-2bc(b^2+c^2-a^2)\overset{?}{\geq}0$$
$$\Leftrightarrow(b^2+c^2-a^2-bc)^2\overset{?}{\geq}0\Leftrightarrow(2bc\cos A-bc)^2\overset{?}{\geq}0\Leftrightarrow4b^2c^2\left(\cos A-\frac{1}{2}\right)^2\overset{?}{\geq}0$$
$$\to\text{true}\therefore r_a+\frac{R}{r}h_a\geq\sqrt{3}s\ \forall\ \Delta\ ABC,''=''\text{ iff }\hat{A}=60^\circ\ (QED)$$

Solution 2 by Tapas Das-India

$$We\ know\ that\ \frac{r}{s-a}=\tan\frac{A}{2},\frac{2R}{a}=\frac{1}{\sin A}(as\ \ a=2R\sin A)=\frac{1+\tan^2\frac{A}{2}}{2\tan\frac{A}{2}}$$
$$We\ need\ to\ show,r_a+\frac{R}{r}h_a\geq\sqrt{3}s\ or\ \frac{r.s}{s-a}+\frac{R}{r}.\frac{2rs}{a}\geq\sqrt{3}s$$
$$or\ \frac{r}{s-a}+\frac{2R}{a}\geq\sqrt{3}\ or\ \tan\frac{A}{2}+\frac{1+\tan^2\frac{A}{2}}{2\tan\frac{A}{2}}\geq\sqrt{3}\ or$$
$$3\tan^2\frac{A}{2}+1\geq2\sqrt{3}\tan\frac{A}{2}\ or\ 3\tan^2\frac{A}{2}-2\sqrt{3}\tan\frac{A}{2}+1\geq0$$
$$or\left(\sqrt{3}\tan\frac{A}{2}-1\right)^2\geq0\ true$$

Equality holds for an equilateral triangle.

3009. In ΔABC the following relationship holds:

$$\sin(A)\cos(B)+\sin(B)\cos(C)+\sin(C)\cos(A)\leq\frac{3\sqrt{3}}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sin(A)\cos(B)+\sin(B)\cos(C)+\sin(C)\cos(A)=\frac{1}{2}(\sin(A+B)+\sin(A-B))+$$
$$\frac{1}{2}(\sin(B+C)+\sin(B-C))+\frac{1}{2}(\sin(A+C)+\sin(A-C))=\frac{1}{2}(\sin(A+B)+\sin(B+C)+$$
$$\sin(A+C))+\frac{1}{2}(\sin(A-B)+\sin(B-C)+\sin(C-A))=\frac{1}{2}(sinA+sinB+sinC)+$$
$$\frac{1}{2}(\sin(A-B)+\sin(B-C)+\sin(C-A))=\frac{1}{2}\left(\sum\nolimits_1+\sum\nolimits_2\right)$$

$$*\sum\nolimits_2=\sin(A-B)+\sin(B-C)+\sin(C-A)=2sin\frac{(A-B)+(B-C)}{2}.cos\frac{(A-B)-(B-C)}{2}+$$
$$\sin(C-A)=2sin\frac{A-C}{2}.cos\frac{(A+C)-2B}{2}-\sin(A-C)=2sin\frac{A-C}{2}.cos\frac{(\pi-b)-2B}{2}-$$
$$2sin\frac{A-C}{2}.cos\frac{A-C}{2}=2sin\frac{A-C}{2}.sin\frac{3B}{2}-2sin\frac{A-C}{2}.cos\frac{A-C}{2}=2sin\frac{A-C}{2}(sin\frac{3B}{2}-$$
$$cos\frac{A-C}{2})=2sin\frac{A-C}{2}\left(sin\frac{3B}{2}-\sin\left(\frac{\pi}{2}-\frac{A-C}{2}\right)\right)=$$
$$2sin\frac{A-C}{2}.2sin\frac{\frac{3B}{2}-\frac{\pi}{2}+\frac{A-C}{2}}{2}.cos\frac{\frac{3B}{2}+\frac{\pi}{2}-\frac{A-C}{2}}{2}=$$
$$4sin\frac{A-C}{2}.sin\frac{3B-\pi+A-C}{4}.cos\frac{3B+\pi-A+C}{4}=$$
$$4sin\frac{A-C}{2}.sin\frac{2B+(A+B)-\pi-C}{4}.cos\frac{2B+(C+B)+\pi-A}{4}=$$
$$4sin\frac{A-C}{2}.sin\frac{2B+(\pi-C)-\pi-C}{4}.cos\frac{2B+(\pi-A)+\pi-A}{4}=$$
$$4sin\frac{A-C}{2}.sin\frac{2B-2C}{4}.cos\frac{2B+2\pi-2A}{4}=4sin\frac{A-C}{2}.sin\frac{B-C}{2}.\cos\left(\frac{\pi}{2}+\frac{B-A}{2}\right)=$$
$$4sin\frac{A-C}{2}.sin\frac{B-C}{2}.\left(-sin\frac{B-A}{2}\right)=-4sin\frac{A-C}{2}.sin\frac{B-C}{2}.sin\frac{B-A}{2}=$$
$$-4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2}$$

$$\boxed{\sum\nolimits_2=-4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2}}$$

$$*\sum\nolimits_1=sinA+sinB+sinC\le\frac{3\sqrt{3}}{2}\quad Let's\ prove\ it.$$

Let $f(x)=\sin(x)\ \ x\in[0;180°]\ \ f''(x)=\sin(x)<0$

According to Jensen's theorem.

$$\frac{sinA+sinB+sinC}{3}\le\sin\left(\frac{A+B+C}{3}\right)\ \ sinA+sinB+sinC\le3\sin\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}\quad(1)$$

In triangle ΔABC ***wlog:*** $A\le B\le C$

$$\sum\nolimits_2=-4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2}\le0\ \ (2)$$

So,

$$\sin(A)\cos(B)+\sin(B)\cos(C)+\sin(C)\cos(A)=\frac{1}{2}(sinA+sinB+sinC)+$$
$$\left(-4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2}\right)\overset{2}{\le}\frac{1}{2}(sinA+sinB+sinC)\overset{1}{\le}\frac{3\sqrt{3}}{4}\quad(Proved)$$

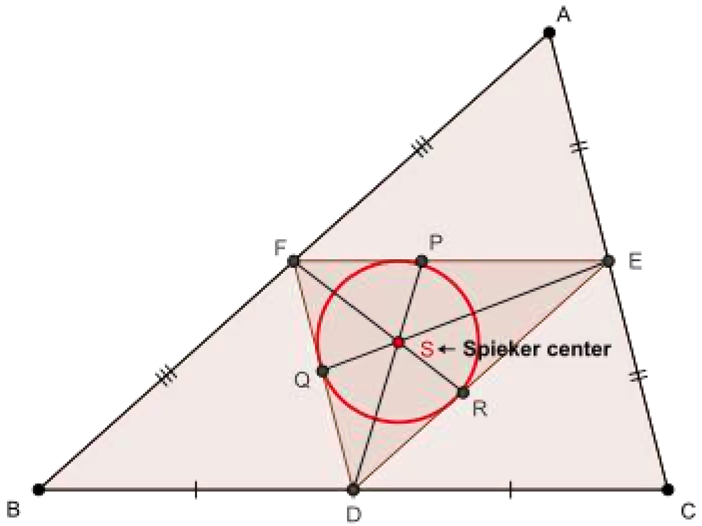
Equality holds if $a=b=c$

3010. In any Δ ABC, the following relationship holds :

$$\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\le 3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$(say)

$$\text{Now, }16[DEF]^2=2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right)-\sum_{cyc}\frac{a^4}{16}=\frac{1}{16}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$

$$=\frac{16r^2s^2}{16}\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because\text{ Spieker center is incenter of }\Delta DEF,\therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and }m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow \mathrm{c}\sin\alpha \overset{(***)}{=} \frac{\mathrm{r}(a+\mathrm{b})}{2\mathrm{AS}} \text{ and via sine law on } \Delta\,\mathrm{AES}, \mathrm{b}\sin\beta \overset{(****)}{=} \frac{\mathrm{r}(a+\mathrm{c})}{2\mathrm{AS}}$$

$$\text{Now, } [\mathrm{BAX}] + [\mathrm{BAX}] = [\mathrm{ABC}] \Rightarrow \frac{1}{2}\mathrm{p}_a\mathrm{c}\sin\alpha + \frac{1}{2}\mathrm{p}_a\mathrm{b}\sin\beta = \mathrm{rs}$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{\mathrm{p}_a(a+\mathrm{b}+a+\mathrm{c})}{4\mathrm{AS}} = \mathrm{s} \Rightarrow \mathrm{p}_a = \frac{4\mathrm{s}}{2\mathrm{s}+a}\mathrm{AS}$$

$$\Rightarrow \mathrm{p}_a^2 \overset{\text{via (ii)}}{=} \frac{16\mathrm{s}^2}{(2\mathrm{s}+a)^2}\cdot\frac{\mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2)}{8\mathrm{s}}$$

$$\therefore \mathrm{p}_a^2 \overset{(\bullet)}{=} \frac{2\mathrm{s}}{(2\mathrm{s}+a)^2}\left(\mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2)\right)$$

$$\text{Now, } \mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2) = \mathrm{b}^3+\mathrm{c}^3+a^3-a\mathrm{bc}+a(2\mathrm{b}^2+2\mathrm{c}^2-a^2)-a^3$$

$$= \sum_{\mathrm{cyc}} a^3 - a\mathrm{bc} + 2a.2\mathrm{bc}\cos\mathrm{A} = 2\mathrm{s}(\mathrm{s}^2-6\mathrm{Rr}-3\mathrm{r}^2) - 4\mathrm{Rrs} + 16\mathrm{Rrs}\cos\mathrm{A}$$

$$\Rightarrow \mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2) \overset{(\bullet\bullet)}{=} 2\mathrm{s}(\mathrm{s}^2-8\mathrm{Rr}-3\mathrm{r}^2+8\mathrm{Rr}\cos\mathrm{A})$$

$$= 2\mathrm{s}\left(\mathrm{s}^2-8\mathrm{Rr}-3\mathrm{r}^2+8\mathrm{Rr}\left(1-2\sin^2\frac{\mathrm{A}}{2}\right)\right) \therefore (\bullet),(\bullet\bullet) \Rightarrow$$

$$\mathrm{p}_a \overset{(\bullet\bullet\bullet)}{=} \frac{2\mathrm{s}}{2\mathrm{s}+a}\cdot\sqrt{\mathrm{s}^2-3\mathrm{r}^2-16\mathrm{Rr}\sin^2\frac{\mathrm{A}}{2}}$$

$$\text{We have : } \frac{\mathrm{p}_a}{\mathrm{w}_a}+\frac{\mathrm{p}_\mathrm{b}}{\mathrm{w}_\mathrm{b}}+\frac{\mathrm{p}_\mathrm{c}}{\mathrm{w}_\mathrm{c}}$$

$$= \sum_{\mathrm{cyc}}\left(\frac{2\mathrm{s}}{2\mathrm{s}+a}\cdot\sqrt{\mathrm{s}^2-3\mathrm{r}^2-16\mathrm{Rr}\sin^2\frac{\mathrm{A}}{2}}\cdot\frac{a(\mathrm{b}+\mathrm{c}).\sqrt{\mathrm{bc}}}{2a\mathrm{bc}.\sqrt{\mathrm{s}(\mathrm{s}-a)}}\right)$$

$$= \frac{2\mathrm{s}}{2\mathrm{s}(9\mathrm{s}^2+6\mathrm{Rr}+\mathrm{r}^2).8\mathrm{Rrs}}\cdot\sum_{\mathrm{cyc}}\left(\frac{(2\mathrm{s}+\mathrm{b})(2\mathrm{s}+\mathrm{c})a(\mathrm{b}+\mathrm{c}).\sqrt{\mathrm{bc}(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})}}{\sqrt{\mathrm{s}(\mathrm{s}-a)(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})}}\cdot\sqrt{\mathrm{s}^2-3\mathrm{r}^2-16\mathrm{Rr}\sin^2\frac{\mathrm{A}}{2}}\right)$$

$$= \frac{1}{(9\mathrm{s}^2+6\mathrm{Rr}+\mathrm{r}^2).8\mathrm{Rr}^2\mathrm{s}^2}\cdot\sum_{\mathrm{cyc}}\left(\sqrt{(2\mathrm{s}+\mathrm{b})(2\mathrm{s}+\mathrm{c})a(\mathrm{b}+\mathrm{c})\mathrm{bc}(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})}.\sqrt{(2\mathrm{s}+\mathrm{b})(2\mathrm{s}+\mathrm{c})a(\mathrm{b}+\mathrm{c})\left(\mathrm{s}^2-3\mathrm{r}^2-16\mathrm{Rr}\sin^2\frac{\mathrm{A}}{2}\right)}\right)$$

$$\overset{\mathrm{CBS}}{\leq} \frac{1}{(9\mathrm{s}^2+6\mathrm{Rr}+\mathrm{r}^2).8\mathrm{Rr}^2\mathrm{s}^2}\cdot\sqrt{x}.\sqrt{\mathrm{y}}$$

$$\left(\begin{array}{l}\text{where } x = 4\mathrm{Rrs}.\sum_{\mathrm{cyc}}\left((2\mathrm{s}+\mathrm{b})(2\mathrm{s}+\mathrm{c})(\mathrm{b}+\mathrm{c})(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})\right) \text{ and} \\ \mathrm{y} = \sum_{\mathrm{cyc}}\left((2\mathrm{s}+\mathrm{b})(2\mathrm{s}+\mathrm{c})a(\mathrm{b}+\mathrm{c})\left(\mathrm{s}^2-3\mathrm{r}^2-16\mathrm{Rr}\sin^2\frac{\mathrm{A}}{2}\right)\right)\end{array}\right)$$

$$\therefore \frac{\mathrm{p}_a}{\mathrm{w}_a}+\frac{\mathrm{p}_\mathrm{b}}{\mathrm{w}_\mathrm{b}}+\frac{\mathrm{p}_\mathrm{c}}{\mathrm{w}_\mathrm{c}} \overset{①}{\leq} \frac{1}{(9\mathrm{s}^2+6\mathrm{Rr}+\mathrm{r}^2).8\mathrm{Rr}^2\mathrm{s}^2}\cdot\sqrt{x}.\sqrt{\mathrm{y}}$$

$$\text{Now, } \sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)=\sum_{cyc}\left((8s^2-2sa+bc)(s-b)(s-c)\right)$$

$$=r^2s.\sum_{cyc}\left(\frac{2s(s-a)+6s^2+bc}{s-a}\right)=r^2s\left(6s+\frac{6s^2(4Rr+r^2)}{r^2s}+s.\frac{s^2+(4R+r)^2}{s^2}\right)$$

$$\Rightarrow \sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)\overset{(\blacksquare)}{=}6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2$$

and also,

$$\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right)=r^2s.\sum_{cyc}\frac{a\left(2s(s-a)+6s^2+bc\right)}{s-a}$$

$$=r^2s.\left(2s(2s)+6s^2.\sum_{cyc}\frac{a-s+s}{s-a}+\frac{4Rrs(4Rr+r^2)}{r^2s}\right)$$

$$=r^2s.\left(4s^2+6s^2.\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)+4R(4R+r)\right)$$

$$\Rightarrow \sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right)\overset{(\blacksquare\blacksquare)}{=}r^2s\left(-8s^2+\frac{24Rs^2}{r}+16R^2+4Rr\right)$$

$$\textit{and moreover}, 4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$$

$$=4Rrs.r^2s.\sum_{cyc}\frac{(2s+b)(2s+c)(s+s-a)}{s-a}$$

$$=\frac{4Rrs.r^2s^2}{r^2s}.\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)+$$

$$4Rrs.r^2s.\sum_{cyc}(8s^2-2sa+bc)\overset{\text{via }(\blacksquare)}{=}4Rrs^2\left(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2\right)$$

$$+4Rr^3s^2(21s^2+4Rr+r^2)\Rightarrow 4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$$

$$=x\overset{(\blacksquare\blacksquare\blacksquare)}{=}8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right)$$

$$\text{Now, } y=\sum_{cyc}\left((s^2-3r^2)(2s+b)(2s+c)a(b+c)\right)$$

$$-\frac{16Rr}{4Rrs}.\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\left(\sum_{cyc}ab-bc\right)\right)$$

$$=(s^2-3r^2)\sum_{cyc}\left(\left(\sum_{cyc}ab-bc\right)(8s^2-2sa+bc)\right)-$$

$$\frac{4}{s}\begin{pmatrix}(s^2+4Rr+r^2).\sum_{cyc}(a(2s+b)(2s+c)(s-b)(s-c))\\ -4Rrs.\sum_{cyc}((2s+b)(2s+c)(s-b)(s-c))\end{pmatrix}$$

$$\overset{via\,(\blacksquare)\,and\,(\blacksquare\blacksquare)}{=}(s^2-3r^2)(s^2+4Rr+r^2)(21s^2+4Rr+r^2)-$$

$$(s^2-3r^2)\left(8s^2(s^2+4Rr+r^2)-24Rrs^2+(s^2+4Rr+r^2)^2-16Rrs^2\right)$$

$$-\frac{4}{s}\begin{pmatrix}(s^2+4Rr+r^2).r^2s\left(-8s^2+\frac{24Rs^2}{r}+16R^2+4Rr\right)\\ -4Rrs.(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2)\end{pmatrix}$$

$$\Rightarrow y\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}4s^2\left(3s^4-(2Rr-2r^2)s^2-r^2(16R^2+10Rr+r^2)\right)$$

$$\therefore ①,(\blacksquare\blacksquare\blacksquare),(\blacksquare\blacksquare\blacksquare\blacksquare)\Rightarrow\left(\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\right)^2\le$$

$$\frac{8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right).4s^2\begin{pmatrix}3s^4-(2Rr-2r^2)s^2\\ -r^2(16R^2+10Rr+r^2)\end{pmatrix}}{(9s^2+6Rr+r^2)^2.64R^2r^4s^4}\rightarrow(a)$$

$$\text{Now},\left(3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\right)^2=9+\frac{32}{9}.\frac{(R-2r)^2}{4r^2}+4\sqrt{2}.\frac{R-2r}{r}$$

$$\ge 9+\frac{8(R-2r)^2}{9r^2}+\frac{28(R-2r)}{5r}\therefore\left(3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\right)^2$$

$$\ge\frac{40(R-2r)^2+252r(R-2r)+405r^2}{45r^2}\rightarrow(b)$$

$\therefore (a),(b)\Rightarrow$ **it suffices to prove :**

$$\frac{8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right).4s^2\begin{pmatrix}3s^4-(2Rr-2r^2)s^2\\ -r^2(16R^2+10Rr+r^2)\end{pmatrix}}{(9s^2+6Rr+r^2)^2.64R^2r^4s^4}$$

$$\le\frac{40(R-2r)^2+252r(R-2r)+405r^2}{45r^2}$$

$$\Leftrightarrow-(1620R+2295r)s^6+(6480R^3+14904R^2r+9522Rr^2-1665r^3)s^4$$

$$+r(8640R^4+30672R^3r+33948R^2r^2+9936Rr^3+675r^4)s^2$$

$$+r^2(2880R^5+13344R^4r+14600R^3r^2+5428R^2r^3+842Rr^4+4r^5)\overset{?}{\underset{②}{\ge}}0$$

$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\ge 0\text{ and }s^2-(m+n)\le 0,\text{ where }m=$$

$$2R^2+10Rr-r^2\text{ and }n=2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0$$

$$\Rightarrow P=-(1620R+2295r)s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$-r(26676R^2+33138Rr-2925r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$\geq 0 \therefore$ in order to prove ②, it suffices to prove : LHS of ② $\geq P \Leftrightarrow$

$$(702R^4-51345R^3r-54270R^2r^2+20484Rr^3-360r^4)s^2+$$

$$r(213768R^5+426828R^4r+217267R^3r^2+36170R^2r^3-140Rr^4-360r^5)\overset{③}{\geq}0$$

and it's trivially true if : $702R^4-51345R^3r-54270R^2r^2+20484Rr^3-360r^4 \geq 0$ and so, we now focus on the case when :

$702R^4-51345R^3r-54270R^2r^2+20484Rr^3-360r^4<0$ and then :

$$\text{LHS of ③} \overset{\text{Gerretsen}}{\geq} \left(\begin{matrix}702R^4-51345R^3r-54270R^2r^2+\\20484Rr^3-360r^4\end{matrix}\right)(4R^2+4Rr+3r^2)$$

$$+r(213768R^5+426828R^4r+217267R^3r^2+36170R^2r^3-140Rr^4-360r^5)\overset{?}{\geq}0$$

$$\Leftrightarrow 1404t^6+5598t^5+3237t^4-35956t^3-23072t^2+29936t-720\overset{?}{\geq}0$$

$$\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1404t^5+8406t^4+16352t^3+3697t(t^2-4)+4142t^2+360)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \Rightarrow ② \text{ is true} \Rightarrow \left(\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\right)^2 \leq \left(3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\right)^2$$

$$\therefore \frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c} \leq 3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\ \forall\, \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3011. If $m \geq 0, x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{xa^{2m}}{(y+z)^{m+1}h_a^2}+\frac{yb^{2m}}{(z+x)^{m+1}h_b^2}+\frac{zc^{2m}}{(x+y)^{m+1}h_c^2} \geq \frac{2^{m-1}(\sqrt{3})^{m+1}F^{m-1}}{(x+y+z)^m}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\frac{xa^{2m}}{(y+z)^{m+1}h_a^2}+\frac{yb^{2m}}{(z+x)^{m+1}h_b^2}+\frac{zc^{2m}}{(x+y)^{m+1}h_c^2}=\sum\frac{xa^{2m}}{(y+z)^{m+1}h_a^2}=$$

$$=\frac{1}{4F^2}\sum\frac{xa^{2m+2}}{(y+z)^{m+1}}=\frac{1}{4F^2}\sum\frac{x^{m+1}a^{2m+2}}{x^m(y+z)^{m+1}}=$$

$$=\frac{1}{4F^2}\sum\frac{\left(\frac{xa^2}{y+z}\right)^{m+1}}{x^m} \overset{Radon}{\geq} \frac{1}{4F^2}\frac{\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)^{m+1}}{(x+y+z)^m}=$$

$$\overset{Tsintsifas}{\geq} \frac{1}{2^2F^2}\frac{\left(2\sqrt{3}F\right)^{m+1}}{(x+y+z)^m}=\frac{2^{m-1}\left(\sqrt{3}\right)^{m+1}F^{m-1}}{(x+y+z)^m}$$

Equality holds for an equilateral triangle.

3012. In ΔABC the following relationship holds:

$$\frac{a^4}{m_bm_c}+\frac{b^4}{m_cm_a}+\frac{c^4}{m_am_b}\geq\frac{16}{3}\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$\frac{a^4}{m_bm_c}+\frac{b^4}{m_cm_a}+\frac{c^4}{m_am_b}=\sum\frac{a^4}{m_bm_c}\overset{CBS}{\geq}$$

$$\geq\frac{(a^2+b^2+c^2)^2}{\sum m_am_b}\geq\frac{(a^2+b^2+c^2)^2}{\sum m_a^2}=\frac{(a^2+b^2+c^2)^2}{\frac{3}{4}(a^2+b^2+c^2)}=$$

$$=\frac{4}{3}(a^2+b^2+c^2)\overset{Ionescu-Weitzenbock}{\geq}\frac{16}{3}\sqrt{3}F$$

Equality holds for an equilateral triangle.

3013.

In any Δ ABC, the following relationship holds :

$$\sqrt{w_ar_a}+\sqrt{w_br_b}+\sqrt{w_cr_c}\geq 2\sqrt[4]{8Rr^3}\left(\frac{r_a}{r+r_a}+\frac{r_b}{r+r_b}+\frac{r_c}{r+r_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\overset{?}{\geq}32Rr\cos^2\frac{A}{2}\overset{\text{via (i)}}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0 \Leftrightarrow (b+c-2a)^2\overset{?}{\geq}0 \to \text{true}$$

$$\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\ \textit{and analogs}\to(m)$$

$$\text{We have}:\frac{r_a}{r+r_a}=\frac{\frac{rs}{s-a}}{\frac{rs}{s}+\frac{rs}{s-a}}=\frac{s}{s+s-a}\Rightarrow\frac{r_a}{r+r_a}=\frac{s}{b+c}\therefore\sqrt{w_a r_a}.\frac{r+r_a}{r_a}$$

$$=\frac{1}{s}.\sqrt{\frac{2bc}{b+c}.\cos\frac{A}{2}.s\tan\frac{A}{2}.(b+c)^2}=\frac{1}{s}.\sqrt{2sbc.\sin\frac{A}{2}.(b+c)}$$

$$\overset{\text{via }(m)}{\geq}\frac{1}{s}.\sqrt{2sbc.\sin\frac{A}{2}\cos\frac{A}{2}.2.\sqrt{8Rr}}=\frac{2}{s}.\sqrt{sbc.\sqrt{8Rr}.\frac{a}{4R}}=\frac{2}{s}.\sqrt{\sqrt{8Rr}.\frac{4Rrs^2}{4R}}$$

$$=\frac{2}{s}.s.\sqrt[4]{8Rr}.\sqrt[4]{r^2}\Rightarrow\sqrt{w_a r_a}.\frac{r+r_a}{r_a}\geq 2.\sqrt[4]{8Rr^3}\Rightarrow\sqrt{w_a r_a}\geq 2.\sqrt[4]{8Rr^3}.\frac{r_a}{r+r_a}$$

$$\textit{and analogs}\therefore\sqrt{w_a r_a}+\sqrt{w_b r_b}+\sqrt{w_c r_c}\geq 2\sqrt[4]{8Rr^3}\left(\frac{r_a}{r+r_a}+\frac{r_b}{r+r_b}+\frac{r_c}{r+r_c}\right)$$

$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3014. In ΔABC the following relationship holds:

$$\frac{1}{\sqrt{h_a}}+\frac{2}{\sqrt{r_a}}\leq\frac{\sqrt{h_a}}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

We need to show:

$$\frac{1}{\sqrt{h_a}}+\frac{2}{\sqrt{r_a}}\leq\frac{\sqrt{h_a}}{r}\ or\ \sqrt{\frac{a}{2F}}+2\sqrt{\frac{s-a}{F}}\leq\frac{1}{r}\sqrt{\frac{2F}{a}}$$

$$or\ \frac{1}{\sqrt{2F}}\left(\sqrt{a}+2\sqrt{2(s-a)}\right)\leq\frac{1}{r}\sqrt{\frac{2F}{a}}\ \ or\left(a+2\sqrt{2(s-a)a}\right)\leq\frac{2F}{r}$$

$$or,\left(a+2\sqrt{2(s-a)a}\right)\leq\frac{2rs}{r}\ or\left(a+2\sqrt{2(s-a)a}\right)\leq 2s\ True$$

$$since:\left(a+2\sqrt{2(s-a)a}\right)\overset{AM-GM}{\leq}a+\frac{2(s-a)+a}{2}=a+(b+c)=2s$$

Equality holds for an equilateral triangle.

3015. In ΔABC the following relationship holds:

$$\frac{1}{w_a}+\frac{4}{h_a+r_a}\le\frac{1}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\frac{1}{w_a}+\frac{4}{h_a+r_a}\overset{w_a\ge h_a}{\le}\frac{1}{h_a}+\frac{4}{h_a+r_a}\overset{AM-HM}{\le}\frac{1}{h_a}+\frac{1}{h_a}+\frac{1}{r_a}=$$

$$=\frac{2}{h_a}+\frac{1}{r_a}=\frac{a}{F}+\frac{s-a}{F}=\frac{s}{F}=\frac{1}{r}$$

Equality holds for an equilateral triangle.

3016. In ΔABC the following relationship holds:

$$\sum w_a(h_b+h_c)\le 6\sqrt{3}F$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$h_b+h_c=\frac{ac}{2R}+\frac{ab}{2R}=\frac{a(b+c)}{2R},w_a=\frac{2bc}{b+c}\cos\frac{A}{2},w_a(h_b+h_c)=\frac{abc}{R}\cos\frac{A}{2}$$

$$\sum\cos\frac{A}{2}=\left(\sqrt{\sum\cos\frac{A}{2}}\right)^2\overset{CBS}{\le}\sqrt{3\sum\cos^2\frac{A}{2}}=\sqrt{3\left(2+\frac{r}{2R}\right)}\overset{Euler}{\le}\sqrt{3\left(2+\frac{1}{4}\right)}=\frac{3\sqrt{3}}{2}$$

$$\sum w_a(h_b+h_c)=\sum\frac{abc}{R}\cos\frac{A}{2}=4F\sum\cos\frac{A}{2}\le 4F\frac{3\sqrt{3}}{2}=6\sqrt{3}F$$

Equality holds for an equilateral triangle.

3017. In ΔABC the following relationship holds:

$$s^2\ge\frac{27Rr}{2}\cdot\frac{(a+b)^2}{4ab}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$s=\frac{a+b+c}{2},\frac{abc}{4s}=Rr$$

$$We\ need\ to\ show: s^2\ge\frac{27Rr}{2}\cdot\frac{(a+b)^2}{4ab}\ or\ s^2\ge\frac{27}{2}\cdot\frac{abc}{4s}\cdot\frac{(a+b)^2}{4ab}$$

$$or 32s^3\ge 27\,c\,(a+b)^2\ or\ 32\left(\frac{a+b+c}{2}\right)^3\ge 27\,c(a+b)^2$$

$$or\ 32\frac{(a+b+c)^3}{8}\ge\ 27\,c(a+b)^2\ \ or\ 4(a+b+c)^3\ge 27\,c(a+b)^2$$

$$true\ since: 4(a+b+c)^3=4\left(c+\frac{a+b}{2}+\frac{a+b}{2}\right)^3\overset{AM-GM}{\ge}4\left(3\sqrt[3]{\frac{c(a+b)^2}{4}}\right)^3=27c(a+b)^2$$

Equality holds for an equilateral triangle.

3018. In ΔABC the following relationship holds:

$$\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c}\ge 4-\frac{2r}{R}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\sum\frac{a}{s-a}=\frac{\sum a(s-b)(s-c)}{(s-a)(s-b)(s-c)}=\frac{\sum a(s^2-s(b+c)+bc)}{sr^2}=$$

$$=\frac{2s^3-2s(ab+bc+ca)+3abc}{sr^2}=$$

$$=\frac{2s^3-2s(s^2+r^2+4Rr)+12Rrs}{sr^2}=\frac{2(2R-r)}{r}$$

$$\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c}\overset{Panaitopol}{\ge}\frac{r_a}{\frac{Rh_a}{2r}}+\frac{r_b}{\frac{Rh_b}{2r}}+\frac{r_c}{\frac{Rh_c}{2r}}=\frac{2r}{R}\left(\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\right)=$$

$$=\frac{r}{R}\sum\frac{a}{s-a}=\frac{r}{R}\cdot\frac{2(2R-r)}{r}=4-\frac{2r}{R}$$

Equality holds for an equilateral triangle.

3019. In ΔABC the following relationship holds:

$$\frac{w_a+h_a+r_a}{R+r+r_a}\le\frac{h_a}{2r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\frac{w_a}{h_a}=\frac{2\sqrt{bcs(s-a)}}{b+c}\frac{2R}{bc}=\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2s-a)}=\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2(s-a)+a)}\overset{AM-GM}{\le}$$

$$\le\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}2\sqrt{2(s-a).a}}=\frac{2R\sqrt{s}}{\sqrt{2abc}}=\frac{2R\sqrt{s}}{\sqrt{8Rrs}}=\sqrt{\frac{R}{2r}}\ (1)$$

$$We\ need\ to\ show:$$

$$\frac{w_a+h_a+r_a}{R+r+r_a}\le\frac{h_a}{2r}\ or\ \frac{w_a+h_a+r_a}{h_a}\le\frac{R+r+r_a}{2r}\ or\ 1+\frac{w_a+r_a}{h_a}\le\frac{R}{2r}+\frac{1}{2}+\frac{r_a}{2r}$$

$$or\ \frac{w_a}{h_a}+\frac{r_a}{h_a}\le\frac{R}{2r}+\frac{r_a}{2r}-\frac{1}{2}\ or\ \frac{w_a}{h_a}+\frac{r_a}{h_a}\le\frac{R}{2r}+\frac{rs}{2r(s-a)}-\frac{1}{2}$$

$$or\ \frac{w_a}{h_a}+\frac{r_a}{h_a}\le\frac{R}{2r}+\frac{a}{2(s-a)}$$

$$or\ \sqrt{\frac{R}{2r}}+\frac{F}{s-a}\frac{a}{2F}\le\frac{R}{2r}+\frac{a}{2(s-a)}\ or\ \sqrt{\frac{R}{2r}}+\frac{a}{2(s-a)}\le\frac{R}{2r}+\frac{a}{2(s-a)}\ or$$

$$\sqrt{\frac{R}{2r}}\le\frac{R}{2r}\ or\ \frac{R}{2r}\le\frac{R^2}{4r^2}\ or\ R\ge 2r\ Euler$$

Equality holds for an equilateral triangle.

3020. In ΔABC the following relationship holds:

$$\frac{(a+b-c)R}{4F\tan\frac{A}{2}\tan\frac{B}{2}}\ge 1$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Tapas Das-India

$$\frac{(a+b-c)R}{4F\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{2(s-c)R}{4F\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}=$$

$$=\frac{2(s-c)R}{4F}\cdot\frac{s\sqrt{s-a}\sqrt{s-b}}{\sqrt{s-b}\sqrt{s-a}(s-c)}=\frac{2Rs}{4rs}=\frac{R}{2r}\geq 1(Euler)$$

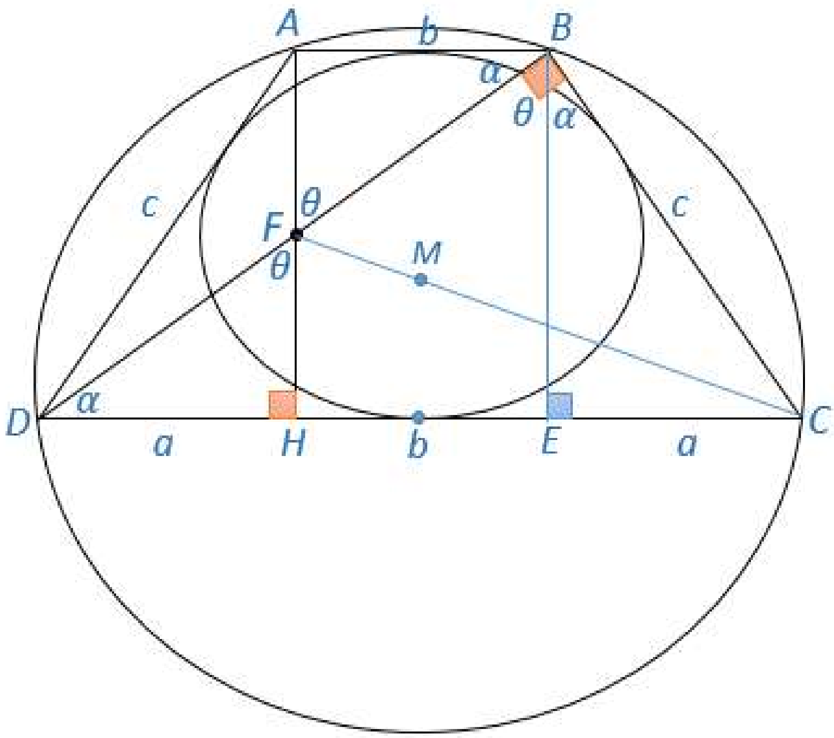
Equality holds for an equilateral triangle.

3021. ***Prove that:***

$$\frac{\overline{DB}}{\overline{AH}}=\frac{\overline{FH}}{\overline{FA}}=\varphi \text{ and } R^2=r^2\varphi^3$$

Proposed by Jafar Nikpour-Iran

Solution by Mirsadix Muzefferov-Azerbaijan



$$Let\ AD = c, AB = b, DH = a. Then\ BC = c, EC = a, HE = b.$$
$$Also, 2a + 2b = 2c \Rightarrow a + b = c$$
$$In\ \Delta CBD\ (B = 90^\circ)\ BE^2 = AH^2 = DE \cdot EC = (a+b)a\ (1)$$
$$In\ \Delta BEC\ (E = 90^\circ)\ AH^2 = BE^2 = BC^2 - CE^2 = c^2 - a^2 = (a+b)^2 - a^2$$
$$= b^2 + 2ab\ (2)$$
$$From\ (1)\ and\ (2)\ we\ have\ (a+b)a = b^2 + 2ab \Rightarrow a^2 - ab - b^2 = 0$$
$$\Rightarrow \left(\frac{a}{b}\right)^2 - \frac{a}{b} - 1 = 0$$
$$Let\ \frac{a}{b} = t, t^2 - t - 1 = 0 \Rightarrow t = \frac{1+\sqrt{5}}{2} = \varphi \Rightarrow \frac{a}{b} = \varphi$$
$$\Delta FHC \equiv \Delta FCB\ (FC\ common\ side, HC = CB\ and\ F\widehat{H}C = F\widehat{B}C = 90^\circ)$$
$$Therefore\ FB = FH, \Delta DBE \sim \Delta FAB \Rightarrow \frac{DB}{BE} = \frac{FB}{AF} \Rightarrow \frac{DB}{AH} = \frac{FH}{AF}$$
$$\Delta DFH \sim \Delta ABF \Rightarrow \frac{FH}{AF} = \frac{a}{b} = \varphi$$
$$(2) \Rightarrow 4r^2 = (2a+b)^2 \Rightarrow \frac{R^2}{r^2} = \frac{2a+b}{b} = 2\varphi + 1$$
$$As\ a\ known\ fact\ \varphi^2 - \varphi - 1 = 0 \Rightarrow \varphi^2 = \varphi + 1, \varphi^3 = \varphi^2 \cdot \varphi = (\varphi+1)\varphi = \varphi^2 + \varphi =$$
$$= \varphi + 1 + \varphi = 2\varphi + 1\ proved$$

3022. **If in $\Delta ABC, A = 108^\circ, B = C = 36^\circ$ then:**

$$w_b = 2m_a$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Using\ the\ known\ results:$$

$$\sin 108^\circ = \cos 18^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}, \sin 36^\circ = \frac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$a = 2R\sin A = 2R\sin 108^\circ = \frac{R}{2}\sqrt{10+2\sqrt{5}}, b = c = 2R\ \sin 36^\circ = \frac{R}{2}\sqrt{10-2\sqrt{5}}$$

$$2b^2 + 2c^2 - a^2 = 4b^2 - a^2 = 4.\frac{R^2}{4}\left(\sqrt{10-2\sqrt{5}}\right)^2 - \frac{R^2}{4}\left(\sqrt{10+2\sqrt{5}}\right)^2 =$$
$$= \frac{R^2}{4}(30 - 10\sqrt{5}) = \frac{R^2}{4}(5-\sqrt{5})^2, 2m_a = \sqrt{2b^2+2c^2-a^2} = \frac{R}{2}(5-\sqrt{5})\ (A)$$

$$\cos\frac{B}{2}=\cos 18^\circ=\frac{1}{4}\sqrt{10+2\sqrt{5}},2ac=2.\frac{R}{2}\sqrt{10+2\sqrt{5}}.\frac{R}{2}\sqrt{10-2\sqrt{5}}=$$

$$=\frac{R^2}{2}\sqrt{100-20}=\frac{R^2}{2}4\sqrt{5}=2\sqrt{5}R^2$$

$$a+c=2R(\sin A+\sin C)=4R\sin\frac{A+C}{2}\cos\frac{A-C}{2}=$$

$$=4R\sin 72^\circ\cos 36^\circ=4R.\frac{1}{4}\sqrt{10+2\sqrt{5}}\frac{\sqrt{5}+1}{4}=\frac{R}{4}(\sqrt{5}+1)\sqrt{10+2\sqrt{5}}$$

Using above result we get:

$$w_b=\frac{2ac}{a+c}\cos\frac{B}{2}=\frac{2\sqrt{5}R}{\sqrt{5}+1}=\frac{R}{2}(5-\sqrt{5})\ (B)$$

$$\textit{from }(A)\&(B)\textit{we get }w_b=2m_a$$

3023. If in $\Delta ABC, A=3B$ then:

$$(a-b)^2(a+b)=bc^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$A+B+C=\pi\ or, C=\pi-(A+B)=\pi-(3B+B)=\pi-4B$$
$$\sin C=\sin(\pi-4B)=\sin 4B$$

$$bc^2=2R\sin B\,.\,4R^2\sin^2 C=8R^3\sin B\,.\sin^2 4B\ \ (i)$$

$$(a-b)^2(a+b)=(a^2-b^2)(a-b)=4R^2(\sin^2 A-\sin^2 B)2R(\sin A-\sin B)$$
$$\overset{A=3B}{=}8R^3\sin(A+B)\sin(A-B)(\sin 3B-\sin B)=8R^3\sin 4B\sin 2B\cdot\ 2\cos 2B\sin B$$

$$=8R^3(2\sin 2B\cos 2B)\sin B\sin 4B=8R^3\sin B\,.\sin^2 4B\ \ (ii)$$

$$\textit{From }(i)\&(ii)\textit{ we get }(a-b)^2(a+b)=bc^2$$

3024. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{(b^3+c^3)\cot\frac{B}{2}\cdot\frac{\sin C}{\sin A}+(c^3+a^3)\cot\frac{C}{2}\cdot\frac{\sin B}{\sin A}}{\sin B+\sin C}\geq 288\sqrt{3}r^3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} = 2R(b^3+c^3)\cot\frac{B}{2}\left(\frac{\frac{c}{a}}{b+c}+\frac{\frac{a}{c}}{a+b}\right)+$$

$$+2R(c^3+a^3)\cot\frac{C}{2}\left(\frac{\frac{b}{a}}{b+c}+\frac{\frac{a}{b}}{a+c}\right)+2R(a^3+b^3)\cot\frac{A}{2}\left(\frac{\frac{c}{b}}{a+c}+\frac{\frac{b}{c}}{a+b}\right)\geq$$

$$\overset{A-G}{\geq}\frac{4R(b^3+c^3)\cot\frac{B}{2}}{\sqrt{(b+c)(a+b)}}+\frac{4R(c^3+a^3)\cot\frac{C}{2}}{\sqrt{(b+c)(c+a)}}+\frac{4R(a^3+b^3)\cot\frac{A}{2}}{\sqrt{(a+b)(c+a)}}\overset{A-G}{\geq}$$

$$\geq 12R\sqrt[3]{\left(\prod_{cyc}\frac{b^3+c^3}{b+c}\right)\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}} = 12R\sqrt[3]{\left(\prod_{cyc}(b^2-bc+c^2)\right).\frac{s}{r}}\overset{A-G}{\geq}$$

$$12R.\sqrt[3]{(a^2b^2c^2).\frac{s}{r}} = 12R.\sqrt[3]{16R^2r^2s^2.\frac{s}{r}}\overset{\text{Euler}}{\geq}12Rs.\sqrt[3]{64r^3}\overset{\substack{\text{Euler}\\ and\\ \text{Mitrinovic}}}{\geq}48r.2r.3\sqrt{3}r$$

$$\Rightarrow \text{LHS}\geq 288\sqrt{3}r^3\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\text{ is equilateral (QED)}$$

3025. If in $\Delta ABC, R = 7r$ then:

$$l_a^2+l_b^2+l_c^2\geq\frac{9}{14}\frac{(abc)^{\frac{4}{3}}}{\max(a^2,b^2,c^2)}$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$l_a^2+l_b^2+l_c^2\geq h_a^2+h_b^2+h_c^2=\frac{b^2c^2+a^2c^2+a^2b^2}{4R^2}\overset{AM-GM}{\geq}\frac{3(abc)^{\frac{4}{3}}}{4R^2}=$$

$$\overset{R=7r}{=}\frac{3(abc)^{\frac{4}{3}}}{4R.7r}=\frac{3}{14}\frac{(abc)^{\frac{4}{3}}}{2Rr}$$

We need to show:

$$\frac{3}{14}\frac{(abc)^{\frac{4}{3}}}{2Rr}\geq\frac{9}{14}\frac{(abc)^{\frac{4}{3}}}{\max(a^2,b^2,c^2)},\quad \max(a^2,b^2,c^2)\geq 6Rr$$

$$\frac{a^2+b^2+c^2}{3}\geq 6Rr,\ \ 2(s^2-r^2-4Rr)\geq 18Rr$$

$$s^2\geq 13Rr+r^2,\quad 16Rr-5r^2\geq 13Rr+r^2$$

$$3Rr\geq 6r^2,\quad R\geq 2r\ Euler$$

Equality holds for: $a=b=c$.

3026. If in $\Delta ABC, s=11r$ then:

$$r^2\leq\frac{\sqrt{3}}{44}\max(a^2,b^2,c^2)$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$\frac{a^2+b^2+c^2}{3}\cdot\frac{\sqrt{3}}{44}\overset{Ionescu-Weitzenbock}{\geq}\frac{4\sqrt{3}F}{3}\cdot\frac{\sqrt{3}}{44}=\frac{r.s}{11}\overset{s=11r}{\geq}r\cdot\frac{11r}{11}=r^2$$

$$Hence\ we\ can\ say\ \ r^2\leq\frac{a^2+b^2+c^2}{3}\cdot\frac{\sqrt{3}}{44}\leq\frac{\sqrt{3}}{44}\max(a^2,b^2,c^2)$$

3027. For any triangle ABC prove that :

$$6\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}\right)+\tan^2\frac{C}{2}\geq 3$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have

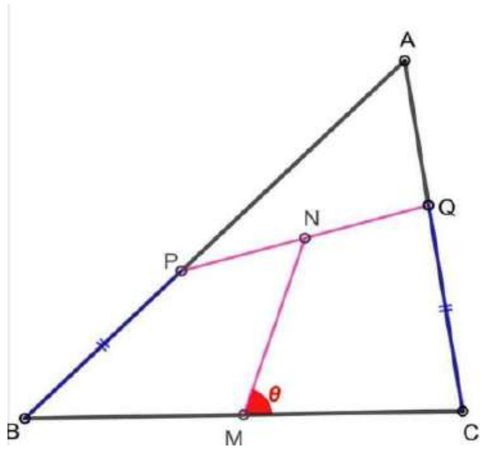
$$6\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}\right)+\tan^2\frac{C}{2}$$
$$=\frac{3}{2}\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}\right)+\frac{1}{2}\left(9\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)+\frac{1}{2}\left(9\tan^2\frac{A}{2}+\tan^2\frac{C}{2}\right)$$
$$\geq 3\tan\frac{A}{2}\tan\frac{B}{2}+3\tan\frac{B}{2}\tan\frac{C}{2}+3\tan\frac{C}{2}\tan\frac{A}{2}=3.$$

Equality holds iff $A = B = 2\tan^{-1}\left(\frac{1}{\sqrt{7}}\right), C = \pi - 4\tan^{-1}\left(\frac{1}{\sqrt{7}}\right)$.

3028. M, N *– are midpoints. Prove :*

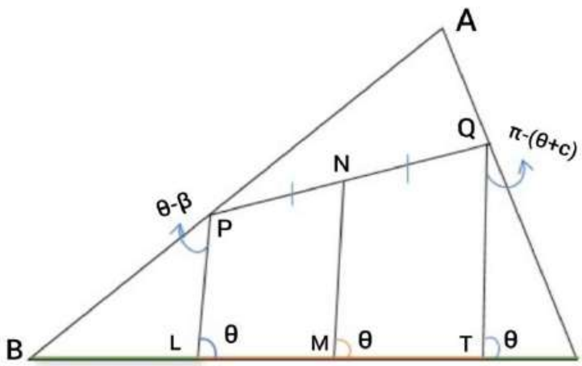
$$\cot\theta = \frac{b\cdot\cot B + c\cdot\cot A - b\cdot\csc A}{b+c}$$

$$\text{if } A = \frac{\pi}{2} \text{ then } \cot\theta = \frac{c-b}{c+b} = \frac{\cot B - 1}{\cot B + 1} = \cot\left(B + \frac{\pi}{4}\right)$$



Proposed by Thanasis Gakopoulos-Greece, Istvan Biro-Romania

Solution by Mirsadix Muzefferov-Azerbaijan



$$By\ construction\ PL \parallel NM \parallel QT. According\ to\ the\ given\ conditions.$$
$$BP = QC\ ;\ PN = NQ\ ;\ BM = MC$$
$$Then\ , according\ to\ Thales'\ theorem\ LM = MT.$$
$$Therefore\ BL = TC$$

$$\left.\begin{array}{l} In\ \Delta BPL\ rule\ sine\ \dfrac{\sin(\theta-B)}{BL}=\dfrac{\sin\theta}{BP} \\ In\ \Delta CQT\ rule\ sine\ \dfrac{\sin(\theta+C)}{TC}=\dfrac{\sin\theta}{QC}\end{array}\right\} \Rightarrow \sin(\theta-B)=\sin(\theta+C)\ (*)$$

$$(*)\Rightarrow \sin(\theta-B)\cdot\sin A=\sin(\theta+C)\ \cdot\sin A$$

$$\Rightarrow\frac{1}{2}\big(\cos(\theta-(A+B))-\cos(\theta+A-B)\big)=$$

$$=\frac{1}{2}\big(\cos(\theta+C-A)-\cos(\theta+(A+C))\big)$$

$$\cos(\theta-(A+B))-\cos(\theta+A-B)=\cos(\theta+C-A)-\cos(\theta+(\pi-B))$$

$$\cos(A+B-\theta)-\cos(\theta+(A-B))=\cos(\theta+(C-A))-\cos(\pi+(\theta-B))$$

$$\cos(A+B-\theta)-\cos(\theta+(A-B))=\cos(\theta+(C-A))-\cos(\theta-B)$$

$$Let's\ add\ (-\cos(B+\theta))\ to\ both\ sides$$

$$\cos(A+B-\theta)-\cos(\theta+(A-B))-\cos(B+\theta)-\cos(\theta+(C-A))=$$

$$=\cos(\theta-B)-\cos(B+\theta);$$

$$Here:-\cos(B+\theta)=-\cos(\pi-(A+C)+\theta)\cos(A+C-\theta)$$

$$\big(\cos(A+B-\theta)-\cos(\theta+(A-B))\big)+\big(\cos(A+C-\theta)-\cos(\theta+(C-A))\big)=$$

$$=\cos(B-\theta)-\cos(B+\theta)$$

$$-2\sin\frac{(A+B-\theta)-(\theta+(A-B))}{2}\cdot\sin\frac{(A+B-\theta)+(\theta+A-B)}{2}-$$

$$-2\sin\frac{(A+C-\theta)-(\theta+(C-A))}{2}\cdot\sin\frac{(A+C-\theta)+(\theta+C-A)}{2}=$$

$$=-2\sin\frac{(B-\theta)-(B+\theta)}{2}\cdot\sin\frac{(B-\theta)+(B+\theta)}{2}$$

$$\sin(B-\theta)\cdot\sin A+\sin(A-\theta)\cdot\sin C=-\sin\theta\cdot\sin B$$

$$Multiply\ both\ sides\ by\ (2R\sin B)$$

$$2R\sin B\cdot\sin(\theta-B)\cdot\sin A+2R\sin B\cdot\sin(\theta-A)\sin C=2R\sin B\sin\theta\sin B\Rightarrow$$

$$\Rightarrow b\sin(\theta-B)\cdot\sin A+c\sin B\cdot\sin(\theta-A)=b\sin B\sin\theta$$

$$b\sin A(\sin\theta\cos B-\sin B\cos\theta)+c\sin B(\sin\theta\cos A-\sin A\cos\theta)$$

$$=b\sin B\sin\theta$$

$$-b\sin A\sin B\cos\theta-c\sin A\sin B\cos\theta+b\sin A\cos B\sin\theta$$

$$+c\sin B\cos A\, sins\,\theta=b\sin B\sin\theta$$

$$(b+c)\sin A\sin B\cos\theta=b\cos B\sin A\sin\theta+c\cos A\sin B\sin\theta-b\sin B\sin\theta$$

$$Let's\ divide\ both\ sides\ by:(\sin\theta\sin A\sin B)$$

$$(b+C)\cot\theta=b\cot B+c\cot A-\frac{b}{\sin A}$$

$$\cot\theta=\frac{b\cot B+c\cot A-b\csc A}{b+c}\quad(proved)$$

$$in\ special\ case\ if\ A=\frac{\pi}{2}\ \Rightarrow\ \cot B=\frac{c}{b}\Rightarrow\cot\theta=\frac{b\cot B+c\cot A-b\csc A}{b+c}=$$

$$=\frac{b\cdot\frac{c}{b}+c\cdot\cot\frac{\pi}{2}-b\csc\frac{\pi}{2}}{b+c}=\frac{c-b}{c+b}=\frac{\frac{c}{b}-1}{\frac{c}{b}+1}=\frac{\cot B-1}{\cot B+1}=\cot\left(B+\frac{\pi}{4}\right)$$

3029. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{m_a m_b m_c}{s h_a h_b h_c}\cdot\sqrt[4]{\frac{w_a w_b w_c}{p_a p_b p_c}}\geq\frac{\sqrt{xy+yz+zx}}{xh_a+yh_b+zh_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have the following known formula (see [1, pp. 1]),

$$p_a^2=s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}.$$

And by the formulas for median and angle bisector of triangle ABC, $m_a^2=\frac{1}{4}(2b^2+2c^2-a^2)$

and $w_a=\frac{2\sqrt{bcs(s-a)}}{b+c}$, we can easily get

$$m_a^2=s(s-a)+\frac{(b-c)^2}{4}\ \text{ and }\ w_a^2=s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}.$$

Using these identities, we have

$$p_a w_a\leq\frac{p_a^2+w_a^2}{2}=s(s-a)+\frac{1}{2}\left(\frac{s(3s+a)}{(2s+a)^2}-\frac{s(s-a)}{(2s-a)^2}\right)(b-c)^2$$

$$=s(s-a)+\frac{s(4s^3-4s^2a+sa^2+a^3)(b-c)^2}{(4s^2-a^2)^2}$$

$$=s(s-a)+\left(\frac{1}{4}-\frac{4sa(s-a)(4s+a)+a^4}{4(4s^2-a^2)^2}\right)(b-c)^2\leq m_a^2.$$

Then

$$\sqrt[4]{\frac{w_aw_bw_c}{p_ap_bp_c}}=\sqrt[4]{\prod_{cyc}\frac{w_a{}^2}{p_aw_a}}\geq\sqrt[4]{\prod_{cyc}\frac{w_a{}^2}{m_a{}^2}}=\frac{\sqrt{\prod_{cyc}m_aw_a}}{m_am_bm_c}\overset{Panaitopol}{\geq}\frac{\sqrt{\prod_{cyc}s(s-a)}}{m_am_bm_c}$$
$$=\frac{s^2r}{m_am_bm_c},$$

and since $h_ah_bh_c=\dfrac{2s^2r^2}{R}$, then we get

$$\frac{m_am_bm_c}{sh_ah_bh_c}\cdot\sqrt[4]{\frac{w_aw_bw_c}{p_ap_bp_c}}\geq\frac{R}{2s^3r^2}\cdot s^2r=\frac{R}{2F}.$$

So it suffices to prove that

$$R(xh_a+yh_b+zh_c)\geq 2F\sqrt{xy+yz+zx}\ \text{ or } xbc+yca+zab\geq 4F\sqrt{xy+yz+zx}.$$

Let $u:=xbc, v:=yca, w:=zab$. The last inequality is equivalent to

$$u+v+w\geq 4F\sqrt{\frac{uv}{abc^2}+\frac{vw}{a^2bc}+\frac{wu}{ab^2c}}$$

$$\overset{squaring}{\Leftrightarrow}\quad u^2+v^2+w^2\geq 2\left(\frac{8F^2}{abc^2}-1\right)uv+2\left(\frac{8F^2}{a^2bc}-1\right)vw+2\left(\frac{8F^2}{ab^2c}-1\right)wu$$

$$\Leftrightarrow u^2+v^2+w^2$$
$$\geq 2(2\sin A\sin B-1)uv+2(2\sin B\sin C-1)vw$$
$$+2(2\sin C\sin A-1)wu,$$

and since $2\sin B\sin C=\cos A+\cos(B-C)$
$\leq\cos A+1$, so it suffices to prove that

$$u^2+v^2+w^2\geq 2\cos C\,.\,uv+2\cos A\,.\,vw+2\cos B\,.\,wu$$

$$\Leftrightarrow u^2+v^2+w^2\geq 2\cos C\,.\,uv+2(\sin B\sin C-\cos B\cos C).\,vw+2\cos B\,.\,wu$$

$$\Leftrightarrow (u-v\cos C-w\cos B)^2+(v\sin C-w\sin B)^2\geq 0,$$

which is true and the proof is complete. Equality holds iff ΔABC is equilateral.

[1] Bogdan Fuștei, Mohamed Amine Ben Ajiba,

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3030. Given triangle ABC such that $A = 2B$, prove that :

$$\frac{109}{64} \le 5\cos^2 A + 5\sin^2 B - \cos C < 6$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $C = \pi - A - B = \pi - 3B > 0$, then $B < \frac{\pi}{3}$. Let $x := \cos B \in \left(\frac{1}{2}, 1\right)$, we have

$$5\cos^2 A + 5\sin^2 B - \cos C = 5\cos^2 2B + 5\sin^2 B + \cos 3B$$

$$= 5(2x^2 - 1)^2 + 5(1 - x^2) + 4x^3 - 3x = 10 - 3x - 25x^2 + 4x^3 + 20x^4$$

$$= 6 - (1 - x)(2x + 1)(10x^2 + 7x - 4) < 6.$$

$$5\cos^2 A + 5\sin^2 B - \cos C = 10 - 3x - 25x^2 + 4x^3 + 20x^4$$

$$= \frac{109}{64} + \frac{531}{64} - 3x - 25x^2 + 4x^3 + 20x^4 = \frac{109}{64} + \left(x - \frac{3}{4}\right)^2\left(20x^2 + 34x + \frac{59}{4}\right) \ge \frac{109}{64},$$

with equality when $x = \frac{3}{4}$ or $B = \cos^{-1}\left(\frac{3}{4}\right), A = 2\cos^{-1}\left(\frac{3}{4}\right), C = \pi - 3\cos^{-1}\left(\frac{3}{4}\right)$.

Therefore

$$\frac{109}{64} \le 5\cos^2 A + 5\sin^2 B - \cos C < 6$$

3031. For a non – obtuse ΔABC, prove that

$$2(\tan A + \tan B) + \tan C \ge \frac{1}{4}\sqrt{5 + \sqrt{17}}(\sqrt{17} + 7)$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $x \to \tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$, then by Jensen's inequality, we have

$$2(\tan A+\tan B)+\tan C\geq 4\tan\left(\frac{A+B}{2}\right)+\tan C=\frac{4}{\tan\frac{C}{2}}+\frac{2\tan\frac{C}{2}}{1-\tan^2\frac{C}{2}}=f\left(\tan\frac{C}{2}\right),$$

where $f(x)=\frac{4}{x}+\frac{2x}{1-x^2}, x\in(0,1)$. We have

$$f'(x)=-\frac{4}{x^2}+\frac{2(1+x^2)}{(1-x^2)^2}=\frac{2(-2+5x^2-x^4)}{(1-x^2)^2x^2}=\frac{2\left(x^2-\frac{5-\sqrt{17}}{2}\right)\left(\frac{5+\sqrt{17}}{2}-x^2\right)}{(1-x^2)^2x^2},$$

then $\min\limits_{x\in(0,1)} f(x)=f\left(\sqrt{\frac{5-\sqrt{17}}{2}}\right)=\frac{1}{4}\sqrt{5+\sqrt{17}}(\sqrt{17}+7)$.

Therefore

$$2(\tan A+\tan B)+\tan C\geq\frac{1}{4}\sqrt{5+\sqrt{17}}(\sqrt{17}+7).$$

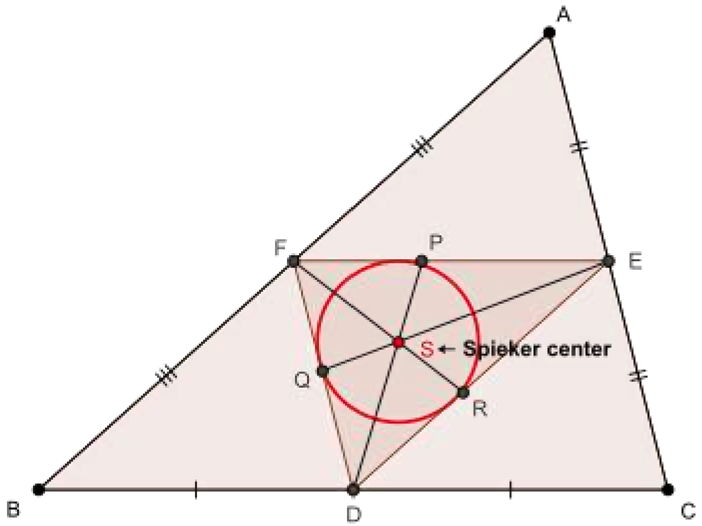
Equality holds iff $C=2\tan^{-1}\sqrt{\frac{5-\sqrt{17}}{2}}, A=B=\frac{\pi-C}{2}$.

3032. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt{(2r_a+h_a)g_a}\geq\sum_{cyc}\left(p_a\cdot\sqrt{\frac{2w_a}{r_b+r_c-4r}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say)

$$\textbf{and inradius of } \Delta\, DEF = r'(say)$$

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right) - \sum_{cyc}\frac{a^4}{16} = \frac{1}{16}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)$$

$$= \frac{16r^2s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore\ m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)$$

$$\text{Now, }b^3+c^3-abc+a\big(4m_a^2\big)=b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$=(b+c)\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2\big)+a\big(b^2+c^2-a^2\big)$$

$$=2s\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2+bc-a^2\big)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}=$$

$$(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}-\frac{a(b-c)^2}{4}$$

$$(a=y+z,b=z+x,c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right) \Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now}, m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow (*)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

and $b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$ and via summation, we get :

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2) =$$
$$2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) +$$
$$a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2$$
$$= 2(n_a^2+g_a^2) \Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2 + g_a^2 \boxed{\overset{(\bullet)}{=}} (b-c)^2 + 2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)}{bc}$$
$$= as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 \boxed{\overset{(\bullet\bullet)}{=}} s(s-a) + \frac{s}{a}(b-c)^2$$

Via $(\bullet)$ and $(\bullet\bullet)$, we get: $g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \dfrac{4s(s-b)(s-c)}{a}$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$
$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$
$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$
$$= (s-a)\left(s - a + \frac{a^2-(b-c)^2}{a}\right) \Rightarrow g_a^2 \overset{(\bullet\bullet\bullet)}{=} (s-a)\left(s - \frac{(b-c)^2}{a}\right)$$
$$\therefore (\bullet\bullet), (\bullet\bullet\bullet) \Rightarrow n_a^2 g_a^2 = s(s-a)\left(s - a + \frac{(b-c)^2}{a}\right)\left(s - \frac{(b-c)^2}{a}\right)$$
$$= s(s-a)\left(s(s-a) + s\frac{(b-c)^2}{a} - \frac{(b-c)^2}{a}(s-a) - \frac{(b-c)^4}{a^2}\right)$$
$$\Rightarrow n_a^2 g_a^2 \boxed{\overset{(\blacksquare)}{=}} s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2}\right)$$

Again, $m_a^2 w_a^2 = \dfrac{(b-c)^2 + 4s(s-a)}{4} \cdot \dfrac{4bcs(s-a)}{(b+c)^2}$

$$\Rightarrow m_a^2 w_a^2 \boxed{\overset{(\blacksquare\blacksquare)}{=}} s(s-a)\frac{bc}{(b+c)^2}\left((b-c)^2 + 4s(s-a)\right)$$
$$\therefore (\blacksquare), (\blacksquare\blacksquare) \Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2$$

$$= s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}-\frac{bc}{(b+c)^2}\left((b-c)^2+4s(s-a)\right)\right)$$

$$= s(s-a)\left(\begin{matrix} s(s-a)+(b-c)^2\left(\dfrac{a^2-(b-c)^2}{a^2}\right) \\ -\dfrac{bc}{(b+c)^2}\left((b-c)^2+(b+c)^2-a^2\right)\end{matrix}\right)$$

$$= s(s-a)\left(s(s-a)-bc+\left(a^2-(b-c)^2\right)\left(\frac{(b-c)^2}{a^2}+\frac{bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(\left((b+c)^2-a^2-4bc\right)+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left((b-c)^2-a^2+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}-1\right)$$

$$= \frac{s(s-a)}{4}.4(s-b)(s-c)\left(\frac{4(b-c)^2}{a^2}-\frac{(b-c)^2}{(b+c)^2}\right)=r^2s^2(b-c)^2\left(\frac{4}{a^2}-\frac{1}{(b+c)^2}\right)$$

$$= r^2s^2(b-c)^2\left(\frac{2}{a}+\frac{1}{b+c}\right)\left(\frac{2b+2c-a}{a(b+c)}\right)\geq 0$$

$$\Rightarrow n_a^2g_a^2\geq m_a^2w_a^2\Rightarrow n_ag_a\geq m_aw_a\to \circledast\circledast$$

We have : $(2r_a+h_a)(r_b+r_c-4r)=\left(\dfrac{2rs}{s-a}+\dfrac{2rs}{a}\right)\left(\dfrac{rs}{s-b}+\dfrac{rs}{s-c}-\dfrac{4rs}{s}\right)$

$$= \frac{2r^2s^2\left(s(s-c)+s(s-b)-4(s-b)(s-c)\right)}{a(s-a)(s-b)(s-c)}$$

$$= \frac{2r^2s^2}{a.r^2s}\left(s(2s-2s+a)-a^2+(b-c)^2\right)=2\left(s(s-a)+\frac{s}{a}(b-c)^2\right)\overset{\text{via }(\bullet\bullet)}{=}2n_a^2$$

$$\Rightarrow (2r_a+h_a)(r_b+r_c-4r).\frac{g_a}{p_a^2.2w_a}=\frac{n_a^2g_a}{p_a^2w_a}\overset{\text{via }\circledast\circledast}{\geq}\frac{n_am_aw_a}{p_a^2w_a}\overset{\text{via }\circledast\circledast}{\geq}1$$

$$\Rightarrow \sqrt{(2r_a+h_a)(r_b+r_c-4r).g_a}\geq p_a.\sqrt{2w_a}$$

$$\Rightarrow \sqrt{(2r_a+h_a)g_a}\geq p_a.\sqrt{\frac{2w_a}{r_b+r_c-4r}}\text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\sqrt{(2r_a+h_a)g_a}\geq\sum_{cyc}\left(p_a.\sqrt{\frac{2w_a}{r_b+r_c-4r}}\right)\forall\Delta ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3033. In any Δ ABC, the following relationship holds :

$$(n_a+n_b+n_c)^2 \le (8R+2r+h_a+h_b+h_c)(4R-5r)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2 \Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-a.\frac{4r^2s^2}{a(s-a)}=as^2-2a.\frac{2rs}{a}.\frac{rs}{s-a}=as^2-2ah_ar_a \Rightarrow n_a^2=s^2-2h_ar_a$$
$$=s^2-\frac{4rs.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}} \Rightarrow n_a^2=s^2-\frac{rs^2}{R}.\sec^2\frac{A}{2} \text{ and analogs}$$
$$\Rightarrow \sum_{cyc} n_a^2=3s^2-\frac{rs^2}{R}.\frac{s^2+(4R+r)^2}{s^2} \Rightarrow (n_a+n_b+n_c)^2 \le 3\sum_{cyc} n_a^2$$
$$=3.\frac{(3R-r)s^2-r(4R+r)^2}{R} \overset{?}{\le} (8R+2r+h_a+h_b+h_c)(4R-5r)$$
$$\Leftrightarrow 6\left((3R-r)s^2-r(4R+r)^2\right)-(4R-5r)(s^2+4Rr+r^2)$$
$$\overset{?}{\le} 2R(8R+2r)(4R-5r) \Leftrightarrow$$
$$(14R-r)s^2 \underset{(*)}{\overset{?}{\le}} 2R(8R+2r)(4R-5r)+6r(4R+r)^2+(4R-5r)(4Rr+r^2)$$
$$\text{Now, } (14R-r)s^2 \overset{\text{Gerretsen}}{\le} (14R-r)(4R^2+4Rr+3r^2) \overset{?}{\le} \text{RHS of } (*)$$
$$\Leftrightarrow 4t^3-2t^2-13t+2 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(4t^2+6t-1) \overset{?}{\ge} 0 \to \text{true}$$
$$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true } \therefore (n_a+n_b+n_c)^2 \le (8R+2r+h_a+h_b+h_c)(4R-5r)$$
$$\forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3034. In ΔABC the following relationship holds:

$$\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}+\frac{1}{b^2}\sqrt{\frac{c^2+a^2}{c^2+ca+a^2}}+\frac{1}{c^2}\sqrt{\frac{a^2+b^2}{a^2+ab+b^2}} \le \frac{1}{4r^2}\cdot\sqrt{\frac{R}{3r}}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}=\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+c^2+bc}}\overset{AM-GM}{\le}\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{3bc}}=$$

$$=\frac{1}{a^2}\sqrt{\frac{1}{3}\left(\frac{b}{c}+\frac{c}{b}\right)}\overset{Bandila}{\le}\frac{1}{a^2}\sqrt{\frac{R}{3r}}\ (1)$$

$$\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}+\frac{1}{b^2}\sqrt{\frac{c^2+a^2}{c^2+ca+a^2}}+\frac{1}{c^2}\sqrt{\frac{a^2+b^2}{a^2+ab+b^2}}=$$

$$=\sum\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}\overset{(1)}{\le}\sum\frac{1}{a^2}\sqrt{\frac{R}{3r}}\overset{Steining}{\le}\sqrt{\frac{R}{3r}}\cdot\frac{1}{4r^2}$$

Equality holds for an equilateral triangle.

3035. **In ΔABC the following relationship holds:**

$$\sum\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}\le\frac{3}{2}\left(1+\frac{R}{r}\right)$$

Proposed by Marian Ursărescu-Romania

Solution by Tapas Das-India

$$\sum\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}=\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)\cos C$$

$$\sum\left(\frac{a}{b}+\frac{b}{a}\right)\cos C=\sum\left(\frac{\sin A}{\sin B}+\frac{\sin B}{\sin A}\right)\cos C=\sum\left(\frac{\sin A\cos C}{\sin B}+\frac{\cos A\sin C}{\sin B}\right)=$$

$$=\sum\frac{\sin(A+C)}{\sin B}\overset{A+B+C=\pi}{=}\sum\frac{\sin B}{\sin B}=3$$

$$\sum\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}=\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)\cos C\overset{Bandila}{\le}\frac{1}{2}\cdot\frac{3R}{r}+\frac{3}{2}=\frac{3}{2}\left(1+\frac{R}{r}\right)$$

Equality holds for an equilateral triangle.

3036. **In acute ΔABC the following relationship holds:**

$$\frac{1}{\prod\cos A}+128\prod\sin\frac{A}{2}\ge 24$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{\prod\cos A}+128\prod\sin\frac{A}{2}=\frac{4R^2}{s^2-(2R+r)^2}+128\frac{r}{4R}\overset{Gerretsen}{\ge}$$

$$\ge\frac{4R^2}{4R^2+4Rr+3r^2-(2R+r)^2}+128\frac{r}{4R}=\frac{4R^2}{2r^2}+32\frac{r}{R}=\frac{2R^2}{r^2}+\frac{32r}{R}$$

We need to show:

$$\frac{2R^2}{r^2}+\frac{32r}{R}\ge 24\ or\ 2x^2+\frac{16}{x}\overset{\frac{R}{r}=x\ge 2}{\ge}12\ or$$

$$x^3-12x+16\ge 0\ or\ (x-2)^2(x+4)\ge 0\ true$$

Equality holds for an equilateral triangle.

3037. In ΔABC the following relationship holds:

$$\sum\frac{bc}{h_a}\le\sum\frac{bc}{r_a}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$In\ \Delta ABC\ wlog\ a\le b\le c\to h_a\ge h_b\ge h_c\to\begin{cases}\frac{1}{h_a}\le\frac{1}{h_b}\le\frac{1}{h_c}\ (1)\\ bc\ge ac\ge ab\ (2)\end{cases}$$

Let us consider conditions (1) *and* (2) *in the Chebyshev's inequality as well.*

$$\sum_{cyc}\frac{bc}{h_a}\le\frac{1}{3}(bc+ac+ab)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{1}{3}\left(\sum_{cyc}bc\right).\frac{1}{r}=\frac{1}{3}\left(\sum_{cyc}bc\right)\left(\sum_{cyc}\frac{1}{r_a}\right)\ (*)$$

$$Again\ wlog\ a\le b\le c\to r_a\le r_b\le r_c\to\begin{cases}\frac{1}{r_a}\ge\frac{1}{r_b}\ge\frac{1}{r_c}\ (3)\\ bc\ge ac\ge ab\ (4)\end{cases}$$

According to Chebyshev's inequality :

$$\left(\sum_{cyc}bc\right)\left(\sum_{cyc}\frac{1}{r_a}\right)\le 3\sum\frac{bc}{r_a}\ (**)$$

Let's use (**) *in* (*):

$$\sum\frac{bc}{h_a}\le 3.\frac{1}{3}\sum\frac{bc}{r_a}\to\sum\frac{bc}{h_a}\le\sum\frac{bc}{r_a}\ (Proved)$$

$$Equality\ \ holds\ \ if\ \ triangle\ \ is\ \ an\ equilateral\ one.$$

3038. In ΔABC the following relationship holds:

$$(1-\cos A)(1-\cos B)(1-\cos C)\geq \cos A\cos B\cos C$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(1-\cos A)(1-\cos B)(1-\cos C)\geq \cos A\cos B\cos C$$

$$1-\sum\cos A+\sum\cos A\cos B-\prod\cos A\geq\prod\cos A$$

$$1-\sum\cos A+\sum\cos A\cos B-2\prod\cos A\geq 0$$

$$1-\sum\cos A+\sum\cos A\cos B-2\prod\cos A=$$

$$=1-\left(1+\frac{r}{R}\right)+\frac{s^2+r^2-4R^2}{4R^2}-2.\frac{s^2-(2R+r)^2}{4R^2}=$$

$$=\frac{1}{4R^2}(2(2R+r)^2-4R^2+r^2-4Rr-s^2)=$$

$$=\frac{1}{4R^2}(4R^2+4Rr+3r^2-s^2)\overset{GERRETSEN}{\geq}\frac{1}{4R^2}(s^2-s^2)=0$$

Equality holds for an equilateral triangle.

3039. In ΔABC the following relationship holds:

$$\frac{1}{\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}}+\frac{1}{\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}}+\frac{1}{\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}}}\geq\frac{3}{\left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum \cos\frac{A}{2} \overset{Jensen}{\leq} 3\cos\left(\frac{A+B+C}{6}\right) = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2} \ (1)$$

$$\sum \sin A = \frac{s}{R} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}}{2} \ (2)$$

$$\sum \cos\frac{A}{2} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \cos\frac{A}{2}} = 3\sqrt[3]{\frac{s}{4R}}$$

$$\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}.\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}.\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}} \overset{AM-GM}{\leq}$$

$$\leq \left(\frac{\sin\frac{A}{2}\cos\frac{A}{2} + \sin\frac{B}{2}\cos\frac{B}{2} + \sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}\right)^{\sum\cos\frac{A}{2}} =$$

$$= \left(\frac{1}{2}\frac{\sin A + \sin B + \sin C}{\sum\cos\frac{A}{2}}\right)^{\sum\cos\frac{A}{2}} \overset{(1),(2),(3)}{\leq} \left(\frac{1}{2}\frac{s}{R}\frac{1}{3}.\sqrt[3]{\left(\frac{4R}{s}\right)}\right)^{\frac{3\sqrt{3}}{2}} =$$

$$= \left(\frac{1}{6}\left(\frac{s}{R}\right)^{\frac{2}{3}} 2^{\frac{2}{3}}\right)^{\frac{3\sqrt{3}}{2}} \overset{(2)}{\leq} \left(\frac{1}{6}\left(\frac{3\sqrt{3}}{2}\right)^{\frac{2}{3}} 2^{\frac{2}{3}}\right)^{\frac{3\sqrt{3}}{2}} = \left(\frac{1}{2}\right)^{\frac{3\sqrt{3}}{2}} \ (4)$$

$$\frac{1}{\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}} + \frac{1}{\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}} + \frac{1}{\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}}} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{1}{\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}.\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}.\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}}}} \overset{(4)}{\geq} 3\left(\frac{1}{\left(\frac{1}{2}\right)^{\frac{3\sqrt{3}}{2}}}\right)^{\frac{1}{3}} = \frac{3}{\left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}}$$

Equality holds for an equilateral triangle.

3040. In ΔABC the following relationship holds:

$$\frac{1+cosA}{sinA} + \frac{1+cosB}{sinB} + \frac{1+cosC}{sinC} \geq 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1+cosA}{sinA}+\frac{1+cosB}{sinB}+\frac{1+cosC}{sinC}=\sum_{cyc}\frac{1+cosA}{sinA}=$$

$$=\sum_{cyc}\frac{1+cos\left(2\cdot\frac{A}{2}\right)}{sin\left(2\cdot\frac{A}{2}\right)}=\sum_{cyc}\frac{1+2cos^2\frac{A}{2}-1}{2sin\frac{A}{2}cos\frac{B}{2}}=$$

$$=\sum_{cyc}cot\frac{A}{2}\overset{JENSEN}{\gtrsim}3cot\left(\frac{A+B+C}{6}\right)=3cot\frac{\pi}{6}=3\sqrt{3}$$

Equality holds for $A=B=C$.

3041. In ΔABC the following relationship holds:

$$\frac{b}{a^2c^2}\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right)\geq\frac{6\sin^3 B}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right)=\left(\sum a^2\right)\left(\sum a\right)-\sum a^3=$$

$$=2s(2s^2-2r^2-8Rr-s^2+3r^2+6Rr)=$$

$$=2s(s^2+r^2-2Rr)\overset{Gerretsen}{\geq}2s(16Rr-5r^2+r^2-2Rr)=2s(14Rr-4r^2)$$

$$\frac{6\sin^3 B}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{6b^3}{8R^3}\cdot\frac{4R}{s}=\frac{3b^3}{R^2s}$$

We need to show:

$$\frac{b}{a^2c^2}2s(14Rr-4r^2)\geq\frac{3b^3}{R^2s}\ or\ \frac{2s^2R^2}{a^2b^2c^2}(14Rr-4r^2)\geq 3$$

$$\frac{2s^2R^2}{16R^2r^2s^2}(14Rr-4r^2)\geq 3\ or\ (14Rr-4r^2)\geq 24r^2\ or$$

$$14Rr\geq 28r^2\ or\ R\geq 2r\ true.$$

Equality holds for an equilateral triangle.

3042. If G –centroid in ΔABC the following relationship holds:

$$\frac{GA^2}{bc}+\frac{GB^2}{ca}+\frac{GC^2}{ab}\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum am_a^2=\frac{1}{4}\sum a(2b^2+2c^2-a^2)=\frac{1}{4}\left(2\sum a(b^2+c^2)-\sum a^3\right)\ (1)$$

$$2\sum a(b^2+c^2)-\sum a^3=2\sum a^2(b+c)-\sum a^3=2\sum a^2(2s-a)-\sum a^3=$$

$$=4s\sum a^2-2\sum a^3-\sum a^3=$$

$$=8s(s^2-r^2-4Rr)-6s(s^2-3r^2-6Rr)=$$

$$=8s^3-8s(4Rr+r^2)-6s^3+6s(6Rr+3r^2)=$$

$$=2s^3-2s(16Rr+4r^2-9r^2-18Rr)=$$

$$=2s(s^2+2Rr+5r^2)\overset{Gerretsen}{\geq}2s(16Rr-5r^2+2Rr+5r^2)=36Rrs\ (2)$$

$$\sum a.m_a^2=\frac{1}{4}\sum a(2b^2+2c^2-a^2)=\frac{1}{4}\left(2\sum a(b^2+c^2)-\sum a^3\right)\overset{(2)}{\geq}$$

$$\geq\frac{1}{4}\cdot 36Rrs=9Rrs$$

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}=\sum\frac{m_a^2}{bc}=\frac{1}{abc}\sum am_a^2\geq\frac{1}{abc}.9Rrs=\frac{9Rrs}{4Rrs}=\frac{9}{4}$$

$$\frac{GA^2}{bc}+\frac{GB^2}{ca}+\frac{GC^2}{ab}=\frac{\left(\frac{2}{3}m_a\right)^2}{bc}+\frac{\left(\frac{2}{3}m_b\right)^2}{ca}+\frac{\left(\frac{2}{3}m_c\right)^2}{ab}=\frac{4}{9}\left(\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\right)\geq\frac{4}{9}.\frac{9}{4}=1$$

Equality holds for an equilateral triangle.

3043. **In any Δ ABC the following relationship holds :**

$$\frac{\cos B\cos C}{\cos\frac{B-C}{2}}+\frac{\cos C\cos A}{\cos\frac{C-A}{2}}+\frac{\cos A\cos B}{\cos\frac{A-B}{2}}\leq\frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\cos B\cos C}{\cos\frac{B-C}{2}}+\frac{\cos C\cos A}{\cos\frac{C-A}{2}}+\frac{\cos A\cos B}{\cos\frac{A-B}{2}}=$$

$$=\frac{1}{\prod_{cyc}\cos\frac{B-C}{2}}\cdot\sum_{cyc}\left(\cos B\cos C\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right)\le$$

$$\le\frac{1}{\prod_{cyc}\left(\frac{b+c}{a}.\sin\frac{A}{2}\right)}\cdot\sum_{cyc}\cos B\cos C$$

$$\left(\because 0<\cos\frac{B-C}{2}\le 1 \text{ and analogs and } \cos B\cos C \text{ and analogs} > 0\right)$$

$$=\frac{4Rrs}{4s(s^2+2Rr+r^2).\frac{r}{4R}}\cdot\left(\left(\frac{R+r}{R}\right)^2-3+\frac{s^2-4Rr-r^2}{2R^2}\right)$$

$$=\frac{4R^2}{s^2+2Rr+r^2}\cdot\frac{s^2-4R^2+r^2}{2R^2}=\frac{2(s^2-4R^2+r^2)}{s^2+2Rr+r^2}\overset{?}{\le}\frac{3}{4}$$

$$\Leftrightarrow 32R^2+6Rr-5r^2\underset{①}{\overset{?}{\ge}}5s^2$$

$$\text{Now, }5s^2\overset{\text{Gerretsen}}{\le}20R^2+20Rr+15r^2\overset{?}{\le}32R^2+6Rr-5r^2$$

$$\Leftrightarrow 6R^2-7Rr-10r^2\overset{?}{\ge}0\Leftrightarrow(6R+5r)(R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\Rightarrow ①\text{ is true}\therefore\frac{\cos B\cos C}{\cos\frac{B-C}{2}}+\frac{\cos C\cos A}{\cos\frac{C-A}{2}}+\frac{\cos A\cos B}{\cos\frac{A-B}{2}}\le\frac{3}{4}$$

$$\forall\,\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3044. In any Δ ABC, the following relationship holds :

$$\frac{a}{h_a+r_a}+\frac{b}{h_b+r_b}+\frac{c}{h_c+r_c}\ge\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{h_a+r_a}+\frac{b}{h_b+r_b}+\frac{c}{h_c+r_c}=\sum_{cyc}\frac{a}{\frac{2rs}{a}+\frac{rs}{s-a}}=\sum_{cyc}\frac{a^2(s-a)}{rs(b+c)}$$

$$=\sum_{cyc}\frac{a^2(2s-a-s)}{rs(b+c)}=\frac{1}{rs}\cdot\sum_{cyc}a^2-\frac{1}{r}\left(\sum_{cyc}\frac{4s^2}{b+c}+\sum_{cyc}\frac{(2s-a)^2}{2s-a}-\sum_{cyc}\frac{4s(2s-a)}{2s-a}\right)$$

$$=\frac{2(s^2-4Rr-r^2)}{rs}-\frac{1}{r}\left(\frac{4s^2(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}+4s-12s\right)$$

$$=\frac{2(s^2-4Rr-r^2)}{rs}-\frac{2s(s^2-4Rr-3r^2)}{r(s^2+2Rr+r^2)}$$

$$=\frac{2\left((2R+3r)s^2-r(8R^2+6Rr+r^2)\right)}{s(s^2+2Rr+r^2)}\overset{?}{\geq}\sqrt{3}$$

$$\Leftrightarrow\frac{4\left((2R+3r)s^2-r(8R^2+6Rr+r^2)\right)^2}{s^2(s^2+2Rr+r^2)^2}\overset{?}{\geq}3$$

$$\Leftrightarrow -3s^6+(16R^2+36Rr+30r^2)s^4-r(128R^3+300R^2r+172Rr^2+27r^3)s^2$$

$$+r^2(256R^4+384R^3r+208R^2r^2+48Rr^3+4r^4)\underset{(1)}{\overset{?}{\geq}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m=$ $2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$

$\Rightarrow -3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0$ *and* so, in order to prove (1), it suffices to prove :

$$\text{LHS of (1)}\overset{?}{\geq}-3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow (R-3r)^2s^4+r(16R^3-39R^2r-34Rr^2-6r^3)s^2$$

$$+r^2(64R^4+96R^3r+52R^2r^2+12Rr^3+r^4)\underset{(2)}{\overset{?}{\geq}}0 \text{ and}$$

$\because (R-3r)^2(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to prove (2), it suffices to prove : LHS of (2) $\overset{?}{\geq}(R-3r)^2(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (48R^3-241R^2r+314Rr^2-96r^3)s^2\underset{(3)}{\overset{?}{\geq}}$$

$$r(192R^4-1792R^3r+3237R^2r^2-1602Rr^3+224r^4)$$

Case 1 $48R^3-241R^2r+314Rr^2-96r^3\geq 0$ *and* then : LHS of (3) $\overset{\text{Gerretsen}}{\geq}$ $(48R^3-241R^2r+314Rr^2-96r^3)(16Rr-5r^2)\overset{?}{\geq}$ RHS of (3)

$$\Leftrightarrow 36t^4-144t^3+187t^2-94t+16\overset{?}{\geq}0\quad\left(t=\frac{R}{r}\right)$$

$\Leftrightarrow (t-2)(36t^2(t-2)+43t-8)\overset{?}{\geq}0\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow$ (3) is true

Case 2 $48R^3-241R^2r+314Rr^2-96r^3<0$ *and* then : LHS of (3) $\overset{\text{Gerretsen}}{\geq}$ $(48R^3-241R^2r+314Rr^2-96r^3)(4R^2+4Rr+3r^2)\overset{?}{\geq}$ RHS of (3)

$$\Leftrightarrow 48t^5-241t^4+557t^3-772t^2+540t-128\overset{?}{\geq}0$$

$\Leftrightarrow (t-2)\left((t-2)(48t^3-49t^2+169t+100)+264\right)\overset{?}{\geq}0\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2$ $\Rightarrow$ (3) is true $\therefore$ combining both *cases*, (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is true $\forall\,\Delta$ ABC

$\therefore\frac{a}{h_a+r_a}+\frac{b}{h_b+r_b}+\frac{c}{h_c+r_c}\geq\sqrt{3}\;\forall\,\Delta$ ABC, " = " iff Δ ABC is equi*lateral* (QED)

3045. **In ΔABC the following relationship holds:**

$$\sum \frac{\cos\frac{A-B}{2}}{12\sin^2\frac{C}{2}} \geq 1$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$Mollweide's: \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}} = \frac{a+b}{c}$$

$$\frac{\cos\frac{A-B}{2}}{12\sin^2\frac{C}{2}} = \frac{1}{12}\cdot\frac{a+b}{c}\cdot\frac{1}{\sin\frac{C}{2}} \overset{AM-GM}{\geq} \frac{2\sqrt{ab}}{12c}\cdot\frac{1}{\sin\frac{C}{2}} = \frac{\sqrt{ab}}{6c}\cdot\frac{1}{\sin\frac{C}{2}}$$

$$\sum \frac{\cos\frac{A-B}{2}}{12\sin^2\frac{C}{2}} \geq \sum \frac{\sqrt{ab}}{6c}\frac{1}{\sin\frac{C}{2}} \overset{AM-GM}{\geq} \frac{3}{6}\sqrt[3]{\prod \csc\frac{C}{2}} = \frac{1}{2}\left(\frac{4R}{r}\right)^{\frac{1}{3}} \overset{Euler}{\geq} \frac{1}{2}(8)^{\frac{1}{3}} = 1$$

Equality holds for an equilateral triangle.

3046. **In any Δ ABC, the following relationship holds :**

$$4h_bh_c - r_br_c \leq 9rr_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$4h_bh_c - r_br_c \leq 9rr_a \Leftrightarrow \frac{16r^2s^2}{bc} - s(s-a) \leq \frac{9r^2s}{s-a}$$

$$\Leftrightarrow 16s(s-a)(s-b)(s-c) - bcs(s-a) \leq 9bc(s-b)(s-c)$$

$$\Leftrightarrow 16s(s-a)(-s(s-a)+bc) - bcs(s-a) \leq 9bc(-s(s-a)+bc)$$

$$\Leftrightarrow 16s^2(s-a)^2 - 24bcs(s-a) + 9b^2c^2 \geq 0 \Leftrightarrow (4s(s-a)-3bc)^2 \geq 0$$

$$\Leftrightarrow ((b+c)^2 - a^2 - 3bc)^2 \geq 0 \Leftrightarrow (b^2+c^2-a^2-bc)^2 \geq 0$$

$$\Leftrightarrow (2bc\cos A - bc)^2 \geq 0 \Leftrightarrow 4b^2c^2\left(\cos A - \frac{1}{2}\right)^2 \geq 0$$

$$\to \text{true} \ \therefore 4h_bh_c - r_br_c \leq 9rr_a$$

$$\forall \Delta ABC,'' ='' \text{ iff } \widehat{A} = 60^\circ \text{ (QED)}$$

3047. In any ΔABC, the following relationship holds :

$$2m_a(n_a + w_a + g_a) \geq 3r_br_c + n_a^2 + w_a^2 + g_a^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$an_a^2 . ag_a^2 \overset{?}{\geq} a^2s^2(s-a)^2 \Leftrightarrow$$

$$\left(b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\right)\begin{pmatrix} b^2(s-b)+c^2(s-c) \\ -a(s-b)(s-c)\end{pmatrix} \underset{(*)}{\overset{?}{\geq}} a^2s^2(s-a)^2$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$ and via such substitutions, $(*) \Leftrightarrow$

$$\left(z(z+x)^2+y(x+y)^2-yz(y+z)\right)\left(y(z+x)^2+z(x+y)^2-yz(y+z)\right)$$
$$\geq x^2(y+z)^2(x+y+z)^2 \Leftrightarrow xy^2+xz^2+y^3+z^3 \geq 2xyz+yz(y+z)$$
$$\Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \geq 0 \to \text{true} \Rightarrow (*) \text{ is true} \Rightarrow n_ag_a \geq s(s-a) \to (1)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2 + a(s-b)(s-c)$

and $b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2 + a(s-b)(s-c)$ and $(m)+(n) \Rightarrow$

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2) = 2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b)$$
$$\Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) + a^2 - (b-c)^2$$
$$\Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2+g_a^2)$$
$$\Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 \overset{(\bullet)}{=} (b-c)^2 + 2s(s-a)$$
$$\Rightarrow n_a^2+g_a^2+2n_ag_a \overset{\text{via } (2)}{\geq} (b-c)^2 + 4s(s-a) \Rightarrow (n_a+g_a)^2 \geq 4m_a^2$$
$$\Rightarrow n_a + g_a \geq 2m_a \to (2)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a)+bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 \overset{(\bullet\bullet)}{=} s(s-a) + \frac{s(b-c)^2}{a}$$

$$\text{Via } (\bullet) \text{ and } (\bullet\bullet), g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + (b-c+a)(b-c-a) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a)^2 + \frac{(s-a)(a^2-(b-c)^2)}{a} \Rightarrow g_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a) - \frac{(s-a)(b-c)^2}{a}$$

$$\therefore \text{via } ②, (\bullet\bullet) \text{ and } (\bullet\bullet\bullet), \text{we have} : 2m_a(n_a + w_a + g_a) - (3r_br_c + n_a^2 + w_a^2 + g_a^2)$$

$$\geq 2m_aw_a + 4m_a^2 - 3s(s-a) - s(s-a) - \frac{s(b-c)^2}{a} - s(s-a)$$

$$+ \frac{s(s-a)(b-c)^2}{(2s-a)^2} - s(s-a) + \frac{(s-a)(b-c)^2}{a}$$

$$\overset{Lascu + A-G}{\geq} 2s(s-a) + 4s(s-a) + (b-c)^2 - 3s(s-a) - s(s-a) - \frac{s(b-c)^2}{a}$$

$$-s(s-a) + \frac{s(s-a)(b-c)^2}{(2s-a)^2} - s(s-a) + \frac{s(b-c)^2}{a} - (b-c)^2$$

$$= \frac{s(s-a)(b-c)^2}{(2s-a)^2} \geq 0 \therefore 2m_a(n_a + w_a + g_a) \geq 3r_br_c + n_a^2 + w_a^2 + g_a^2$$

$$\forall \Delta ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$

3048. In any Δ ABC the following relationship holds :

$$s^2 + 12Rr + 30r^2 \leq (w_a + w_b + w_c)^2 \leq s^2 + 21Rr + 12r^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(a\sin\frac{A}{2}\right) = 4R\sum_{cyc}\left(\sin^2\frac{A}{2}\cos\frac{A}{2}\right) = R.\sum_{cyc}\left(2\sin^2\frac{A}{2}.\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\right)$$

$$\overset{0<\cos\frac{B-C}{2}\leq 1 \text{ and analogs}}{\geq} R.\sum_{cyc}\left((1-\cos A).\left(2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\right)\right)$$

$$= R.\sum_{cyc}\left((1-\cos A).(\sin B+\sin C)\right)$$

$$= R.\sum_{cyc}(\sin B+\sin C) - R\sum_{cyc}\left(\cos A\left(\sum_{cyc}\sin A - \sin A\right)\right)$$

$$= 2R.\frac{s}{R} - R.\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\sin A\right) + \frac{R}{2}.\sum_{cyc}\sin 2A$$

$$= 2s - R\left(\frac{R+r}{R}\right)\left(\frac{s}{R}\right) + 2R.\frac{4Rrs}{8R^3} = 2s - s - \frac{rs}{R} + \frac{rs}{R} \Rightarrow \sum_{cyc}\left(a\sin\frac{A}{2}\right) \geq s \rightarrow (m)$$

$$\sum_{cyc}\frac{a}{(b+c)^2} = \sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2} = 2s.\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2} - \sum_{cyc}\frac{1}{b+c}$$

$$= \frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2 - 2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2} - \frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)} =$$

$$\frac{\left(\left(\sum_{cyc}a^2 + 2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)^2 - 16s^2(s^2+2Rr+r^2) - (s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\Rightarrow \sum_{cyc}\frac{a}{(b+c)^2} \overset{(\bullet)}{=} \frac{(5s^2+4Rr+r^2)^2 - 16s^2(s^2+2Rr+r^2) - (s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

Now, $\sum_{cyc}w_a^2 = \sum_{cyc}\frac{4bcs(s-a)}{(b+c)^2} = \sum_{cyc}\frac{bc((b+c)^2-a^2)}{(b+c)^2} = \sum_{cyc}\left(bc - \frac{a^2bc}{(b+c)^2}\right)$

$$\overset{\text{via }(\bullet)}{=} s^2+4Rr+r^2 + 2Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$= \frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2} \Rightarrow (w_a+w_b+w_c)^2 =$$

$$\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2} + \frac{2.16Rr^2s^2}{s^2+2Rr+r^2}.\sum_{cyc}\frac{b+c}{2bc\cos\frac{A}{2}}$$

$$= \frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2} + \frac{4rs}{s^2+2Rr+r^2}.\sum_{cyc}\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}.(2(s-a)+a)}{\cos\frac{A}{2}}$$

$$= \frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\sum_{cyc}\frac{2r\tan\frac{A}{2}\cos\frac{A}{2}}{\tan\frac{A}{2}}+\sum_{cyc}\left(a\sin\frac{A}{2}\right)\right)$$

$$\overset{via\ (m)}{\geq}\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\sum_{cyc}\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}+s\right)$$

$$\overset{0<\cos\frac{B-C}{2}\leq 1\ and\ analogs}{\geq}\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\sum_{cyc}(\sin B+\sin C)+s\right)$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\frac{2s}{R}+s\right)\overset{?}{\geq}s^2+12Rr+30r^2$$

$$\Leftrightarrow\frac{16(R+2r)s^2}{s^2+2Rr+r^2}\overset{?}{\geq}\frac{(16R+29r)s^4+r(20R^2+108Rr+58r^2)s^2+r^2(48R^3+152R^2r+124Rr^2+29r^3)}{(s^2+2Rr+r^2)^2}$$

$$\Leftrightarrow 3s^4+(12R^2-28Rr-26r^2)s^2-r(48R^3+152R^2r+124Rr^2+29r^3)\ \boxed{\overset{?}{\underset{①}{\geq}}}\ 0$$

$$and\ \because P=3(s^2-16Rr+5r^2)^2+4(3R^2+17Rr-14r^2)(s^2-16Rr+5r^2)$$

$$\overset{Gerretsen}{\geq}0\ \therefore \text{in order to prove ①, it suffices to prove : LHS of ①}\overset{?}{\geq}P$$

$$\Leftrightarrow 36t^3+27t^2-220t+44\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(36t^2+99t-22)\overset{?}{\geq}0\rightarrow true$$

$$\because t\overset{Euler}{\geq}2\Rightarrow ①\ \text{is true}\ \therefore(w_a+w_b+w_c)^2\geq s^2+12Rr+30r^2$$

$$\text{We shall now evaluate : }\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}\ and\ it's=\sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$=3-\sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$

$$=4s\sum_{cyc}\frac{1}{b+c}-4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2-\frac{2}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}(b+c)\right)$$

$$=\frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}+\frac{16s^2}{s^2+2Rr+r^2}$$

$$\Rightarrow\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}\ \boxed{\overset{(\bullet\bullet)}{=}}\ \frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m=$

$$2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$
$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$
$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0 \Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(*)}{\le} 0$$
$$\text{Now, } w_a+w_b+w_c=\frac{\sqrt{bc}}{b+c}.\sqrt{4s(s-a)} \overset{CBS}{\le} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}}$$
$$\overset{via\ (\bullet\bullet)}{=} \sqrt{(s^2+4Rr+r^2).\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}} \overset{?}{\le} \sqrt{s^2+21Rr+12r^2}$$
$$\Leftrightarrow (R-5r)s^4+r(8R^2-2Rr+6r^2)s^2+$$
$$r^2(84R^3+116R^2r+61Rr^2+11r^3) \overset{?}{\underset{②}{\ge}} 0 \text{ and it's trivially true if : } R-5r\ge 0$$

and so, we now focus on the case when : $R-5r<0$ and then, via $(*)$,
$$(R-5r)\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)\ge 0$$
$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\ge$
$$(R-5r)\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)\Leftrightarrow$$
$$(R^3+2R^2r-26Rr^2+4r^3)s^2 \overset{?}{\underset{③}{\ge}} r(16R^4-89R^3r-86R^2r^2-30Rr^3-4r^4)$$

Case 1 $R^3+2R^2r-26Rr^2+4r^3\ge 0$ and then : LHS of ③ $\overset{Gerretsen}{\ge}$
$$(R^3+2R^2r-26Rr^2+4r^3)(16Rr-5r^2) \overset{?}{\ge} \text{RHS of ③}$$
$$\Leftrightarrow 29t^3-85t^2+56t-4\overset{?}{\ge}0 \Leftrightarrow (t-2)(29t^2-27t+2)\overset{?}{\ge}0 \to \text{true} \because t \overset{Euler}{\ge} 2$$
$\Rightarrow$ ③ is true

Case 2 $R^3+2R^2r-26Rr^2+4r^3<0$ and then : LHS of ③ $\overset{Gerretsen}{\ge}$
$$(R^3+2R^2r-26Rr^2+4r^3)(4R^2+4Rr+3r^2)\overset{?}{\ge} \text{RHS of ③}$$
$$\Leftrightarrow t^5-t^4-t^3+t^2-8t+4\overset{?}{\ge}0 \Leftrightarrow (t-2)(t^4+t^3+t^2+3t-2)\overset{?}{\ge}0 \to \text{true}$$
$\because t \overset{Euler}{\ge} 2 \Rightarrow$ ③ is true $\therefore$ combining both cases, ③ $\Rightarrow$ ② is true $\forall\ \Delta ABC$
$$\therefore (w_a+w_b+w_c)^2\le s^2+21Rr+12r^2 \text{ and so,}$$
$$s^2+12Rr+30r^2\le (w_a+w_b+w_c)^2\le s^2+21Rr+12r^2\ \forall\ ABC,$$
with equality iff ΔABC is equilateral (QED)

3049. In any ΔABC, the following relationship holds :

$$\frac{1}{w_aw_b}+\frac{1}{w_bw_c}+\frac{1}{w_cw_a}\ge\frac{4}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{w_a w_b}+\frac{1}{w_b w_c}+\frac{1}{w_c w_a}=\frac{s^2+2Rr+r^2}{16Rr^2s^2}\cdot\sum_{cyc} w_a \geq \frac{s^2+2Rr+r^2}{16Rr^2s^2}\cdot\sum_{cyc} h_a =$$

$$=\frac{s^2+2Rr+r^2}{16Rr^2s^2}\cdot\frac{s^2+4Rr+r^2}{2R}\overset{?}{\geq}\frac{4}{9}\cdot\frac{(s^2+4Rr+r^2)^2}{16R^2r^2s^2}\Leftrightarrow s^2\overset{?}{\geq}14Rr-r^2\rightarrow \text{true}$$

$$\because s^2 \overset{\text{Gerretsen}}{\geq} 14Rr-r^2+2r(R-2r) \overset{\text{Euler}}{\geq} 14Rr-r^2$$

$$\therefore \frac{1}{w_a w_b}+\frac{1}{w_b w_c}+\frac{1}{w_c w_a}\geq\frac{4}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \ \forall\, ABC,$$

with equality iff Δ ABC is equilateral (QED)

3050. In ΔABC the following relationships holds:

$$R\geq\frac{1}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}\cdot r$$

$$\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\cdot r\leq s\leq 2\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)\cdot R$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

We need to show:

$$R\geq\frac{1}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}\cdot r$$

$$\frac{4R}{r}\geq\frac{4}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}$$

$$\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\geq\frac{4}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)} \ or\ 1-\sin\frac{A}{2}\geq 2\sin\frac{B}{2}\sin\frac{C}{2}$$

$$1-\sin\frac{A}{2}\geq\cos\frac{B-C}{2}-\cos\frac{B+C}{2} \ or$$

$$1-\sin\frac{A}{2}\overset{A+B+C=\pi}{\geq}\cos\frac{B-C}{2}-\sin\frac{A}{2} \ or\ \cos\frac{B-C}{2}\leq 1 \ True$$

We need to show:

$$\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\cdot r\leq s$$

$$\frac{s}{r}\ge\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\ or,\prod\cot\frac{A}{2}\ge\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\cot\frac{B}{2}\cot\frac{C}{2}\ge\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}}\ge\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\ or\ \frac{\left(\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\right)}{\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)}\ge\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\frac{\left(\sin\frac{A}{2}+\cos\frac{B-C}{2}\right)}{\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)}\overset{A+B+C=\pi}{\ge}\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\frac{x+y}{y-x}\overset{\sin\frac{A}{2}=x,\cos\frac{B-C}{2}=y}{\ge}\frac{1+x}{1-x}$$

$$x+y-x^2-xy\ge y-x+xy-x^2\ or\ 2x\ge 2xy\ or\ y\le 1\ or$$

$$\cos\frac{B-C}{2}\le 1\ True\ (A)$$

$$We\ need\ to\ show:$$

$$s\le 2\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)R\ or,\frac{s}{2R}\le\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)$$

$$\frac{s}{4R}\le\frac{1}{2}\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)$$

$$\prod\cos\frac{A}{2}\le\frac{1}{2}\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)$$

$$2\cos\frac{B}{2}\cos\frac{C}{2}\le\left(1+\sin\frac{A}{2}\right)\ or\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\le 1+\sin\frac{A}{2}$$

$$or,\sin\frac{A}{2}+\cos\frac{B-C}{2}\overset{A+B+C=\pi}{\le}1+\sin\frac{A}{2}\ or\cos\frac{B-C}{2}\le 1\ True\ (B)$$

$$From\ (A)\&(B):$$

$$\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\cdot r\le s\le 2\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)\cdot R$$

3051. In any Δ ABC, the following relationship holds :

$$m_a+w_a+h_a\le\left(\frac{1}{\sin\frac{A}{2}}+1\right)\left(2r+\frac{R}{2}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$w_a = \frac{2bc}{b+c}.\cos\frac{A}{2} = \frac{4R^2(\cos(B-C)+\cos A).\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$= \frac{R\left(2\cos^2\frac{B-C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right)}{\cos\frac{B-C}{2}}$$

$$\Rightarrow w_a = \frac{2R(c^2-s^2)}{c}\ \left(c = \cos\frac{B-C}{2}\ \textit{and}\ s = \sin\frac{A}{2}\right) \to (m)$$

Case 1 $A < \frac{\pi}{2}$ *and then, via* $(m), m_a + w_a + h_a \le 2R\cos^2\frac{A}{2} + \frac{2R(c^2-s^2)}{c}$

$$+2R(c^2-s^2) = 2R\left(1 - s^2 + \frac{c^2-s^2}{c} + c^2 - s^2\right) = 2R.\frac{c + c^2 + c^3 - s^2(2c+1)}{c}$$

$$\overset{?}{\le} \left(\frac{s+1}{s}\right)\left(4Rs(c-s) + \frac{R}{2}\right)$$

$$\Leftrightarrow \frac{2\left(c + c^2 + c^3 - s^2(2c+1)\right)}{c} \overset{?}{\le} \frac{(1+s)(1+8sc-8s^2)}{2s}$$

$$\Leftrightarrow -4c^3s + 8c^2s^2 + 4c^2s - 8cs^2 + 4s^3 - 3cs + c \overset{?}{\ge} 0$$

$$\Leftrightarrow (4c^2s - 8cs^2 + 4s^3) - (4c^3s - 8c^2s^2 + 4cs) + cs + c \overset{?}{\ge} 0$$

$$\Leftrightarrow 4s(c^2 - 2cs + s^2) - 4sc\left((c^2 - 2cs + s^2) - s^2 + 1\right) + cs + c \overset{?}{\ge} 0$$

$$\Leftrightarrow 4s(c-s)^2 - 4sc(c-s)^2 - 4sc(1+s)(1-s) + c(1+s) \overset{?}{\ge} 0$$

$$\Leftrightarrow (4s - 4sc)(c-s)^2 + c(1+s)\left(1 - 4s(1-s)\right) \overset{?}{\ge} 0$$

$$\Leftrightarrow 4s(1-c)(c-s)^2 + c(1+s)(2s-1)^2 \overset{?}{\ge} 0 \to \text{true} \because 1 - c = 1 - \cos\frac{B-C}{2}$$

$$\ge 0, \text{with equality iff } \cos\frac{B-C}{2} = 1\ \textit{and}\ s = \sin\frac{A}{2} = \frac{1}{2} \Rightarrow \text{equality iff}$$

$$B = C\ \textit{and}\ A = \frac{\pi}{3} \Rightarrow \text{equality iff } \Delta\, ABC \text{ is equilateral}$$

Case 2 $A \ge \frac{\pi}{2}$ *and then* : $4m_a^2 = 2b^2 + 2c^2 - a^2 \overset{?}{\le} a^2 \Leftrightarrow b^2 + c^2 - a^2 \overset{?}{\le} 0$

$$\Leftrightarrow 2bc.\cos A \overset{?}{\le} 0 \to \text{true} \Rightarrow m_a \le \frac{a}{2} = 2R\cos\frac{A}{2}\sin\frac{A}{2} = 2Rs.\sqrt{1-s^2}$$

and then, via $(m), m_a + w_a + h_a \le$

$$2Rs.\sqrt{1-s^2}+\frac{2R(c^2-s^2)}{c}+2R(c^2-s^2)\overset{?}{\leq}\left(\frac{1}{\sin\frac{A}{2}}+1\right)\left(2r+\frac{R}{2}\right)$$
$$=R.\frac{(1+s)(1+8sc-8s^2)}{2s}$$
$$\Leftrightarrow\frac{(1+s)(1+8sc-8s^2)}{2s}-\frac{2(c^2-s^2)}{c}-2(c^2-s^2)\overset{?}{\geq}2s.\sqrt{1-s^2}$$
$$\Leftrightarrow\frac{c(1+s)(1+8sc-8s^2)-4s(c^2-s^2)-4sc(c^2-s^2)}{2sc}\overset{?}{\geq}2s.\sqrt{1-s^2}$$
$$\Leftrightarrow\frac{-4c^3s+8c^2s^2-4cs^3+4c^2s-8cs^2+4s^3+c(1+s)}{2sc}\overset{?}{\geq}2s.\sqrt{1-s^2}$$
$$\Leftrightarrow\frac{-4sc(c^2-2cs+s^2)+4s(c^2-2cs+s^2)+c(1+s)}{2sc}\overset{?}{\geq}2s.\sqrt{1-s^2}$$
$$\Leftrightarrow\frac{4s(1-c)(c-s)^2+c(1+s)}{2sc}\ \boxed{\overset{?}{\geq}(*)}\ 2s.\sqrt{1-s^2}$$

$$\text{Now, }1-c=1-\cos\frac{B-C}{2}\geq 0\ \therefore\ \text{LHS of }(*)\geq\frac{1+s}{2s}\overset{?}{\geq}2s.\sqrt{1-s^2}$$
$$\Leftrightarrow(1+s)^2-16s^4(1-s^2)\overset{?}{\geq}0\Leftrightarrow 16s^6-16s^4+s^2+2s+1\overset{?}{\geq}0$$
$$\Leftrightarrow\frac{1}{16}\left((4s-3)^2\left(16s^4+24s^3+11s^2+3s-\frac{11}{16}\right)+\frac{1}{16}(355-184s)\right)\ \boxed{\overset{?}{\geq}(**)}\ 0$$

$$\text{Now, }A\geq\frac{\pi}{2}\Rightarrow\frac{A}{2}\geq\frac{\pi}{4}\Rightarrow 1>s\geq\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\ \textit{and}\ \because\ 3s-\frac{11}{16}\text{ is }\uparrow\text{ on }\left[\frac{1}{\sqrt{2}},1\right)$$
$$\therefore\ 3-\frac{11}{16}>3s-\frac{11}{16}\geq 3.\frac{1}{\sqrt{2}}-\frac{11}{16}>0\ \forall\ s\in\left[\frac{1}{\sqrt{2}},1\right)\ \textit{and}\ \because\ s\in\left[\frac{1}{\sqrt{2}},1\right)$$
$$\therefore\ 355-184s>0\ \therefore\ \textit{LHS of }(**)>\frac{1}{16}.(4s-3)^2(16s^4+24s^3+11s^2)>0$$

$\Rightarrow (**)$ is true (strict inequality) $\Rightarrow (*)$ is true $\Rightarrow m_a+w_a+h_a<\left(\frac{1}{\sin\frac{A}{2}}+1\right)\left(2r+\frac{R}{2}\right)$ and combining both cases, $m_a+w_a+h_a\leq\left(\frac{1}{\sin\frac{A}{2}}+1\right)\left(2r+\frac{R}{2}\right)$ $\forall$ ABC, with equality iff Δ ABC is equilateral (QED)

3052. In any ΔABC the following relationship holds :

$$\sum_{cyc}\left(\frac{g_a}{h_a}+\frac{m_a}{w_a}+\frac{n_a}{p_a}\right)\leq\ 5+\frac{2R}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $h_a \le g_a \le w_a \le m_a \le p_a \le n_a$ (see [1, pp. 2]), then we have

$$\frac{g_a}{h_a}+\frac{m_a}{w_a}+\frac{n_a}{p_a}=\frac{n_a}{h_a}+\frac{g_a}{h_a}+\frac{m_a}{w_a}-n_a\left(\frac{1}{h_a}-\frac{1}{p_a}\right)\le\frac{n_a}{h_a}+\frac{g_a}{h_a}+\frac{m_a}{w_a}-p_a\left(\frac{1}{h_a}-\frac{1}{p_a}\right)$$

$$=\frac{n_a}{h_a}+1+\frac{g_a}{h_a}+\frac{m_a}{w_a}-\frac{p_a}{h_a}\le\frac{n_a}{h_a}+1+\frac{g_a}{h_a}+\frac{m_a}{g_a}-\frac{m_a}{h_a}=\frac{n_a}{h_a}+2-\left(\frac{m_a}{g_a}-1\right)\left(\frac{g_a}{h_a}-1\right)$$
$$\le\frac{n_a}{h_a}+2,$$

then

$$\sum_{cyc}\left(\frac{g_a}{h_a}+\frac{m_a}{w_a}+\frac{n_a}{p_a}\right)\le\sum_{cyc}\left(\frac{n_a}{h_a}+2\right)\overset{CBS}{\le}6+\sqrt{\sum_{cyc}\frac{1}{h_a}\cdot\sum_{cyc}\frac{{n_a}^2}{h_a}}$$
$$=6+\sqrt{\frac{1}{r}\cdot\sum_{cyc}\frac{a}{2sr}\cdot s\left(s-a+\frac{(b-c)^2}{a}\right)}$$

$$=6+\frac{1}{\sqrt{2}r}\cdot\sqrt{\sum_{cyc}[a(s-a)+(b-c)^2]}=6+\frac{1}{\sqrt{2}r}\cdot\sqrt{2r(4R+r)+2(s^2-3r^2-12Rr)}$$

$$=6+\frac{\sqrt{s^2-8Rr-2r^2}}{r}\overset{Gerretsen}{\le}6+\frac{\sqrt{4R^2-4Rr+r^2}}{r}=5+\frac{2R}{r}.$$

as desired. Equality holds iff ΔABC is equilateral.

[1] Bogdan Fuştei, Mohamed Amine Ben Ajiba,

"SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE" – www.ssmrmh.ro

3053. In any ΔABC the following relationship holds :

$$\frac{m_a}{w_a}+\frac{w_a}{h_a}+\frac{h_a}{m_a}\le\frac{3}{2}+\frac{3R}{4r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $h_a \le w_a \le m_a$, then we have

$$\frac{m_a}{w_a}+\frac{w_a}{h_a}+\frac{h_a}{m_a}=\left(\frac{m_a}{w_a}-1\right)\left(1-\frac{w_a}{h_a}\right)+1+\frac{m_a}{h_a}+\frac{h_a}{m_a}\le 1+\frac{m_a}{h_a}+\frac{h_a}{m_a}\le$$

$$\overset{Panaitopol}{\le} 1+\frac{R}{2r}+1\overset{Euler}{\le}\frac{3}{2}+\frac{3R}{4r}$$

as desired. Equality holds iff ΔABC is equilateral.

3054. In any Δ ABC, the following relationship holds :

$$\prod_{cyc}\sqrt{w_a^2-h_a^2}\cdot\prod_{cyc}\sqrt{m_a^2-h_a^2}=\prod_{cyc}\sqrt{g_a^2-h_a^2}\cdot\prod_{cyc}\sqrt{n_a^2-h_a^2}\le 4Rs^2(R-2r)^3$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)$$
$$=an_a^2+a(s-b)(s-c)\text{ and }b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)\text{ and}$$
$$\text{via summation, we get : }(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$$
$$=2(n_a^2+g_a^2)+a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)$$
$$\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$
$$\text{Again, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)}{bc}$$
$$=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2\overset{(**)}{=}s(s-a)+\frac{s}{a}(b-c)^2$$
$$\text{Via }(*)\text{ and }(**), g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$
$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$
$$=(s-a)^2+\frac{(s-a)(a^2-(b-c)^2)}{a}=s(s-a)-\frac{(s-a)(b-c)^2}{a}$$

$$\Rightarrow g_a^2-h_a^2=s(s-a)-\frac{(s-a)(b-c)^2}{a}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{(s-a)^2}{a^2}.(b-c)^2\Rightarrow\sqrt{g_a^2-h_a^2}=\frac{s-a}{a}.|b-c|\text{ and analogs}\rightarrow ①$$

$$\text{Via }(**),n_a^2-h_a^2=s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{s^2}{a^2}.(b-c)^2\Rightarrow\sqrt{n_a^2-h_a^2}=\frac{s}{a}.|b-c|\text{ and analogs}\rightarrow ②$$

$$\text{Also},m_a^2-h_a^2=s(s-a)+\frac{(b-c)^2}{4}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{4s^2-4sa+a^2}{4a^2}.(b-c)^2=\frac{(b+c)^2}{4a^2}.(b-c)^2$$

$$\Rightarrow\sqrt{m_a^2-h_a^2}=\frac{b+c}{2a}.|b-c|\text{ and analogs}\rightarrow ③$$

$$\text{Finally},w_a^2-h_a^2=s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{4s^2(s-a)^2}{a^2(b+c)^2}.(b-c)^2\Rightarrow\sqrt{w_a^2-h_a^2}=\frac{2s(s-a)}{a(b+c)}.|b-c|\text{ and analogs}\rightarrow ④$$

$$\text{We have}: R-2r\overset{?}{\geq}\frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\overset{?}{\geq}\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}$$

$$\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\overset{?}{\geq}\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\overset{?}{\geq}\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\overset{?}{\geq}4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\overset{?}{\geq}4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\overset{?}{\geq}0\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\overset{?}{\geq}0$$

$\rightarrow$ true $\Rightarrow 4R(R-2r)\geq(b-c)^2$ *and analogs* $\rightarrow$ (m) *and via* ③ *and* ④,

$$\prod_{cyc}\sqrt{w_a^2-h_a^2}.\prod_{cyc}\sqrt{m_a^2-h_a^2}=\prod_{cyc}\left(\frac{b+c}{2a}.|b-c|\right).\prod_{cyc}\left(\frac{2s(s-a)}{a(b+c)}.|b-c|\right)$$

$$=\frac{s^3.r^2s}{16R^2r^2s^2}.\prod_{cyc}(b-c)^2\Rightarrow\prod_{cyc}\sqrt{w_a^2-h_a^2}.\prod_{cyc}\sqrt{m_a^2-h_a^2}=\frac{s^2}{16R^2}.\prod_{cyc}(b-c)^2$$

$\rightarrow$ (i) *and via* ① *and* ②, $\prod_{cyc}\sqrt{g_a^2-h_a^2}.\prod_{cyc}\sqrt{n_a^2-h_a^2}=$

$$\prod_{cyc}\left(\frac{s-a}{a}.|b-c|\right).\prod_{cyc}\left(\frac{s}{a}.|b-c|\right)=\frac{s^2.r^2s^2}{16R^2r^2s^2}.\prod_{cyc}(b-c)^2$$

$$\Rightarrow \prod_{cyc}\sqrt{g_a^2-h_a^2}\cdot\prod_{cyc}\sqrt{n_a^2-h_a^2}=\frac{s^2}{16R^2}\cdot\prod_{cyc}(b-c)^2 \rightarrow \text{(ii)}$$

$$\therefore \text{(ii) and (ii)} \Rightarrow \prod_{cyc}\sqrt{w_a^2-h_a^2}\cdot\prod_{cyc}\sqrt{m_a^2-h_a^2}=\prod_{cyc}\sqrt{g_a^2-h_a^2}\cdot\prod_{cyc}\sqrt{n_a^2-h_a^2}$$

$$=\frac{s^2}{16R^2}\cdot\prod_{cyc}(b-c)^2 \overset{\text{via (m)}}{\leq} \frac{s^2}{16R^2}\cdot 64R^3(R-2r)^3=4Rs^2(R-2r)^3\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3055. In any Δ ABC, the following relationship holds :

$$m_a h_a \leq w_a\left(\frac{n_a+h_a}{2}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality} \Rightarrow g_a \leq AI+r \overset{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r \overset{?}{\leq} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r \overset{?}{\leq} \frac{8Rrs\cos\frac{A}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}(b+c)} \Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1 \overset{?}{\leq} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1 \overset{?}{\leq} \frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\leq} a \Leftrightarrow$$

$$4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\leq} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \leq 1 \rightarrow \text{true} \therefore g_a \leq w_a \rightarrow (1)$$

$$\text{Now, } an_a^2.ag_a^2 \overset{?}{\geq} a^2s^2(s-a)^2 \Leftrightarrow$$

$$\left(b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\right)\left(\begin{matrix}b^2(s-b)+c^2(s-c)\\-a(s-b)(s-c)\end{matrix}\right) \underset{(*)}{\overset{?}{\geq}} a^2s^2(s-a)^2$$

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z \therefore s=x+y+z \Rightarrow a=y+z,$$
$$b=z+x \text{ and } c=x+y \text{ and via such substitutions, } (*) \Leftrightarrow$$

$$\left(z(z+x)^2+y(x+y)^2-yz(y+z)\right)\left(y(z+x)^2+z(x+y)^2-yz(y+z)\right)$$
$$\geq x^2(y+z)^2(x+y+z)^2 \Leftrightarrow xy^2+xz^2+y^3+z^3 \geq 2xyz+yz(y+z)$$
$$\Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \geq 0 \rightarrow \text{true} \Rightarrow (*) \text{ is true} \Rightarrow n_a g_a \geq s(s-a) \rightarrow (2)$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2+a(s-b)(s-c)$$
$$\text{and } b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2+a(s-b)(s-c) \text{ and } (m)+(n) \Rightarrow$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2)+a^2-(b-c)^2$$

$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2+(b-c)^2 = 2(n_a^2+g_a^2)$$

$$\Rightarrow 2(b-c)^2+4s(s-a) = 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 = (b-c)^2+2s(s-a)$$

$$\Rightarrow n_a^2+g_a^2+2n_a g_a \overset{\text{via }②}{\geq} (b-c)^2+4s(s-a) \Rightarrow (n_a+g_a)^2 \geq 4m_a^2$$

$$\Rightarrow m_a \leq \frac{n_a+g_a}{2} \Rightarrow m_a h_a \leq h_a.\frac{n_a+g_a}{2} \overset{?}{\leq} w_a\left(\frac{n_a+h_a}{2}\right)$$

$$\Leftrightarrow n_a(w_a-h_a)+h_a(w_a-g_a) \overset{?}{\geq} 0 \to \text{true via } ①$$

$$\therefore m_a h_a \leq w_a\left(\frac{n_a+h_a}{2}\right)\ \forall\, \Delta\, ABC,'' ='' \text{ iff } b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $n_a \geq m_a$ and $m_a w_a \geq s(s-a)$, then we have

$$(n_a+w_a)^2 \geq {n_a}^2+2m_a w_a+{w_a}^2 \geq s\left(s-a+\frac{(b-c)^2}{a}\right)+2s(s-a)+\frac{4bcs(s-a)}{(b+c)^2}$$

$$= 4s(s-a)+\frac{s(b-c)^2}{a}-\frac{s(s-a)(b-c)^2}{(b+c)^2}$$

$$= 4s(s-a)+(b-c)^2+\frac{(s-a)^2(4s-a)(b-c)^2}{(b+c)^2}$$

$$\geq 4s(s-a)+(b-c)^2 = 2(b^2+c^2)-a^2 = 4{m_a}^2 \ \Rightarrow\ n_a+w_a \geq 2m_a.$$

Therefore

$$m_a h_a \leq \frac{(n_a+w_a)h_a}{2} = \frac{h_a n_a+w_a h_a}{2} \leq \frac{w_a n_a+w_a h_a}{2} = w_a\left(\frac{n_a+h_a}{2}\right).$$

Equality holds iff $b=c$.

3056. In ΔABC the following relationship holds:

$$\frac{b^2+c^2}{b+c}\cos\frac{A}{2} \leq n_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$n_a^2 = s(s-a)+\frac{s(b-c)^2}{a}, \qquad g_a^2 = s(s-a)-\frac{(s-a)(b-c)^2}{a}$$

$$m_a^2 = s(s-a) + \frac{(b-c)^2}{4} \text{ or } 4m_a^2 = 4s(s-a) + (b-c)^2$$

Using above result we get:

$$n_a^2 + g_a^2 = (b-c)^2 + 2s(s-a)$$

$$n_a w_a \overset{n_a \ge m_a}{\ge} m_a w_a \overset{\text{Known inequality } m_a \ge \frac{b+c}{2}\cos\frac{A}{2}}{\ge} \frac{b+c}{2}\cos\frac{A}{2} \cdot \frac{2bc}{b+c}\cos\frac{A}{2} =$$

$$= bc\cos^2\frac{A}{2} = s(s-a)$$

$$(n_a + w_a)^2 = n_a^2 + g_a^2 + 2\,n_a w_a \ge (b-c)^2 + 2s(s-a) + 2\,m_a w_a \ge$$

$$\ge (b-c)^2 + 2s(s-a) + 2s(s-a) =$$

$$= 4s(s-a) + (b-c)^2 = 4m_a^2 \text{ or}, n_a + w_a \ge 2m_a \ (1)$$

$$\frac{b^2+c^2}{b+c}\cos\frac{A}{2} = \left((b+c) - \frac{2bc}{b+c}\right)\cos\frac{A}{2} = (b+c)\cos\frac{A}{2} - \frac{2bc}{b+c}\cos\frac{A}{2} \le$$

$$\overset{m_a \ge \frac{b+c}{2}\cos\frac{A}{2}}{\le} 2\,m_a - w_a \overset{(1)}{\le} n_a + w_a - w_a = n_a$$

3057. **In any Δ ABC, the following relationship holds :**

$$\frac{g_a}{h_a} + \frac{g_b}{h_b} + \frac{g_c}{h_c} \le \sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\text{and } b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c) \text{ and via summation, we get :}$$
$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2) =$$
$$2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) +$$
$$a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 =$$
$$2(n_a^2+g_a^2) \Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2 + g_a^2 \overset{(*)}{=} (b-c)^2 + 2s(s-a)$$
$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2\overset{(**)}{=}s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$\text{Via }(*)\text{ and }(**), g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$

$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)$$

$$=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}=(s-a)^2+\frac{4r^2s}{a}=(s-a)^2+2rh_a$$

$$\Rightarrow\frac{g_a^2}{h_a}=\frac{a(s-a)^2}{2rs}+2r\text{ and analogs}\Rightarrow\sum_{cyc}\frac{g_a^2}{h_a}=\frac{1}{2rs}.\sum_{cyc}a(s^2-2sa+a^2)+6r$$

$$=\frac{1}{2rs}.\left(s^2(2s)-4s(s^2-4Rr-r^2)+2s(s^2-6Rr-3r^2)\right)+6r$$

$$=\frac{4Rrs-2r^2s}{2rs}+6r\therefore\sum_{cyc}\frac{g_a^2}{h_a}=2R+5r\therefore\frac{g_a}{h_a}+\frac{g_b}{h_b}+\frac{g_c}{h_c}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{g_a^2}{h_a}}.\sqrt{\sum_{cyc}\frac{1}{h_a}}$$

$$=\sqrt{\frac{2R+5r}{r}}\overset{?}{\le}\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}=\sum_{cyc}\sqrt{\frac{s\tan\frac{A}{2}.4R\cos^2\frac{A}{2}\tan\frac{A}{2}}{2rs}}=\sqrt{\frac{2R}{r}}.\sum_{cyc}\sin\frac{A}{2}$$

$$\Leftrightarrow\left(\frac{2R}{r}\right)\left(\sum_{cyc}\sin^2\frac{A}{2}+2\left(\prod_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\mathrm{cosec}\frac{A}{2}\right)\right)\underset{①}{\overset{?}{\ge}}\frac{2R+5r}{r}$$

$$\because\sum_{cyc}\mathrm{cosec}\frac{A}{2}\overset{Jensen}{\ge}6\therefore\text{LHS of ①}\ge\left(\frac{2R}{r}\right)\left(\frac{2R-r}{2R}+\frac{r}{2R}.6\right)=\frac{2R+5r}{r}$$

$$\Rightarrow \text{①} \text{ is true} \Rightarrow \frac{g_a}{h_a}+\frac{g_b}{h_b}+\frac{g_c}{h_c} \le \sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3058. In any Δ ABC the following relationship holds :

$$\left(\frac{R}{r}+2024\right)\left(\sum_{cyc} h_a\right)+2025\sum_{cyc} r_a \ge \sum_{cyc} m_a + 4050\sum_{cyc} w_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} w_a = \sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right) \overset{CBS}{\le} 2\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}}$$

$$\overset{A-G}{\le} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}} = \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}} = \sqrt{\sum_{cyc} bc}.\sqrt{\frac{4R+r}{2R}}$$

$$= \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \Rightarrow \sum_{cyc} w_a \le \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \to (a)$$

$$\text{Now, } \left(\frac{R}{r}+2024\right)\left(\sum_{cyc} h_a\right)+2025\sum_{cyc} r_a$$

$$= \left(\frac{R}{r}-1\right)\left(\sum_{cyc} h_a\right)+2025\left(\sum_{cyc} h_a+\sum_{cyc} r_a\right) \overset{A-G}{\ge}$$

$$\left(\frac{R}{2r}-1+\frac{R}{2r}\right)\left(\sum_{cyc} h_a\right)+4050.\sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \overset{\substack{\text{Euler + Panaitopol} \\ \text{and} \\ \text{via } (a)}}{\ge}$$

$$\sum_{cyc} m_a + 4050\sum_{cyc} w_a \therefore \left(\frac{R}{r}+2024\right)\left(\sum_{cyc} h_a\right)+2025\sum_{cyc} r_a \ge$$

$$\sum_{cyc} m_a + 4050\sum_{cyc} w_a\ \forall\,\Delta ABC, '' = ''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

3059. In any Δ ABC the following relationship holds :

$$\prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}} \geq \frac{\sqrt{3}}{2}\cdot\sum_{sym}\sqrt{\frac{a}{b}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \prod_{cyc}\sqrt{\frac{2rs(b+c).as}{bc.rs(2s-a)}} = \sqrt{\frac{8.4Rrs.s^3}{16R^2r^2s^2}}$$

$$\Rightarrow \prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \sqrt{\frac{2s^2}{Rr}} \rightarrow (i)$$

$$\sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \sum_{cyc}\sqrt{\frac{2rs(b+c).as}{bc.rs(2s-a)}} = \sqrt{\frac{2s}{4Rrs}}\cdot\sum_{cyc} a \Rightarrow \sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \sqrt{\frac{2s^2}{Rr}} \rightarrow (ii)$$

$$\prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}} = \sqrt{\frac{2r^2s^2}{R}\cdot\frac{1}{r^3}} \Rightarrow \prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}} = \sqrt{\frac{2s^2}{Rr}} \rightarrow (iii)$$

$$\text{Now, }\frac{\sqrt{3}}{2}\cdot\sum_{sym}\sqrt{\frac{a}{b}} = \frac{\sqrt{3}}{2}\cdot\sum_{cyc}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right) = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{4Rrs}}\sum_{cyc}\left(\sqrt{a(b+c)}.\sqrt{b+c}\right)$$

$$\overset{CBS}{\leq} \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{4Rrs}}\cdot\sqrt{2\sum_{cyc} ab}\,.\sqrt{4s} \leq \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{Rr}}\cdot\sqrt{\frac{2}{3}.4s^2} \therefore \frac{\sqrt{3}}{2}\cdot\sum_{sym}\sqrt{\frac{a}{b}} \leq \sqrt{\frac{2s^2}{Rr}} \rightarrow (m)$$

$$\therefore (i),(ii),(iii),(m) \Rightarrow \prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}} = \prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}} \geq \frac{\sqrt{3}}{2}\cdot\sum_{sym}\sqrt{\frac{a}{b}}$$

$$\forall\,\Delta ABC, '' = ''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

3060. In any Δ ABC the following relationship holds :

$$\frac{1}{w_a}+\frac{1}{r_a}+\frac{1}{h_a}\geq\frac{3}{\sqrt[3]{s^2r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{w_a}+\frac{1}{r_a}+\frac{1}{h_a}\overset{A-G}{\geq}3.\sqrt[3]{\frac{1}{w_a r_a h_a}}=3.\sqrt[3]{\frac{b+c}{2bc\cos\frac{A}{2}.\frac{s\tan\frac{A}{2}.2rs}{4R\cos^2\frac{A}{2}.\tan\frac{A}{2}}}}\overset{?}{\geq}\frac{3}{\sqrt[3]{s^2r}}$$

$$\Leftrightarrow\frac{Ra(b+c)}{4Rrs\cos\frac{A}{2}.\sec^2\frac{A}{2}}\overset{?}{\geq}1$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\sin\frac{A}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\geq}4.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\sec\frac{A}{2}$$

$$\Leftrightarrow\cos^2\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\geq}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(2\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$\Leftrightarrow\cos\frac{B-C}{2}-\sin^2\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\geq}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{A}{2}+\cos\frac{B-C}{2}\right)$$

$$\Leftrightarrow\cos\frac{B-C}{2}-\cos^2\frac{B-C}{2}+\sin^2\frac{A}{2}-\sin^2\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\geq}0$$

$$\Leftrightarrow\left(\cos\frac{B-C}{2}+\sin^2\frac{A}{2}\right)\left(1-\cos\frac{B-C}{2}\right)\overset{?}{\geq}0\rightarrow\text{true}\because 0<\cos\frac{B-C}{2}\leq 1$$

$$\therefore\frac{1}{w_a}+\frac{1}{r_a}+\frac{1}{h_a}\geq\frac{3}{\sqrt[3]{s^2r}}\ \forall\,\Delta\text{ ABC (QED)}$$

3061. In any Δ ABC, the following relationship holds :

$$n_a\leq h_a+4R\left(\frac{b-c}{a}\right)^2\leq h_a+\frac{16}{3}(R-2r)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s(a^2-(b-c)^2)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2\rightarrow \text{(i)}$$

$$\text{Now, } n_a\overset{?}{\ge}\frac{b^2-bc+c^2}{2R}\Leftrightarrow\frac{n_a}{h_a}\overset{?}{\ge}\frac{b^2-bc+c^2}{bc}\Leftrightarrow\frac{n_a^2}{h_a^2}-1\overset{?}{\ge}\left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$
$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}$$
$$\overset{\text{via (1)}}{\Leftrightarrow} s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$
$$\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}.\frac{b^2c^2}{4R^2}\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{4R^2}$$
$$\Leftrightarrow\frac{s^2}{a^2}\overset{?}{\ge}\frac{b^2+c^2}{4R^2}\ (\because (b-c)^2\ge 0)\Leftrightarrow 4R^2s^2\overset{?}{\ge}a^2b^2+c^2a^2\rightarrow \text{true}$$
$$(\text{strict inequality})\because 4R^2s^2\overset{\text{Goldstone}}{\ge}\sum_{\text{cyc}}a^2b^2>a^2b^2+c^2a^2\therefore n_a\ge\frac{b^2-bc+c^2}{2R}$$

$$\Rightarrow n_a-h_a=\frac{n_a^2-h_a^2}{n_a+h_a}\le\frac{s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}}{\frac{b^2+c^2}{2R}}$$
$$\le\frac{\frac{s^2}{a^2}(b-c)^2}{\frac{(b+c)^2}{4R}}\le\frac{\frac{s^2}{a^2}(b-c)^2}{\frac{s^2}{4R}}\ (\because b+c=s+s-a>s)\Rightarrow n_a-h_a\le 4R\left(\frac{b-c}{a}\right)^2$$
$$\Rightarrow\boxed{n_a\le h_a+4R\left(\frac{b-c}{a}\right)^2}\overset{?}{\le}h_a+\frac{16}{3}(R-2r)\Leftrightarrow\frac{4}{3}(R-2r).a^2\overset{?}{\ge}R(b-c)^2$$
$$\Leftrightarrow\frac{4}{3}.R\left(1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\right).16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}$$
$$\overset{?}{\ge}R.16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}$$
$$\Leftrightarrow\frac{4}{3}\left(1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\right)\left(1-\sin^2\frac{A}{2}\right)\overset{?}{\ge}1-\cos^2\frac{B-C}{2}$$
$$\Leftrightarrow 3\cos^2\frac{B-C}{2}-16(x-x^3)\cos\frac{B-C}{2}+1+12x^2-16x^4\ \boxed{\overset{?}{\underset{①}{\ge}}}\ 0\ \left(x=\sin\frac{A}{2}\right)$$

Now, LHS of ① is a quadratic polynomial in $\cos\frac{B-C}{2}$ with discriminant $=$

$$256(x-x^3)^2-12(1+12x^2-16x^4)=256x^6-320x^4+112x^2-12$$

$=4(4x^2-1)^2(4x^2-3)\le 0$ iff $x\le\frac{\sqrt{3}}{2}$ and so, **when $x\le\frac{\sqrt{3}}{2}$**, discriminant ≤ 0

$\Rightarrow$ LHS of ① $\ge 0\Rightarrow$ ① is true and we now focus on the scenario :

when $x > \frac{\sqrt{3}}{2}$ and then, in order to prove ①, it suffices to prove : $\cos\frac{B-C}{2} >$

$$\frac{8(x-x^3)+(4x^2-1).\sqrt{4x^2-3}}{3}\left(\because x > \frac{\sqrt{3}}{2} > \frac{1}{2} \Rightarrow 4x^2-1 > 0\right)$$

$$\text{and} \because \cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2} \overset{\frac{b+c}{a}>1}{>} x \therefore \text{it suffices to prove} : x >$$

$$\Leftrightarrow (8x^3-5x)^2 > (4x^2-3)(4x^2-1)^2 \left(\because x > \frac{\sqrt{3}}{2} \Rightarrow 8x^2 > 6 > 5 \Rightarrow 8x^3 > 5x\right)$$

$$\Leftrightarrow 3-3x^2 > 0 \Rightarrow 1 > x^2 \to \text{true} \Rightarrow ① \text{ is true and combining both cases,}$$

$$① \text{ is true } \forall \Delta ABC \Rightarrow \boxed{h_a + 4R\left(\frac{b-c}{a}\right)^2 \le h_a + \frac{16}{3}(R-2r)} \text{ and so,}$$

$$n_a \le h_a + 4R\left(\frac{b-c}{a}\right)^2 \le h_a + \frac{16}{3}(R-2r)\ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3062. In any ΔABC the following relationship holds :

$$\frac{n_a}{w_a} + \frac{n_b}{w_b} + \frac{n_c}{w_c} \le \frac{13R}{8r} - \frac{1}{4}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality we have

$$\frac{n_a}{w_a} + \frac{n_b}{w_b} + \frac{n_c}{w_c} \le \sqrt{(a{n_a}^2 + b{n_b}^2 + c{n_c}^2)\left(\frac{1}{a{w_a}^2} + \frac{1}{b{w_b}^2} + \frac{1}{c{w_c}^2}\right)},$$

with

$$a{n_a}^2 + b{n_b}^2 + c{n_c}^2 = \sum_{cyc} as\left(s-a+\frac{(b-c)^2}{a}\right) = s\sum_{cyc} a(s-a) + s\sum_{cyc}(b-c)^2 =$$

$$= 2sr(4R+r) + 2s(s^2-3r^2-12Rr)$$

$$= 2s(s^2-2r^2-8Rr) \overset{Gerretsen}{\le} 2s(4R^2-4Rr+r^2).$$

$$\frac{1}{a{w_a}^2} + \frac{1}{b{w_b}^2} + \frac{1}{c{w_c}^2} = \sum_{cyc}\frac{(b+c)^2}{4abcs(s-a)} = \frac{1}{16s^2Rr}\sum_{cyc}\frac{(2s-a)^2}{s-a} =$$

$$= \frac{1}{16s^2Rr}\sum_{cyc}\left(\frac{s^2}{s-a}+3s-a\right) = \frac{1}{16s^2Rr}\left(\frac{s(4R+r)}{r}+7s\right) = \frac{R+2r}{4sRr^2}.$$

Then

$$\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c} \le \sqrt{\frac{(4R^2-4Rr+r^2)(R+2r)}{2Rr^2}} \overset{?}{\lesssim} \frac{13R}{8r}-\frac{1}{4}$$

$$\Leftrightarrow 32(4R^2-4Rr+r^2)(R+2r) \le R(13R-2r)^2 \;\Leftrightarrow\; (41R-16r)(R-2r)^2 \ge 0,$$

which is true by Euler inequality $R \ge 2r$.

The proof is complete. Equality holds iff ΔABC is equilateral.

3063. In any ΔABC the following relationship holds :

$$\frac{g_a}{h_a} \le \frac{4R+r}{\sqrt{3}s}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sqrt{3}.\frac{g_a}{h_a} = \frac{a\sqrt{3(s-a)\left(s-\frac{(b-c)^2}{a}\right)}}{2sr} \le \frac{3a(s-a)+a\left(s-\frac{(b-c)^2}{a}\right)}{4sr} =$$

$$= \frac{a(4s-3a)-(b-c)^2}{4sr} = \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4sr} = \frac{4R+r}{s}$$

3064. In any ΔABC the following relationship holds :

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c} \ge \sqrt[4]{\frac{2r}{R}}\left(\frac{1}{\sqrt[4]{r_a h_a{}^3}}+\frac{1}{\sqrt[4]{r_a h_a{}^3}}+\frac{1}{\sqrt[4]{r_a h_a{}^3}}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}=\sum_{cyc}\frac{b+c}{2\sqrt{bcs(s-a)}}=\sum_{cyc}\left(\frac{a}{2\sqrt{abs(s-c)}}+\frac{a}{2\sqrt{cas(s-b)}}\right)\geq$$

$$\geq\sum_{cyc}\sqrt[4]{\frac{a^2}{bcs^2(s-b)(s-c)}}=\sum_{cyc}\sqrt[4]{\frac{a^3(s-a)}{4Rsr.s^2.sr^2}}=\sqrt[4]{\frac{2r}{R}}.\sum_{cyc}\sqrt[4]{\frac{a^3(s-a)}{8F^4}}=$$

$$=\sqrt[4]{\frac{2r}{R}}\left(\frac{1}{\sqrt[4]{r_ah_a{}^3}}+\frac{1}{\sqrt[4]{r_ah_a{}^3}}+\frac{1}{\sqrt[4]{r_ah_a{}^3}}\right)$$

Equality holds iff ΔABC is equilateral.

3065. In any ΔABC the following relationship holds :

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\geq\sqrt{\frac{3\sqrt{3}}{F}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Morocco

We will first prove the following lemma, that for any $x,y,z,u,v,w>0$, we have

$$(v+w)x+(w+u)y+(u+v)z\geq 2\sqrt{(uv+vw+wu)(xy+yz+zx)}.$$

By CBS inequality, we have

$$(v+w)x+(w+u)y+(u+v)z=(u+v+w)(x+y+z)-(ux+vy+wz)$$

$$=\sqrt{[2(uv+vw+wu)+u^2+v^2+w^2][2(xy+yz+zx)+x^2+y^2+z^2]}-(ux+vy+wz)$$

$$\geq 2\sqrt{(uv+vw+wu)(xy+yz+zx)}+ux+vy+wz-(ux+vy+wz)$$

$$=2\sqrt{(uv+vw+wu)(xy+yz+zx)},$$

which completes the proof of the lemma. Equality holds iff $\frac{u}{x}=\frac{v}{y}=\frac{w}{z}$.

Now, we have

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}=\sum_{cyc}\frac{b+c}{2bc\cos\frac{A}{2}}=\sum_{cyc}\frac{4R\left(\sin\frac{B}{2}\cos\frac{B}{2}+\sin\frac{C}{2}\cos\frac{C}{2}\right)}{bc\sin A}.\sin\frac{A}{2}$$

$$=\frac{4R}{2F}\sum_{cyc}\left(\sin\frac{B}{2}\cos\frac{B}{2}+\sin\frac{C}{2}\cos\frac{C}{2}\right)\sin\frac{A}{2}=\frac{2R}{F}\sum_{cyc}\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\overset{Lemma}{\geq}\frac{4R}{F}\sqrt{\sum_{cyc}\cos\frac{B}{2}\cos\frac{C}{2}.\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2}.\sin\frac{A}{2}\sin\frac{C}{2}}$$

$$=\frac{4R}{F}\sqrt{\frac{s}{4R}\sum_{cyc}\frac{1}{\cos\frac{A}{2}}.\frac{r}{4R}\sum_{cyc}\sin\frac{A}{2}}$$

$$=\sqrt{\frac{1}{F}\sum_{cyc}\sin\frac{A}{2}.\sum_{cyc}\frac{1}{\cos\frac{A}{2}}}\overset{H\ddot{o}lder}{\geq}\sqrt{\frac{1}{F}\sum_{cyc}\sin\frac{A}{2}.\sqrt{\frac{27}{\sum_{cyc}\cos\frac{B}{2}\cos\frac{C}{2}}}}.$$

So it suffices to prove that

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2\geq\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}.\quad(*)$$

Using the substitution $(A,B,C)\to(\pi-2A',\pi-2B',\pi-2C')$,

where $A',B',C'\in\left(0,\frac{\pi}{2}\right)$ $(*)\Leftrightarrow$

$$(\cos A'+\cos B'+\cos C')^2$$
$$\geq\sin A'\sin B'+\sin B'\sin C'+\sin C'\sin A',\ \forall acute\Delta A'B'C'$$

$$\Leftrightarrow\left(1+\frac{r'}{R'}\right)^2\geq\frac{s'^2+r'^2+4R'r'}{4R'^2}\Leftrightarrow 4R'^2+4R'r'+3r'^2\geq s'^2,$$

which is Gerretsen's inequality. So the proof is complete.

Equality holds iff ΔABC **is equilateral.**

3066. **In any ΔABC the following relationship holds :**

$$\sqrt{\frac{m_a}{a}}+\sqrt{\frac{w_b}{b}}+\sqrt{\frac{h_c}{c}}\le\sqrt[6]{\frac{3\sqrt{3}}{32}}\left(\sqrt[3]{\frac{b^2+c^2}{a^2}}+\sqrt[3]{\frac{c^2+a^2}{b^2}}+\sqrt[3]{\frac{a^2+b^2}{c^2}}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By $\mathrm{AM}-\mathrm{GM}$ inequality, we have

$$\sqrt{\frac{m_a}{a}}=\sqrt[4]{\frac{2(b^2+c^2)-a^2}{4a^2}}=\sqrt[12]{\frac{3a^2[2(b^2+c^2)-a^2]^3}{192a^8}}$$
$$\le\sqrt[12]{\frac{(3a^2+3[2(b^2+c^2)-a^2])^4}{4^4.192a^8}}=$$
$$=\sqrt[6]{\frac{3\sqrt{3}}{32}}.\sqrt[3]{\frac{b^2+c^2}{a^2}}.$$

Then

$$\sqrt{\frac{w_b}{b}}\le\sqrt{\frac{m_b}{b}}\le\sqrt[6]{\frac{3\sqrt{3}}{32}}.\sqrt[3]{\frac{c^2+a^2}{b^2}}\ \text{ and }\ \sqrt{\frac{h_c}{c}}\le\sqrt{\frac{m_c}{c}}\le\sqrt[6]{\frac{3\sqrt{3}}{32}}.\sqrt[3]{\frac{a^2+b^2}{c^2}}.$$

Therefore

$$\sqrt{\frac{m_a}{a}}+\sqrt{\frac{w_b}{b}}+\sqrt{\frac{h_c}{c}}\le\sqrt[6]{\frac{3\sqrt{3}}{32}}\left(\sqrt[3]{\frac{b^2+c^2}{a^2}}+\sqrt[3]{\frac{c^2+a^2}{b^2}}+\sqrt[3]{\frac{a^2+b^2}{c^2}}\right).$$

Equality holds iff ΔABC is equilateral.

3067. In any Δ ABC, the following relationship holds :

$$\frac{1}{m_a w_b w_c}+\frac{1}{m_b w_c w_a}+\frac{1}{m_c w_a w_b}\le\frac{\sqrt{3}}{s}\left(\frac{1}{4r^2}+\frac{h_a+h_b+h_c}{4r_a r_b r_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } m_a\ge w_a \text{ and analogs} \therefore \frac{1}{m_a w_b w_c}+\frac{1}{m_b w_c w_a}+\frac{1}{m_c w_a w_b}\le\frac{3}{w_a w_b w_c}$$

$$=\frac{3(s^2+2Rr+r^2)}{16Rr^2s^2}\overset{?}{\le}\frac{\sqrt{3}}{s}\left(\frac{1}{4r^2}+\frac{h_a+h_b+h_c}{4r_a r_b r_c}\right)=\frac{\sqrt{3}}{s}\left(\frac{1}{4r^2}+\frac{s^2+4Rr+r^2}{8Rrs^2}\right)$$

$$=\frac{\sqrt{3}}{s}\left(\frac{2Rs^2+r(s^2+4Rr+r^2)}{8Rr^2s^2}\right)$$

$$\Leftrightarrow\frac{9(s^2+2Rr+r^2)^2}{4}\overset{?}{\le}\frac{3\left(2Rs^2+r(s^2+4Rr+r^2)\right)^2}{s^2}\Leftrightarrow$$

$$-3s^6+(16R^2+4Rr-2r^2)s^4+r^2(52R^2+36Rr+5r^2)s^2+4r^4(4R+r)^2\overset{?}{\ge}0 \quad (1)$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\ge 0 \text{ and } s^2-(m+n)\le 0, \text{ where } m = 2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(a)}{\le}0$$

$$\Rightarrow -3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\ge 0 \text{ and so,}$$

in order to prove (1), it suffices to prove : LHS of (1) $\ge$ $-3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$

$$\Leftrightarrow (R^2-14Rr+r^2)s^4+r(48R^3+49R^2r+18Rr^2+2r^3)s^2+r^4(4R+r)^2\overset{?}{\ge}0 \quad (2)$$

We note that (2) is trivially true if : $R^2-14Rr+r^2\ge 0$ and so we now focus on the case when : $R^2-14Rr+r^2<0$ and then :

$$(R^2-14Rr+r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\overset{\text{via }(a)}{\ge}0 \text{ and so,}$$

in order to prove (2), it suffices to prove :

$$\text{LHS of (2)}\ge(R^2-14Rr+r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow(4R^3+12R^2r-229Rr^2+66r^3)s^2$$

$$-r(64R^4-848R^3r-596R^2r^2-135Rr^3-10r^4)\overset{(3)}{\ge}0$$

$$\boxed{\text{Case 1}}\ 4R^3+12R^2r-229Rr^2+66r^3\ge 0 \text{ and then : LHS of ③} \overset{\text{Gerretsen}}{\ge} (4R^3+12R^2r-229Rr^2+66r^3)(16Rr-5r^2)$$

$$-r(64R^4-848R^3r-596R^2r^2-135Rr^3-10r^4)\overset{?}{\ge}0$$

$$\Leftrightarrow 255t^3-782t^2+584t-80\overset{?}{\ge}0\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(255t^2-272t+40)\overset{?}{\ge}0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow \text{③ is true}$$

$$\boxed{\text{Case 2}}\ 4R^3+12R^2r-229Rr^2+66r^3<0 \text{ and then : LHS of ③} \overset{\text{Gerretsen}}{\ge} (4R^3+12R^2r-229Rr^2+66r^3)(4R^2+4Rr+3r^2)$$

$$-r(64R^4-848R^3r-596R^2r^2-135Rr^3-10r^4)\overset{?}{\ge}0$$

$$\Leftrightarrow 4t^5-2t^3-5t^2-72t+52\overset{?}{\ge}0\Leftrightarrow(t-2)(4t^4+8t^3+14t^2+23t-26)\overset{?}{\ge}0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow \text{③ is true} \therefore \text{combining both cases, ③} \Rightarrow \text{②} \Rightarrow \text{① is true}$$

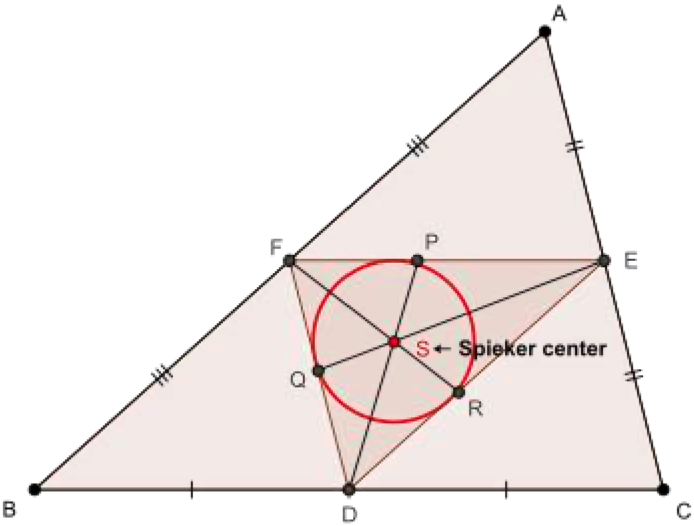
$$\forall\,\Delta\,ABC \therefore \frac{1}{m_a w_b w_c}+\frac{1}{m_b w_c w_a}+\frac{1}{m_c w_a w_b}\le \frac{\sqrt{3}}{s}\left(\frac{1}{4r^2}+\frac{h_a+h_b+h_c}{4r_a r_b r_c}\right)\ \forall\,\Delta\,ABC,\text{"="} \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3068. In any $\Delta\,ABC$, the following relationship holds :

$$p_a p_b p_c \ge s^2 r\sqrt{\frac{7R-4r}{5R}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakaraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say)
and inr$a$dius of Δ DEF $= r'(say)$

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right) - \sum_{cyc}\frac{a^4}{16} = \frac{1}{16}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)$$

$$= \frac{16r^2s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $arrive$ at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=\mathrm{Rr}\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=\mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again},\ \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2\mathrm{Rr}\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\ \mathrm{AFS},\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\ \text{and via sine law on }\Delta\ \mathrm{AES},\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2\overset{(\cdot)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\text{We have}:\prod_{\text{cyc}}(2s+a)=8s^3+4s^2\sum_{\text{cyc}}a+2s\sum_{\text{cyc}}ab+4\mathrm{Rrs}=$$

$$8s^3+4s^2.2s+2s\left(s^2+4Rr+r^2\right)+4Rrs \Rightarrow \prod_{cyc}(2s+a) \overset{(\bullet\bullet)}{=} 2s\left(9s^2+6Rr+r^2\right)$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right)=b^3+c^3+a^3-abc+a\left(2b^2+2c^2-a^2\right)-a^3$$
$$=\sum_{cyc} a^3-abc+2a.2bc\cos A=2s\left(s^2-6Rr-3r^2\right)-4Rrs+16Rrs\cos A$$
$$\Rightarrow b^3+c^3-abc+a\left(4m_a^2\right)=2s(Q+8Rr\cos A) \text{ and analogs}$$
$$\left(Q=s^2-8Rr-3r^2\right) \Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)$$

$$=8s^3\left(Q^3+Q^2.8Rr\sum_{cyc}\cos A+Q.64R^2r^2.\sum_{cyc}\cos B\cos C+512R^3r^3\prod_{cyc}\cos A\right)$$

$$=8s^3\left(\begin{array}{c}Q^3+Q^2.8Rr.\dfrac{R+r}{R}+Q.32R^2r^2.\left(\left(\dfrac{R+r}{R}\right)^2-\left(3-\dfrac{2\left(s^2-4Rr-r^2\right)}{4R^2}\right)\right)\\ +512R^3r^3.\dfrac{s^2-(2R+r)^2}{4R^2}\end{array}\right)$$

$$\Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right) \overset{(\bullet\bullet\bullet)}{=}$$

$$8s^3\left(\begin{array}{c}\left(s^2-8Rr-3r^2\right)^3+\left(s^2-8Rr-3r^2\right)^2.8r(R+r)+\\ \left(s^2-8Rr-3r^2\right).16r^2\left(s^2-4R^2+r^2\right)+128Rr^3\left(s^2-(2R+r)^2\right)\end{array}\right)$$

$$\therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet) \Rightarrow \prod_{cyc} p_a^2=$$

$$\frac{8s^3.8s^3\left(\begin{array}{c}\left(s^2-8Rr-3r^2\right)^3+\left(s^2-8Rr-3r^2\right)^2.8r(R+r)+\\ \left(s^2-8Rr-3r^2\right).16r^2\left(s^2-4R^2+r^2\right)+128Rr^3\left(s^2-(2R+r)^2\right)\end{array}\right)}{4s^2\left(9s^2+6Rr+r^2\right)^2} \overset{?}{\geq} s^4r^2.\frac{7R-4r}{5R}$$

$$\Leftrightarrow 80Rs^6-r\left(1280R^2-633Rr-324r^2\right)s^4-r^3\left(3316R^2+3934Rr-72r^2\right)s^2$$
$$-r^4\left(252R^3+1220R^2r+199Rr^2-4r^3\right) \underset{①}{\overset{?}{\geq}} 0 \text{ and } \because$$

$$P=80R\left(s^2-16Rr+5r^2\right)^3+r\left(2560R^2-567Rr+324r^2\right)\left(s^2-16Rr+5r^2\right)^2+$$
$$4r^2\left(5120R^3-2165R^2r+1526Rr^2-792r^3\right)\left(s^2-16Rr+5r^2\right) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } ①, \text{it suffices to prove : LHS of } ① \overset{?}{\geq} P$$
$$\Leftrightarrow 1585t^3-480t^2-6348t+1936 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left(1585t^2+2690t-968\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow ① \text{ is true} \therefore \prod_{cyc} p_a^2 \geq s^4r^2.\frac{7R-4r}{5R} \Rightarrow p_ap_bp_c \geq s^2r.\sqrt{\frac{7R-4r}{5R}} \ \forall \Delta ABC,$$
$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3069. In any Δ ABC the following relationship holds :

$$2\sum_{cyc}\frac{m_a}{a}\le\sum_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{s}{2r}\overset{?}{\ge}\sum_{cyc}\frac{m_a}{a}\Leftrightarrow\frac{s}{2r}\overset{?}{\ge}\frac{1}{4Rrs}\sum_{cyc}bcm_a\Leftrightarrow(2Rs^2)^2\overset{?}{\ge}\left(\sum_{cyc}bcm_a\right)^2$$

$$=\sum_{cyc}\left(b^2c^2\left(\frac{2b^2+2c^2-a^2}{4}\right)\right)+2\sum_{cyc}(bc.ca.m_am_b)$$

$$\Leftrightarrow 16R^2s^4\overset{?}{\ge}\sum_{cyc}\left(b^2c^2\left(2\sum_{cyc}a^2-3a^2\right)\right)+32Rrs\sum_{cyc}cm_am_b$$

$$\Leftrightarrow 16R^2s^4\ \boxed{\overset{?}{\ge}\ (1)}\ 4(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$-144R^2r^2s^2+32Rrs\sum_{cyc}cm_am_b$$

$$\text{Now, } m_am_b\overset{?}{\le}\frac{2c^2+ab}{4}\Leftrightarrow\left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right)\overset{?}{\le}\frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2\overset{?}{\ge}0$$

$$\Leftrightarrow(a+b)^2(a-b)^2-c^2(a-b)^2\overset{?}{\ge}0$$

$$\Leftrightarrow(a-b)^2(a+b+c)(a+b-c)\overset{?}{\ge}0\to\text{true}\Rightarrow m_am_b\le\frac{2c^2+ab}{4}\ \text{and analogs}$$

$$\therefore\ \text{RHS of (1)}\le 4(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-144R^2r^2s^2$$

$$+32Rrs\sum_{cyc}\left(c.\frac{2c^2+ab}{4}\right)\overset{?}{\le}16R^2s^4$$

$$\Leftrightarrow(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-36R^2r^2s^2$$

$$+8Rrs(4(s^2-6Rr-3r^2)+12Rrs)\overset{?}{\le}4R^2s^4$$

$$\Leftrightarrow s^6-(4R^2+4Rr-r^2)s^4-(12R^2+16Rr+r^2)r^2s^2-r^3(4R+r)^3\ \boxed{\overset{?}{\le}\ (2)}\ 0$$

$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\ge 0\ \text{and}\ s^2-(m+n)\le 0,\ \text{where}\ m=$$
$$2R^2+10Rr-r^2\ \text{and}\ n=2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0$$

$$\Rightarrow s^6-(4R^2+20Rr-2r^2)s^4+r(4R+r)^3.s^2\le 0$$

$\therefore$ in order to prove ②, it suffices to prove :

$$s^6-(4R^2+4Rr-r^2)s^4-(12R^2+16Rr+r^2)r^2s^2-r^3(4R+r)^3\overset{?}{\le}$$

$$s^6-(4R^2+20Rr-2r^2)s^4+r(4R+r)^3.s^2$$

$$\Leftrightarrow (16R-5r)s^4-(64R^3+60R^2r+28Rr^2+2r^3)s^2-r^2(4R+r)^3\underset{③}{\overset{?}{\le}}0$$

Again, LHS of ③ $\overset{\text{Gerretsen}}{\le}$

$$\left((16R-5r)(4R^2+4Rr+3r^2)-(64R^3+60R^2r+28Rr^2+2r^3)\right)s^2$$

$$-r^2(4R+r)^3\overset{?}{\le}0\Leftrightarrow (16R-5r)s^2\underset{④}{\overset{?}{\le}}(4R+r)^3$$

Finally, LHS of ④ $\overset{\text{Gerretsen}}{\le}(16R-5r)(4R^2+4Rr+3r^2)\overset{?}{\le}(4R+r)^3$

$$\Leftrightarrow 4r(R-2r)^2\overset{?}{\ge}0\to \text{true}\Rightarrow ④\Rightarrow ③\Rightarrow ②\Rightarrow ① \text{ is true}\therefore 2\sum_{cyc}\frac{m_a}{a}\le\frac{s}{r}\to (m)$$

We have : $\sum_{cyc}\frac{r_b+r_c}{a}=\sum_{cyc}\frac{4R\cos^2\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}=s.\sum_{cyc}\frac{1}{r_a}\Rightarrow\sum_{cyc}\frac{r_b+r_c}{a}=\frac{s}{r}\to(*)$

and also, $\prod_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\frac{4R\cos^2\frac{A}{2}}{a}=\frac{64R^3.\frac{s^2}{16R^2}}{4Rrs}\Rightarrow\prod_{cyc}\frac{r_b+r_c}{a}=\frac{s}{r}\to(**)$

Moreover, $\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\prod_{cyc}\sqrt{\frac{4R\cos^2\frac{A}{2}.(s-a)}{ra}}=\sqrt{\frac{64R^3.\frac{s^2}{16R^2}.r^2s}{r^3.4Rrs}}$

$\Rightarrow\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\frac{s}{r}\to(***)$ and finally, $\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\sum_{cyc}\sqrt{\frac{4R\cos^2\frac{A}{2}.(s-a)}{ra}}$

$$=\sum_{cyc}\sqrt{\frac{4R\cos^2\frac{A}{2}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\sin\frac{A}{2}}}=s\sum_{cyc}\sqrt{\frac{1}{r_a^2}}\Rightarrow\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}\frac{s}{r}\to(****)$$

$\therefore$ (m), (*), (**), (***) and (****) $\Rightarrow$

$$2\sum_{cyc}\frac{m_a}{a}\le\sum_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

3070. In ΔABC the following relationship holds:

$$\sum \frac{1}{2r+h_a} \le \frac{3}{5r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x=\frac{r}{h_a}, y=\frac{r}{h_b}, z=\frac{r}{h_c}\ then\ x+y+z=r\sum \frac{1}{h_a}=\frac{r}{r}=1$$

$$We\ need\ to\ show:$$

$$\sum \frac{1}{2r+h_a} \le \frac{3}{5r}\ or \sum \frac{r}{2r+h_a} \le \frac{3}{5}\ or \sum \frac{\frac{r}{h_a}}{2\frac{r}{h_a}+1} \le \frac{3}{5}\ or\ \sum \frac{x}{2x+1} \le \frac{3}{5}$$

Lemma :

$$\frac{x}{2x+1} \le \frac{2+9x}{25}\ \forall\ x>0$$

Proof:

$$\frac{x}{2x+1} \le \frac{2+9x}{25}\ or\ 25x \le (2x+1)(2+9x)\ or\ 18x^2-12x+2 \ge 0\ or$$
$$2(3x-1)^2 \ge 0\ (true)$$

$$\sum \frac{x}{2x+1} \overset{Lemma}{\le} \sum \frac{2+9x}{25} = \frac{1}{25}\big(3\times 2+9(x+y+z)\big) \overset{x+y+z=1}{=} \frac{1}{25}(6+9) = \frac{15}{25} = \frac{3}{5}$$

Equality holds for an equilateral triangle.

3071. In ΔABC the following relationship holds:

$$\sum \frac{1}{2r+r_a} \le \frac{3}{5r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x=\frac{r}{r_a}, y=\frac{r}{r_b}, z=\frac{r}{r_c}\ then\ x+y+z=r\sum \frac{1}{r_a}=\frac{r}{r}=1$$

$$We\ need\ to\ show:$$

$$\sum \frac{1}{2r+r_a} \le \frac{3}{5r} \; or, \sum \frac{r}{2r+r_a} \le \frac{3}{5} \; or \sum \frac{\frac{r}{r_a}}{2\frac{r}{r_a}+1} \le \frac{3}{5} \; or, \sum \frac{x}{2x+1} \le \frac{3}{5}$$

Lemma :

$$\frac{x}{2x+1} \le \frac{2+9x}{25} \; \forall x > 0$$

Proof:

$$\frac{x}{2x+1} \le \frac{2+9x}{25} \; or \; 25x \le (2x+1)(2+9x) \; or \; 18x^2 - 12x + 2 \ge 0 \; or$$

$$2(3x-1)^2 \ge 0 \; (true)$$

$$\sum \frac{x}{2x+1} \overset{Lemma}{\le} \sum \frac{2+9x}{25} = \frac{1}{25}\big(3 \times 2 + 9(x+y+z)\big) \overset{x+y+z=1}{=} \frac{1}{25}(6+9) = \frac{15}{25} = \frac{3}{5}$$

Equality holds for an equilateral triangle.

3072. In ΔABC the following relationship holds:

$$A \cdot \frac{a^2}{bc} + B \cdot \frac{b^2}{ac} + C \cdot \frac{c^2}{ab} \ge 4\pi \; \min\left(sin^2\frac{A}{2}, sin^2\frac{B}{2}, sin^2\frac{C}{2}\right)$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$A \cdot \frac{a^2}{bc} + B \cdot \frac{b^2}{ac} + C \cdot \frac{c^2}{ab} = A \cdot \frac{a^3}{abc} + B \cdot \frac{b^3}{abc} + C \cdot \frac{c^3}{abc} =$$

$$= \frac{1}{abc}(A.a^3 + B.b^3 + C.c^3) \overset{Chebyshev}{\ge} \frac{1}{abc} \cdot \frac{1}{3}(A+B+C)(a^3+b^3+c^3) =$$

$$= \frac{1}{12Rrs}\pi \cdot 2s(s^2 - 3r^2 - 6Rr) \overset{Gerretsen}{\ge} \frac{\pi}{6Rr}(16Rr - 5r^2 - 3r^2 - 6Rr) =$$

$$= \frac{\pi}{6Rr}(10Rr - 8r^2)$$

We need to show:

$$\frac{\pi}{6Rr}(10Rr - 8r^2) \ge 4\pi \min\left(sin^2\frac{A}{2}, sin^2\frac{B}{2}, sin^2\frac{C}{2}\right)$$

$$\frac{\pi}{6Rr}(10Rr - 8r^2) \ge 4\pi \cdot \frac{1}{3}\sum \sin^2\frac{A}{2}$$

$$or \; \frac{10Rr - 8r^2}{6Rr} \ge \frac{4}{3}\frac{(2R-r)}{2R} \; or \; 10Rr - 8r^2 \ge 8Rr - 4r^2$$

$$2Rr \ge 4r^2 \; or \; R \ge 2r \; (Euler)$$

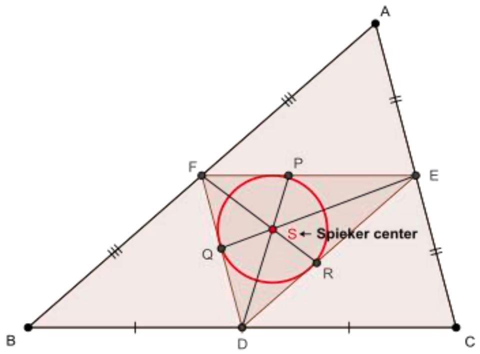
Equality holds for an equilateral triangle.

3073. In any Δ ABC with $r_a = 3r$, the following relationship holds :

$$w_a^2 + m_a^2 + p_a^2 = h_a^2 + g_a^2 + n_a^2 + \frac{21r(R-2r)}{8}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right) - \sum_{cyc}\frac{a^4}{16} = \frac{1}{16}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)$$

$$= \frac{16r^2s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$
$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$
$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$
$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$
$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$
$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$
$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}=$$
$$(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)((z+x)+(x+y)-2(y+z))}{4}-\frac{a(b-c)^2}{4}$$
$$(a=y+z, b=z+x, c=x+y)$$
$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$ and $b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)$ and via summation, we get:

$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)=$$
$$2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+$$
$$a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 4m_a^2+(b-c)^2$$
$$=2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)}{bc}$$

$$= as^2 - as\left(\frac{a^2 - (b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - \frac{a^2 - (b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 \overset{\circledast\circledast}{=} s(s-a) + \frac{s}{a}(b-c)^2$$

Via $\circledast$ and $\circledast\circledast$, we get: $g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a)\left(s - a + \frac{a^2 - (b-c)^2}{a}\right) = (s-a)\left(s - \frac{(b-c)^2}{a}\right)$$

$$\Rightarrow g_a^2 \overset{\circledast\circledast\circledast}{=} s(s-a) - \frac{s-a}{a}(b-c)^2$$

$\therefore$ via $(\bullet\bullet\bullet)$, $\circledast\circledast$ and $\circledast\circledast\circledast$, $w_a^2 + m_a^2 + p_a^2 - h_a^2 - g_a^2 - n_a^2$

$$= s(s-a) - \frac{s(s-a)}{(2s-a)^2}.(b-c)^2 + s(s-a) + \frac{(b-c)^2}{4} + s(s-a)$$

$$+\frac{s(3s+a)}{(2s+a)^2}.(b-c)^2 - s(s-a) + \frac{s(s-a)}{a^2}.(b-c)^2 + \frac{s-a}{a}.(b-c)^2 - s(s-a)$$

$$-\frac{s}{a}.(b-c)^2 \Rightarrow w_a^2 + m_a^2 + p_a^2 - h_a^2 - g_a^2 - n_a^2$$

$$\overset{(\blacksquare)}{=} \frac{64s^6 - 64s^5a - 48s^4a^2 + 36s^2a^4 + 4sa^5 - 3a^6}{4a^2(4s^2 - a^2)^2}.(b-c)^2$$

$$\text{Now, } r_a = 3r \Rightarrow \frac{rs}{s-a} = 3r \Rightarrow 2s \overset{(\blacksquare\blacksquare)}{=} 3a \Rightarrow 2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$= 3.4R\sin\frac{A}{2}\cos\frac{A}{2} \Rightarrow \sin\frac{A}{2} + \cos\frac{B-C}{2} = 3\sin\frac{A}{2}$$

$$\Rightarrow 2S \overset{(\blacksquare\blacksquare\blacksquare)}{=} C\left(S = \sin\frac{A}{2}, C = \cos\frac{B-C}{2}\right)$$

Via $(\blacksquare)$ and $(\blacksquare\blacksquare)$, we have : $w_a^2 + m_a^2 + p_a^2 - h_a^2 - g_a^2 - n_a^2 =$

$$\frac{729a^6 - 486a^6 - 243a^6 + 81a^6 + 6a^6 - 3a^6}{4a^2(9a^2 - a^2)^2}.16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}$$

$$=\frac{21}{4}.R^2S^2(1-C^2)\therefore \text{via } (\blacksquare\blacksquare\blacksquare), \text{we have}:$$

$$w_a^2+m_a^2+p_a^2-h_a^2-g_a^2-n_a^2 \overset{\circledast}{=} \frac{21}{16}.R^2C^2(1-C^2)$$

$$\text{Finally}, \frac{21r(R-2r)}{8}=\frac{21.4R^2.\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.(1-4SC+4S^2)}{8}$$

$$=\frac{21.2R^2.S(C-S).(1-4SC+4S^2)}{8} \overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=} \frac{21.2R^2.\frac{C}{2}\left(C-\frac{C}{2}\right).(1-2C^2+C^2)}{8}$$

$$\Rightarrow \frac{21r(R-2r)}{8} \overset{\circledast\circledast}{=} \frac{21}{16}.R^2C^2(1-C^2)\therefore \circledast \text{ and } \circledast\circledast$$

$$\Rightarrow w_a^2+m_a^2+p_a^2=h_a^2+g_a^2+n_a^2+\frac{21r(R-2r)}{8}\ \forall\,\Delta\, ABC \text{ with } r_a=3r \text{ (QED}$$

3074. In ΔABC the following relationship holds:

$$\sum\frac{a}{2R+r-r_a}=\prod\frac{a}{2R+r-r_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum ar_a=F\sum\frac{a}{s-a}=\frac{F}{\prod(s-a)}\left(\sum(a(s-b)(s-c))\right)=$$
$$=\frac{F}{sr^2}(s^2(a+b+c)-2s(ab+bc+ca)+3abc)$$
$$=\frac{r.s}{sr^2}(2s^3-2s(s^2+r^2+4Rr)+12Rrs)=\frac{2s(2Rr-r^2)}{r}=2s(2R-r)\ (1)$$

$$\sum a(r_b+r_c)=\left(\sum a\right)\left(\sum r_a\right)-\sum ar_a=$$
$$=2s(4R+r)-2s(2R-r)=2s(2R+2r)\ (2)$$

$$\sum ar_br_c=F^2\sum\frac{a}{(s-b)(s-c)}=\frac{F^2}{\prod(s-a)}\sum a(s-a)=$$
$$=\frac{r^2s^2}{sr^2}\left(s(a+b+c)-\sum a^2\right)=s(2s^3-2(s^2-r^2-4Rr)=2s(4Rr+r^2)\ (3)$$

$$\sum a(2R+r-r_b)(2R+r-r_c)=$$

$$=\sum a(4R^2+4Rr-2R(r_b+r_c)+r^2-r(r_b+r_c)+r_br_c)=$$

$$=4R^2\sum a+4Rr\sum a-2R\sum a(r_b+r_c)+r^2\sum a-r\sum a(r_b+r_c)+\sum ar_br_c$$

$$\overset{using\ (2)\&\ (3)}{=}2s(2Rr)=4Rrs=abc$$

$$\sum\frac{a}{2R+r-r_a}=\frac{\sum a(2R+r-r_b)(2R+r-r_c)}{\prod(2R+r-r_a)}=\frac{abc}{\prod(2R+r-r_a)}=$$

$$=\frac{a}{2R+r-r_a}\cdot\frac{b}{2R+r-r_b}\frac{c}{2R+r-r_c}=\prod\frac{a}{2R+r-r_a}$$

3075. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{2a+x}{-a+b+c+y}+\frac{2b+x}{a-b+c+y}+\frac{2c+x}{a+b-c+y}\geq\frac{3(4s+3x)}{2s+3y}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ 2a+x\geq 2b+x\geq 2c+x\ and$$
$$2s-2a+y\leq 2s-2b+y\leq 2s-2c+y$$

$$\frac{2a+x}{-a+b+c+y}+\frac{2b+x}{a-b+c+y}+\frac{2c+x}{a+b-c+y}\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\left(\sum(2a+x)\right)\left(\sum\frac{1}{2s-2a+y}\right)\overset{CBS}{\geq}\frac{1}{3}(2(a+b+c)+3x)\left(\frac{(1+1+1)^2}{6s-4s+3y}\right)=$$

$$=\frac{1}{3}(4s+3x).\frac{9}{2s+3y}\geq\frac{3(4s+3x)}{2s+3y}$$

Equality holds for an equilateral triangle and x=y=0.

3076. In ΔABC the following relationship holds:

$$\frac{1}{h_a^2\sin A}+\frac{1}{h_b^2\sin B}+\frac{1}{h_c^2\sin C}\ge\frac{2}{F}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\frac{1}{h_a^2\sin A}+\frac{1}{h_b^2\sin B}+\frac{1}{h_c^2\sin C}=\sum\frac{1}{h_a^2\sin A}=\sum a^2.\frac{2R}{a.4F^2}=$$

$$=\frac{R}{2F^2}\sum a=\frac{R}{2F^2}.2s=\frac{Rs}{F^2}\overset{Euler}{\ge}\frac{2rs}{F^2}=\frac{2}{F}$$

Equality holds for an equilateral triangle.

3077. In ΔABC the following relationship holds:

$$\frac{a^mb}{h_a.h_b^m}+\frac{b^mc}{h_b.h_c^m}+\frac{c^ma}{h_c.h_a^m}\ge 2^{m+1}\left(\sqrt{3}\right)^{1-m}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das

$$\frac{a^mb}{h_a.h_b^m}+\frac{b^mc}{h_b.h_c^m}+\frac{c^ma}{h_c.h_a^m}=\sum\frac{a^mb}{h_a.h_b^m}=\sum\frac{a^m.b.a.b^m}{(2F)(2F)^m}=\sum\frac{a^{m+1}b^{m+1}}{2^{m+1}F^{m+1}}=$$

$$=\frac{1}{(2F)^{m+1}}\sum(ab)^{m+1}\overset{CBS}{\ge}\frac{1}{(2F)^{m+1}}\frac{(\sum ab)^{m+1}}{3^m}\ge$$

$$\overset{Gordon}{\ge}\frac{1}{(2F)^{m+1}}\frac{1}{3^m}\left(4\sqrt{3}F\right)^{m+1}=\frac{1}{(2F)^{m+1}}\frac{1}{\sqrt{3}^{2m}}\left(2^2\sqrt{3}F\right)^{m+1}=2^{m+1}\left(\sqrt{3}\right)^{1-m}$$

Equality holds for an equilateral triangle.

3078. In ΔABC the following relationship holds:

$$13(a^2+b^2+c^2)\geq 48\sqrt{3}F+\sum_{cyc}(2a-3b)^2$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das-India

$$13(a^2+b^2+c^2)-\left(48\sqrt{3}F+\sum_{cyc}(2a-3b)^2\right)=$$

$$=13\sum a^2-48\sqrt{3}F-\sum(4a^2+9b^2-12ab)=$$

$$=13\sum a^2-48\sqrt{3}F-4\sum a^2-9\sum b^2+12\sum ab=$$

$$=13\sum a^2-13\sum a^2+12\sum ab-48\sqrt{3}F\overset{Gordon}{\geq}12.\left(4\sqrt{3}F\right)-48\sqrt{3}F=0$$

Equality holds for an equilateral triangle.

3079. In ΔABC the following relationship holds:

$$((r_a+r_b)^2+2)((r_b+r_c)^2+2)((r_c+r_a)^2+2)\geq 108\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution by Tapas Das-India

$$((r_a+r_b)^2+2)((r_b+r_c)^2+2)((r_c+r_a)^2+2)=\prod((r_a+r_b)^2+2)=$$

$$=\prod((r_a+r_b)^2+1+1)\geq$$

$$\overset{AM-GM}{\geq}\prod\left(3(r_a+r_b)^{\frac{2}{3}}\right)=27\sqrt[3]{(r_a+r_b)^2(r_b+r_c)^2(r_c+r_a)^2}\geq$$

$$\overset{AM-GM}{\geq}27\sqrt[3]{4r_ar_b.4r_br_c.4r_cr_a}=108\sqrt[3]{(r_ar_br_c)^2}=108\sqrt[3]{s^4r^2}\geq$$

$$\overset{Mitrinovic}{\geq}108\sqrt[3]{s^3.3\sqrt{3}r.r^2}=108\sqrt[3]{F^3\left(\sqrt{3}\right)^3}=108\sqrt{3}F$$

Equality holds for an equilateral triangle.

3080. In any Δ ABC, the following relationship holds:

$$\frac{(w_a+w_b+w_c)(m_a+m_b+m_c)}{(h_a+h_b+h_c)(r_a+r_b+r_c)} \le \frac{(a+b+c)^2}{3(ab+bc+ca)} \le \frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{(r_a+r_b+r_c)(m_a+m_b+m_c)}{(w_a+w_b+w_c)(h_a+h_b+h_c)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} w_a = \sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right) \overset{CBS}{\le} 2\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}}$$

$$\overset{A-G}{\le} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}} = \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}} = \sqrt{\sum_{cyc} bc}.\sqrt{\frac{4R+r}{2R}}$$

$$= \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \Rightarrow \sum_{cyc} w_a \le \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)}$$

$$\therefore \frac{(r_a+r_b+r_c)(m_a+m_b+m_c)}{(w_a+w_b+w_c)(h_a+h_b+h_c)} \overset{Chu-Yang}{\ge} \frac{2R(4R+r).\sqrt{4s^2-28Rr+29r^2}}{\sqrt{(\sum_{cyc} h_a)(\sum_{cyc} r_a)}.\sum_{cyc} ab} \overset{?}{\ge} \frac{\sum_{cyc} a^2}{\sum_{cyc} ab}$$

$$\Leftrightarrow 2R^3(4R+r)(4s^2-28Rr+29r^2) \ge (s^2+4Rr+r^2)(s^2-4Rr-r^2)^2$$
$$\Leftrightarrow -s^6+(4Rr+r^2)s^4+(32R^4+8R^3r+16R^2r^2+8Rr^3+r^4)s^2$$
$$-r(224R^5-176R^4r+6R^3r^2+48R^2r^3+12Rr^4+r^5) \overset{(1)}{\ge} 0$$

Now, via Gerretsen, $P = -s^4(s^2-4R^2-4Rr-3r^2)$
$$-(4R^2+2r^2)s^2(s^2-4R^2-4Rr-3r^2)$$
$$-(16R^4-8R^3r-4R^2r^2-5r^4)(s^2-16Rr+5r^2) \ge 0$$

$\therefore$ in order to prove (1), it suffices to prove : LHS of (1) $\ge P$

$$\Leftrightarrow 16t^5-16t^4-15t^3-14t^2-46t+12 \ge 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(16t^4+16t^3+17t^2+20t-6) \ge 0 \to \text{true} \because t \overset{Euler}{\ge} 2 \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{(r_a+r_b+r_c)(m_a+m_b+m_c)}{(w_a+w_b+w_c)(h_a+h_b+h_c)} \to (i)$$

We shall now evaluate : $\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}$ and it's $= \sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2}$

$$= 3-\sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$

$$= 4s\sum_{cyc}\frac{1}{b+c} - 4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2 - \frac{2}{2s(s^2+2Rr+r^2)}.\sum_{cyc}(b+c)\right)$$

$$=\frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}+\frac{16s^2}{s^2+2Rr+r^2}$$

$$\Rightarrow \sum_{cyc}\frac{4s(s-a)}{(b+c)^2}\overset{(*)}{=}\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(**)}{\le}0$$

$$\text{Now, } w_a+w_b+w_c=\frac{\sqrt{bc}}{b+c}.\sqrt{4s(s-a)}\overset{CBS}{\le}\sqrt{\sum_{cyc}bc}.\sqrt{\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}}\overset{\text{via }(*)}{=}$$

$$\sqrt{(s^2+4Rr+r^2).\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}}\overset{?}{\le}\sqrt{s^2+21Rr+12r^2}$$

$$\Leftrightarrow (R-5r)s^4+r(8R^2-2Rr+6r^2)s^2+$$

$$r^2(84R^3+116R^2r+61Rr^2+11r^3)\overset{?}{\underset{②}{\ge}}0 \text{ and it's trivially true if : } R-5r\ge 0$$

and so, we now focus on the case when : $R-5r<0$ and then, via $(**)$,

$$(R-5r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\ge 0$$

$\therefore$ in order to prove ②, it suffices to prove :

$$\text{LHS of ②}\ge (R-5r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow (R^3+2R^2r-26Rr^2+4r^3)s^2\overset{?}{\underset{③}{\ge}}r(16R^4-89R^3r-86R^2r^2-30Rr^3-4r^4)$$

Case 1 $R^3+2R^2r-26Rr^2+4r^3\ge 0$ and then : LHS of ③ $\overset{Gerretsen}{\ge}$

$$(R^3+2R^2r-26Rr^2+4r^3)(16Rr-5r^2)\overset{?}{\ge}\text{RHS of ③}$$

$$\Leftrightarrow 29t^3-85t^2+56t-4\overset{?}{\ge}0\Leftrightarrow (t-2)(29t^2-27t+2)\overset{?}{\ge}0\to \text{true}\because t\overset{Euler}{\ge}2$$

$\Rightarrow$ ③ is true

Case 2 $R^3+2R^2r-26Rr^2+4r^3<0$ and then : LHS of ③ $\overset{Gerretsen}{\ge}$

$$(R^3+2R^2r-26Rr^2+4r^3)(4R^2+4Rr+3r^2)\overset{?}{\ge}\text{RHS of ③}$$

$$\Leftrightarrow t^5-t^4-t^3+t^2-8t+4\overset{?}{\ge}0\Leftrightarrow (t-2)(t^4+t^3+t^2+3t-2)\overset{?}{\ge}0\to \text{true}$$

$\because t\overset{Euler}{\ge}2\Rightarrow$ ③ is true $\therefore$ combining both cases, ③ $\Rightarrow$ ② is true $\forall\,\Delta ABC$

$$\therefore w_a+w_b+w_c\le\sqrt{s^2+21Rr+12r^2}\Rightarrow\frac{(w_a+w_b+w_c)(m_a+m_b+m_c)}{(h_a+h_b+h_c)(r_a+r_b+r_c)}$$

$$\overset{Chu-Yang}{\le}\frac{2R.\sqrt{(s^2+21Rr+12r^2)(4s^2-16Rr+5r^2)}}{(s^2+4Rr+r^2)(4R+r)}\overset{?}{\le}\frac{(a+b+c)^2}{3(ab+bc+ca)}$$

$$=\frac{4s^2}{3(s^2+4Rr+r^2)}\Leftrightarrow 4(4R+r)^2s^4\overset{?}{\ge}9R^2(s^2+21Rr+12r^2)(4s^2-16Rr+5r^2)$$

$$\Leftrightarrow (28R^2+32Rr+4r^2)s^4-R^2r(612R+477r)s^2$$

$$+R^2r^2(3024R^2+783Rr-540r^2)\overset{?}{\underset{④}{\geq}}0$$

$$\text{Now, via Gerretsen, } Q=(28R^2+32Rr+4r^2)(s^2-16Rr+5r^2)^2$$
$$+r(284R^3+267R^2r-192Rr^2-40r^3)(s^2-16Rr+5r^2)\geq 0$$
$$\therefore \text{ in order to prove ④, it suffices to prove : LHS of ④} \geq Q$$
$$\Leftrightarrow 400t^4-77t^3-1551t^2+160t+100\geq 0$$
$$\Leftrightarrow (t-2)(400t^3+657t^2+53t(t-2)+13(t^2-4)+t+2)\geq 0$$
$$\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow ④ \text{ is true} \therefore \frac{(w_a+w_b+w_c)(m_a+m_b+m_c)}{(h_a+h_b+h_c)(r_a+r_b+r_c)}\leq$$
$$\frac{(a+b+c)^2}{3(ab+bc+ca)}\to \text{(ii)} \therefore \text{(i) and (ii) combined with } 3\sum_{cyc}a^2\geq\left(\sum_{cyc}a\right)^2 \text{ gives :}$$
$$\frac{(w_a+w_b+w_c)(m_a+m_b+m_c)}{(h_a+h_b+h_c)(r_a+r_b+r_c)}\leq\frac{(a+b+c)^2}{3(ab+bc+ca)}\leq\frac{a^2+b^2+c^2}{ab+bc+ca}$$
$$\leq\frac{(r_a+r_b+r_c)(m_a+m_b+m_c)}{(w_a+w_b+w_c)(h_a+h_b+h_c)}\ \forall\ ABC, \text{with equality iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3081. In any Δ ABC, the following relationship holds :

$$w_a+w_b+w_c\leq h_a+h_b+h_c+\frac{38}{25}(R-2r)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}=\sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2}=3-\sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$
$$=4s\sum_{cyc}\frac{1}{b+c}-4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2-\frac{2}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}(b+c)\right)$$
$$=\frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}+\frac{16s^2}{s^2+2Rr+r^2}$$
$$\Rightarrow\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}\overset{(*)}{=}\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}$$
$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\geq 0 \text{ and } s^2-(m+n)\leq 0, \text{where } m=$$
$$2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$
$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$
$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(**)}{\leq}0$$

$$\text{Now, } w_a+w_b+w_c=\frac{\sqrt{bc}}{b+c}\cdot\sqrt{4s(s-a)}\overset{CBS}{\le}\sqrt{\sum_{cyc}bc}\cdot\sqrt{\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}}\overset{via\,(*)}{=}$$

$$\sqrt{(s^2+4Rr+r^2)\cdot\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}}\overset{?}{\le}\sqrt{s^2+21Rr+12r^2}$$

$$\Leftrightarrow (R-5r)s^4+r(8R^2-2Rr+6r^2)s^2+$$

$$r^2(84R^3+116R^2r+61Rr^2+11r^3)\underset{①}{\overset{?}{\ge}}0 \text{ and it's trivially true if : } R-5r\ge 0$$

$$\text{and so, we now focus on the case when : } R-5r<0 \text{ and then, via } (**),$$

$$(R-5r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\ge 0$$

$$\therefore \text{ in order to prove ①, it suffices to prove :}$$

$$\text{LHS of ①}\ge (R-5r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow (R^3+2R^2r-26Rr^2+4r^3)s^2\underset{②}{\overset{?}{\ge}}r(16R^4-89R^3r-86R^2r^2-30Rr^3-4r^4)$$

Case 1 $R^3+2R^2r-26Rr^2+4r^3\ge 0$ and then : LHS of ② $\overset{Gerretsen}{\ge}$

$$(R^3+2R^2r-26Rr^2+4r^3)(16Rr-5r^2)\overset{?}{\ge}\text{RHS of ②}$$

$$\Leftrightarrow 29t^3-85t^2+56t-4\overset{?}{\ge}0\Leftrightarrow(t-2)(29t^2-27t+2)\overset{?}{\ge}0\to\text{true} \because t\overset{Euler}{\ge}2$$

$$\Rightarrow ② \text{ is true}$$

Case 2 $R^3+2R^2r-26Rr^2+4r^3<0$ and then : LHS of ② $\overset{Gerretsen}{\ge}$

$$(R^3+2R^2r-26Rr^2+4r^3)(4R^2+4Rr+3r^2)\overset{?}{\ge}\text{RHS of ②}$$

$$\Leftrightarrow t^5-t^4-t^3+t^2-8t+4\overset{?}{\ge}0\Leftrightarrow(t-2)(t^4+t^3+t^2+3t-2)\overset{?}{\ge}0\to\text{true}$$

$$\because t\overset{Euler}{\ge}2\Rightarrow ② \text{ is true} \therefore \text{combining both cases, } ②\Rightarrow ① \text{ is true } \forall\,\Delta ABC$$

$$\therefore w_a+w_b+w_c\le\sqrt{s^2+21Rr+12r^2}\overset{?}{\le}h_a+h_b+h_c+\frac{38}{25}(R-2r)$$

$$\Leftrightarrow s^2+21Rr+12r^2\overset{?}{\le}\frac{(s^2+4Rr+r^2)^2}{4R^2}+\frac{1444}{625}(R-2r)^2$$

$$+\left(\frac{s^2+4Rr+r^2}{R}\right)\cdot\frac{38}{25}(R-2r)$$

$$\Leftrightarrow 625(s^2+4Rr+r^2)^2+5776R^2(R-2r)^2+3800R(R-2r)(s^2+4Rr+r^2)$$

$$-2500R^2(s^2+21Rr+12r^2)\underset{\circledast}{\overset{?}{\ge}}0 \text{ and } \because 625(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\ge}0$$

$$\therefore \text{ in order to prove } \circledast, \text{ it suffices to prove : LHS of } \circledast\overset{?}{\ge}625(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(325R^2+4350Rr-1250r^2)s^2+1444R^4-15101R^3r-45874R^2r^2$$

$$+24350Rr^3-3750r^4\underset{\circledast\circledast}{\overset{?}{\ge}}0$$

$$\text{We have : LHS of } \circledast\circledast \overset{\text{Rouche}}{\geq}$$

$$(325R^2+4350Rr-1250r^2)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$+1444R^4-15101R^3r-45874R^2r^2+24350Rr^3-3750r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(2094R^3+1037R^2r-3125Rr^2+1250r^3) \boxed{\overset{?}{\underset{\circledast\circledast\circledast}{\geq}}}$$

$$2(R-2r).\sqrt{R^2-2Rr}.(325R^2+4350Rr-1250r^2) \text{ and } \because R-2r \overset{\text{Euler}}{\geq} 0$$

$$\therefore \text{ in order to prove } \circledast\circledast\circledast, \text{it suffices to prove :}$$

$$(2094R^3+1037R^2r-3125Rr^2+1250r^3)^2 \overset{?}{>}$$

$$4(R^2-2Rr)(325R^2+4350Rr-1250r^2)^2$$

$$\Leftrightarrow 3962336t^6-6122044t^5-61832131t^4+187133750t^3-80891875t^2$$

$$+4687500t+1562500 \overset{?}{>} 0$$

$$\Leftrightarrow (t-2)\left((t-2)\left((t-2)((t-2).T+81568702)+19136601\right)+219101800\right)$$

$$+252810000 \overset{?}{>} 0 \text{ (with } T=3962336t^2+25576644t+47684957)$$

$$\rightarrow \text{true } \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \circledast\circledast\circledast \Rightarrow \circledast\circledast \Rightarrow \circledast \text{ is true } \Rightarrow w_a+w_b+w_c \leq$$

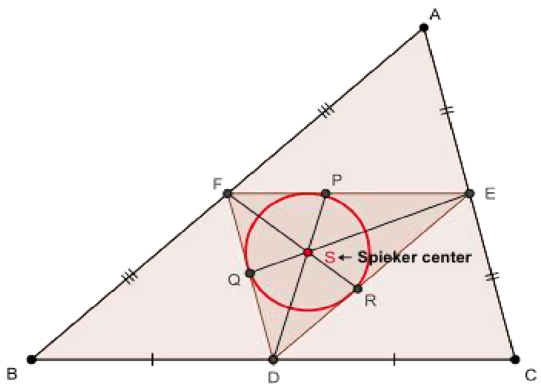
$$h_a+h_b+h_c+\frac{38}{25}(R-2r)\ \forall\ ABC, \text{with equality iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3082. In any Δ ABC, the following relationship holds :

$$s\sum_{cyc}\frac{\sqrt{4r^2+(b-c)^2}}{p_a} \geq \sum_{cyc}\left(a\sqrt{\frac{w_a}{g_a}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say)

$\boldsymbol{and}$ **inradius of** Δ **DEF** $= r'(say)$

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right) - \sum_{cyc}\frac{a^4}{16} = \frac{1}{16}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)$$

$$= \frac{16r^2s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\stackrel{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\stackrel{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\text{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\stackrel{(\text{ii})}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\stackrel{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\stackrel{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\stackrel{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\stackrel{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore p_a^2\stackrel{(\bullet)}{=}\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)$$

$$\text{Now, }b^3+c^3-abc+a\big(4m_a^2\big)=b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$=(b+c)\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2\big)+a\big(b^2+c^2-a^2\big)$$

$$=2s\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2+bc-a^2\big)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}=$$

$$(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}-\frac{a(b-c)^2}{4}$$

$$(a=y+z,b=z+x,c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right) \Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$
$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$
$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$
$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$
$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq} 0 \to \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \to \circledast$$

$$\text{Also, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\text{and } b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c) \text{ and via summation, we get :}$$

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2) =$$

$$2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) +$$

$$a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2$$

$$= 2(n_a^2+g_a^2) \Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2+g_a^2)$$

$$\Rightarrow n_a^2 + g_a^2 \boxed{\overset{(\bullet)}{=}} (b-c)^2 + 2s(s-a)$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)}{bc}$$

$$= as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 \boxed{\overset{(\bullet\bullet)}{=}} s(s-a) + \frac{s}{a}(b-c)^2$$

$$\text{Via } (\bullet) \text{ and } (\bullet\bullet)\text{, we get: } g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a)\left(s - a + \frac{a^2-(b-c)^2}{a}\right) \Rightarrow g_a^2 \overset{(\bullet\bullet\bullet)}{=} (s-a)\left(s - \frac{(b-c)^2}{a}\right)$$

$$\therefore (\bullet\bullet), (\bullet\bullet\bullet) \Rightarrow n_a^2 g_a^2 = s(s-a)\left(s - a + \frac{(b-c)^2}{a}\right)\left(s - \frac{(b-c)^2}{a}\right)$$

$$= s(s-a)\left(s(s-a) + s\frac{(b-c)^2}{a} - \frac{(b-c)^2}{a}(s-a) - \frac{(b-c)^4}{a^2}\right)$$

$$\Rightarrow n_a^2 g_a^2 \boxed{\overset{(\blacksquare)}{=}} s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2}\right)$$

$$\text{Again, } m_a^2 w_a^2 = \frac{(b-c)^2 + 4s(s-a)}{4} \cdot \frac{4bcs(s-a)}{(b+c)^2}$$

$$\Rightarrow m_a^2 w_a^2 \boxed{\overset{(\blacksquare\blacksquare)}{=}} s(s-a)\frac{bc}{(b+c)^2}\left((b-c)^2 + 4s(s-a)\right)$$

$$\therefore (\blacksquare), (\blacksquare\blacksquare) \Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2$$

$$= s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2} - \frac{bc}{(b+c)^2}\left((b-c)^2 + 4s(s-a)\right)\right)$$

$$= s(s-a)\begin{pmatrix} s(s-a)+(b-c)^2\left(\frac{a^2-(b-c)^2}{a^2}\right) \\ -\frac{bc}{(b+c)^2}\left((b-c)^2+(b+c)^2-a^2\right)\end{pmatrix}$$

$$= s(s-a)\left(s(s-a)-bc+\left(a^2-(b-c)^2\right)\left(\frac{(b-c)^2}{a^2}+\frac{bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(\left((b+c)^2-a^2-4bc\right)+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left((b-c)^2-a^2+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}-1\right)$$

$$= \frac{s(s-a)}{4}.4(s-b)(s-c)\left(\frac{4(b-c)^2}{a^2}-\frac{(b-c)^2}{(b+c)^2}\right) = r^2s^2(b-c)^2\left(\frac{4}{a^2}-\frac{1}{(b+c)^2}\right)$$

$$= r^2s^2(b-c)^2\left(\frac{2}{a}+\frac{1}{b+c}\right)\left(\frac{2b+2c-a}{a(b+c)}\right) \geq 0$$

$$\Rightarrow n_a^2 g_a^2 \geq m_a^2 w_a^2 \Rightarrow n_a g_a \geq m_a w_a \to \circledast\circledast$$

$$\text{Now, via } \circledast \text{ and } \circledast\circledast,\ m_a n_a . n_a g_a \geq p_a^2 . m_a w_a \Rightarrow \frac{n_a^2}{p_a^2} \geq \frac{w_a}{g_a} \Rightarrow \frac{n_a}{p_a} \geq \sqrt{\frac{w_a}{g_a}} \to \circledast\circledast\circledast$$

$$\text{We have proved}: a n_a^2 = a s^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$\Rightarrow a^2 n_a^2 = a^2 s^2 - sa\left(a^2-(b-c)^2\right)$$

$$\Rightarrow a^2 n_a^2 = a^2s^2 - sa^3 + sa(b-c)^2 \stackrel{?}{=} 4r^2s^2 + s^2(b-c)^2$$

$$\Leftrightarrow a^2s^2 - sa^3 + sa(b-c)^2 \stackrel{?}{=} s(s-a)\left(a^2-(b-c)^2\right) + s^2(b-c)^2$$

$$= a^2s^2 - sa^3 - s(s-a)(b-c)^2 + s^2(b-c)^2$$

$$= a^2s^2 - sa^3 - s^2(b-c)^2 + sa(b-c)^2 + s^2(b-c)^2$$

$$\Leftrightarrow a^2s^2 - sa^3 + sa(b-c)^2 \stackrel{?}{=} a^2s^2 - sa^3 + sa(b-c)^2 \to \text{true}$$

$$\therefore \frac{n_a^2}{(b-c)^2+4r^2} = \frac{s^2}{a^2} \Rightarrow \frac{s}{a}.\frac{\sqrt{4r^2+(b-c)^2}}{p_a} = \frac{n_a}{p_a} \stackrel{\text{via } \circledast\circledast\circledast}{\geq} \sqrt{\frac{w_a}{g_a}}$$

$$\therefore s.\frac{\sqrt{4r^2+(b-c)^2}}{p_a} \geq a.\sqrt{\frac{w_a}{g_a}} \text{ and analogs}$$

$$\Rightarrow s\sum_{cyc}\frac{\sqrt{4r^2+(b-c)^2}}{p_a} \geq \sum_{cyc}\left(a.\sqrt{\frac{w_a}{g_a}}\right)\ \forall\ \Delta\ ABC, '' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3083. In ΔABC the following relationship holds:

$$\sum \frac{a^2}{(b+c)h_a} \geq \sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\sum \frac{a^2}{(b+c)h_a} = \frac{1}{2F}\sum \frac{a^3}{(b+c)} \overset{Holder}{\geq} \frac{1}{3}\cdot\frac{1}{2F}\cdot\frac{(\sum a)^3}{2(a+b+c)} =$$

$$= \frac{1}{12F}\left(\sum a\right)^2 \geq \frac{1}{12F}\left(3\sum ab\right) \overset{Gordon}{\geq} \frac{1}{4F}4\sqrt{3}F = \sqrt{3}$$

Equality holds for an equilateral triangle.

3084. In ΔABC the following relationship holds:

$$\prod (m^2a^4+2) \geq 144\, m^2F^2, m \in (0,\infty)$$

Proposed by D.M.Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution by Tapas Das-India

$$\prod (m^2a^4+2) = \prod (m^2a^4+1+1) \overset{AM-GM}{\geq} \prod 3(m^2a^4)^{\frac{1}{3}} =$$

$$= 27\,(m^2a^4.m^2b^4.m^2c^4)^{\frac{1}{3}} = 27\,m^2\,((abc)^2)^{\frac{2}{3}} \overset{Carlitz}{\geq} 27m^2\left(\frac{4F}{\sqrt{3}}\right)^{3\times\frac{2}{3}}$$

$$= 27m^2.\frac{16F^2}{3} = 144m^2F^2$$

Equality holds for $a=b=c=m=1$.

3085. In ΔABC the following relationship holds:

$$(a+b+c^2)(a^2+b+c)(a+b^2+c) \geq 144F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$(a+b+c^2)(a^2+b+c)(a+b^2+c) \geq$$

$$\overset{AM-GM}{\geq} 3(abc^2)^{\frac{1}{3}}.3(ab^2c)^{\frac{1}{3}}.3(a^2bc)^{\frac{1}{3}} = 27(a^4b^4c^4)^{\frac{1}{3}} = 27((abc)^2)^{\frac{2}{3}} \overset{Carlitz}{\geq}$$

$$\geq 27.\left(\frac{4F}{\sqrt{3}}\right)^{3\times\frac{2}{3}} = 144F^2$$

Equality holds for an equilateral triangle.

3086. In ΔABC the following relationship holds:

$$\sum \frac{a^{m+1}}{(bx+cy)^{m+1}} \geq \frac{3}{(x+y)^{m+1}}, \qquad m \geq 0, x, y \geq 0$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$\sum \frac{a}{bx+cy} = \sum \frac{a^2}{abx+acy} \overset{CBS}{\geq} \frac{(\sum a)^2}{x\sum ab + y\sum ab} =$$

$$= \frac{(\sum a)^2}{(x+y)\sum ab} \geq \frac{3\sum ab}{(x+y)\sum ab} = \frac{3}{x+y} \ (1)$$

$$\sum \frac{a^{m+1}}{(bx+cy)^{m+1}} \overset{CBS}{\geq} \frac{1}{3^m}\left(\sum \frac{a}{bx+cy}\right)^{m+1} \overset{(1)}{\geq} \frac{1}{3^m}\left(\frac{3}{x+y}\right)^{m+1} = \frac{3}{(x+y)^{m+1}}$$

Equality holds for an equilateral triangle and m=0.

3087. In any Δ ABC, the following relationship holds :

$$\frac{ab}{a^2+b^2} + \frac{bc}{b^2+c^2} + \frac{ca}{c^2+a^2} \geq \frac{1}{2} + \frac{2r}{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{ab}{a^2+b^2} + \frac{bc}{b^2+c^2} + \frac{ca}{c^2+a^2} =$$

$$= \frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \cdot \sum_{cyc}\left(bc\left(\sum_{cyc} a^2b^2 + a^4\right)\right)$$

$$=\frac{(\sum_{cyc}a^2b^2)(\sum_{cyc}ab)-abc\sum_{cyc}a^3}{(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)-a^2b^2c^2}$$

$$=\frac{\left((s^2+4Rr+r^2)^2-16Rrs^2\right)(s^2+4Rr+r^2)-8Rrs^2(s^2-6Rr-3r^2)}{2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-16R^2r^2s^2}\overset{?}{\geq}\frac{R+4r}{2R}$$

$$\Leftrightarrow -2s^6+(8R^2+25Rr-2r^2)s^4-r(52R^3+92R^2r+14Rr^2-2r^3)s^2$$

$$+r^2(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)\underset{①}{\overset{?}{\geq}}0$$

$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\geq 0 \text{ and } s^2-(m+n)\leq 0,$$

$$\text{where } m=2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(*)}{\leq}0$$

$$\text{Via } (*) \text{ and Gerretsen, } P=-2s^2\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)$$

$$-r(15R-2r)s^2(s^2-4R^2-4Rr-3r^2)\geq 0 \therefore \text{ in order to prove } ①,$$

$$\text{it suffices to prove : LHS of } ① \overset{?}{\geq} P\Leftrightarrow (16R^3-48R^2r-27Rr^2+10r^3)s^2$$

$$+r(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)\underset{②}{\overset{?}{\geq}}0$$

Case 1 $16R^3-48R^2r-27Rr^2+10r^3\geq 0$ and then : LHS of ② $\geq$ $r(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)>0\Rightarrow$ ② is true

Case 2 $16R^3-48R^2r-27Rr^2+10r^3<0$ and then : LHS of ②

$$\overset{\text{Gerretsen}}{\geq}(16R^3-48R^2r-27Rr^2+10r^3)(4R^2+4Rr+3r^2)$$

$$+r(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow 16t^5-16t^4-19t^3-26t^2-4t+8\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)(16t^4+16t^3+13t^2-4)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow ② \text{ is true}$$

$$\therefore \text{combining both cases, } ②\Rightarrow ① \text{ is true } \forall\,\Delta ABC \therefore \frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}$$

$$\geq\frac{1}{2}+\frac{2r}{R}\ \forall\,ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3088. In ΔABC the following relationship holds:

$$\frac{\tan\frac{A}{2}}{\sin B+\sin C}+\frac{\tan\frac{B}{2}}{\sin C+\sin A}+\frac{\tan\frac{C}{2}}{\sin A+\sin B}\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \sin A\ge\sin B\ge\sin C\ \ and\ \tan\frac{A}{2}\ge\tan\frac{B}{2}\ge\tan\frac{C}{2}$$

$$\frac{\tan\frac{A}{2}}{\sin B+\sin C}+\frac{\tan\frac{B}{2}}{\sin C+\sin A}+\frac{\tan\frac{C}{2}}{\sin A+\sin B}\ge$$

$$\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum\tan\frac{A}{2}\right)\left(\sum\frac{1}{\sin B+\sin C}\right)\overset{CBS}{\ge}\frac{1}{3}\cdot\frac{4R+r}{s}\cdot\frac{(1+1+1)^2}{2\sum\sin A}=$$

$$=\frac{1}{3}\cdot\frac{4R+r}{s}\cdot\frac{9R}{2s}=\frac{3R(4R+r)}{2s^2}\overset{Gerretsen}{\ge}\frac{12R^2+3Rr}{2(4R^2+4Rr+3r^2)}=\frac{12R^2+3Rr}{(8R^2+8Rr+6r^2)}$$

We need to show,:

$$\frac{12R^2+3Rr}{(8R^2+8Rr+6r^2)}\ge 1\ or\ 12R^2+3Rr\ge 8R^2+8Rr+6r^2$$

$$or 4R^2-5Rr-6r^2\ge 0\ or\ (R-2r)(4R+3r)\ge 0\ Euler$$

Equality holds for an equilateral triangle.

3089. In ΔABC the following relationship holds:

$$2\sum\frac{r_a+r_b}{h_c}=\sum\frac{h_a+h_b+r_a+r_b}{r_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

Known results:

$$\sum a^2=2(s^2-r^2-4Rr)\ (1),\sum a^3=2s(s^2-3r^2-6Rr)\ (2),\sum a=2s\ (3)$$

$$2\sum\frac{r_a+r_b}{h_c}=2\sum\frac{\frac{F}{s-a}+\frac{F}{s-b}}{\frac{2F}{c}}=\sum\frac{c(2s-a-b)}{(s-a)(s-b)}=$$

$$=\sum\frac{c^2}{(s-a)(s-b)}=\frac{1}{\prod(s-a)}\sum c^2(s-c)=$$

$$=\frac{1}{sr^2}\left(s\sum c^2-\sum c^3\right)\overset{(1),(2),(3)}{=}\frac{2s}{sr^2}2r(R+r)=\frac{4(R+r)}{r}\ (A)$$

$$\sum\frac{h_a+h_b+r_a+r_b}{r_c}=\sum\frac{\frac{2F}{a}+\frac{2F}{b}+\frac{F}{s-a}+\frac{F}{s-b}}{\frac{F}{s-c}}=$$

$$=\sum\left(\frac{2}{a}+\frac{2}{b}+\frac{c}{(s-a)(s-b)}\right)(s-c)=2\sum\frac{(a+b)c(s-c)}{abc}+\sum\frac{c(s-c)^2}{\prod(s-a)}=$$

$$=2\sum\frac{(2s-c)c(s-c)}{4Rrs}+\sum\frac{c(s^2-2sc+c^2)}{sr^2}=$$

$$=2\sum\frac{(2s^2c-3sc^2+c^3)}{4Rrs}+\sum\frac{(s^2c-2sc^2+c^3)}{sr^2}=$$

$$=\frac{2}{4Rrs}\left(2s^2\sum c-3s\sum c^2+\sum c^3\right)+\frac{1}{sr^2}\left(s^2\sum c-2s\sum c^2+\sum c^3\right)=$$

$$\overset{(1),(2),(3)}{=}\frac{2}{4Rrs}(12Rrs)+\frac{1}{sr^2}\left(s(4Rr-2r^2)\right)=6+\frac{4R-2r}{r}=\frac{4(R+r)}{r}\quad(B)$$

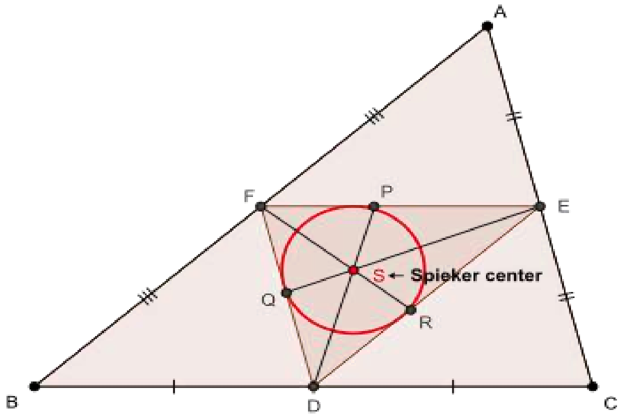
$$\text{From }(A)\&(B)\text{ we get }2\sum\frac{r_a+r_b}{h_c}=\sum\frac{h_a+h_b+r_a+r_b}{r_c}$$

3090. In any Δ ABC, the following relationship holds :

$$\frac{1}{h_a}+\frac{1}{w_a}+\frac{1}{m_a}\le\left(\frac{5R}{3r}-\frac{1}{3}\right)\frac{1}{p_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right)-\sum_{cyc}\frac{a^4}{16}=\frac{1}{16}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$

$$=\frac{16r^2s^2}{16}\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2}\ \text{and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2 s}\big(ca(s-b) + ab(s-c)\big) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\big(b^2+c^2+ab+ca\big) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)$$

$$\text{Now, } b^3+c^3-abc+a\big(4m_a^2\big) = b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$= (b+c)\big(b^2-bc+c^2\big) + a\big(b^2-bc+c^2\big) + a\big(b^2+c^2-a^2\big)$$

$$= 2s\big(b^2-bc+c^2\big) + a\big(b^2-bc+c^2+bc-a^2\big)$$

$$= (2s+a)\big(b^2-bc+c^2\big) + a\left(\frac{(b+c)^2-(b-c)^2}{4} - a^2\right)$$

$$= (2s+a)\big(b^2-bc+c^2\big) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\big(b^2-bc+c^2\big) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4} - \frac{a(b-c)^2}{4} =$$

$$(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y) + (y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4} - \frac{a(b-c)^2}{4}$$

$$(a = y+z, b = z+x, c = x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow \circledast$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 = s^2 - 2r_a h_a$$

$$\therefore a^2 n_a^2 \overset{?}{\leq} 4(R-r)^2 s^2 \Leftrightarrow a^2(s^2 - 2h_a r_a) \overset{?}{\leq} 4(R-r)^2 s^2$$

$$\Leftrightarrow (4R^2\sin^2 A)s^2 - 4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right) \overset{?}{\leq} 4(R^2 - 2Rr + r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2 A) - 2Rr\left(1 - 2\sin^2\frac{A}{2}\right) + r^2 \overset{?}{\geq} 0 \Leftrightarrow R^2\cos^2 A - 2Rr\cos A + r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R\cos A - r)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \therefore an_a \leq 2Rs - 2rs \Rightarrow \frac{n_a}{h_a} \leq \frac{2Rs}{a\left(\frac{2rs}{a}\right)} - \frac{2rs}{a\left(\frac{2rs}{a}\right)}$$

$$\Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r} - 1 \rightarrow \circledast\circledast$$

$$\text{Now, } (2m_a + n_a)^2 - 9p_a^2 = 4m_a^2 + n_a^2 + 4m_a n_a - 9p_a^2 \overset{\text{via } \circledast}{\geq}$$

$$4s(s-a) + (b-c)^2 + s(s-a) + \frac{s(b-c)^2}{a} + 4p_a^2 + \frac{2(b-c)^2}{9} - 9p_a^2$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{=} 5s(s-a) + \frac{11(b-c)^2}{9} + \frac{s(b-c)^2}{a} - 5s(s-a) - \frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$= \left(\frac{11}{9} + \frac{s}{a} - \frac{5s(3s+a)}{(2s+a)^2}\right).(b-c)^2 = \frac{36s^3 - 55s^2a + 8sa^2 + 11a^3}{9a(2s+a)^2}.(b-c)^2$$

$$= \frac{(s-a)\left((s-a)(36s+17a) + 6a^2\right)}{9a(2s+a)^2}.(b-c)^2 \geq 0 \therefore (2m_a + n_a)^2 \geq 9p_a^2$$

$$\Rightarrow 2m_a + n_a \geq 3p_a \therefore \frac{p_a}{h_a} + \frac{p_a}{w_a} + \frac{p_a}{m_a} \leq \frac{1}{3}\left(\frac{2m_a}{h_a} + \frac{n_a}{h_a} + \frac{2m_a}{w_a} + \frac{n_a}{w_a} + \frac{2m_a}{m_a} + \frac{n_a}{m_a}\right)$$

$$\leq \frac{1}{3}\left(\frac{2m_a}{h_a} + \frac{n_a}{h_a} + \frac{2m_a}{h_a} + \frac{n_a}{h_a} + \frac{2m_a}{m_a} + \frac{n_a}{h_a}\right) \overset{\substack{\text{via } \circledast\circledast \\ \text{and} \\ \text{Panaitopol}}}{\leq}$$

$$\frac{1}{3}\left(\frac{R}{r} + \frac{R}{r} - 1 + \frac{R}{r} + \frac{R}{r} - 1 + 2 + \frac{R}{r} - 1\right) = \frac{5R}{3r} - \frac{1}{3}$$

$$\therefore \frac{1}{h_a} + \frac{1}{w_a} + \frac{1}{m_a} \leq \left(\frac{5R}{3r} - \frac{1}{3}\right)\frac{1}{p_a} \ \forall \Delta ABC \text{ (QED)}$$

3091. In ΔABC the following relationship holds:

$$6 \leq \sum \frac{r_a + r}{r_a - r} \leq \frac{3R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{r_a+r}{r_a-r}=\frac{r\left(\frac{s}{s-a}+1\right)}{r\left(\frac{s}{s-a}-1\right)}=\frac{2s-a}{a}=\frac{b+c}{a}$$

$$\sum\frac{r_a+r}{r_a-r}=\sum\frac{b+c}{a}=\sum\left(\frac{b}{a}+\frac{a}{b}\right)\overset{AM-GM}{\geq}2+2+2=6$$

$$\sum\frac{r_a+r}{r_a-r}=\sum\frac{b+c}{a}=\sum\left(\frac{b}{a}+\frac{a}{b}\right)\overset{Bandila}{\leq}\sum\frac{R}{r}=\frac{3R}{r}$$

Equality holds for an equilateral triangle.

3092. In ΔABC the following relationship holds:

$$6\leq\sum\frac{h_a+r}{h_a-r}\leq\frac{3R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{h_a+r}{h_a-r}=\frac{r\left(\frac{2s}{a}+1\right)}{r\left(\frac{2s}{a}-1\right)}=\frac{2s+a}{2s-a}=\frac{2a+(b+c)}{b+c}=\frac{2a}{b+c}+1$$

$$\sum\frac{h_a+r}{h_a-r}=\sum\left(\frac{2a}{b+c}+1\right)=3+2\sum\left(\frac{a}{b+c}\right)\overset{Nesbitt}{\geq}3+2.\frac{3}{2}=6$$

$$\sum\frac{h_a+r}{h_a-r}=\sum\left(\frac{2a}{b+c}+1\right)=3+2\sum\left(\frac{a}{b+c}\right)\overset{AM-HM}{\leq}3+\frac{2}{4}\sum\left(\frac{a}{b}+\frac{a}{c}\right)=$$

$$=3+\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)\overset{Bandila}{\leq}3+\frac{1}{2}\sum\frac{R}{r}=3+\frac{3R}{2r}=\frac{3R}{R}+\frac{3R}{2r}\overset{Euler}{\leq}\frac{3R}{2r}+\frac{3R}{2r}=\frac{3R}{r}$$

Equality holds for an equilateral triangle.

3093. In ΔABC the following relationship holds:

$$\frac{r_a^5}{r_b^2}+\frac{r_b^5}{r_c^2}+\frac{r_c^5}{r_a^2}\geq\frac{\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5}{\left(r_a^3+r_b^3+r_c^3\right)^4}\geq\frac{24^4r^{15}}{(9R^3-64r^3)^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{r_a^5}{r_b^2}+\frac{r_b^5}{r_c^2}+\frac{r_c^5}{r_a^2}=\sum\frac{r_a^5}{r_b^2}$$

$$\left(\sum\frac{r_a^5}{r_b^2}\right)\left(\sum r_b^3\right)^4\overset{Holder}{\geq}\left(\sum\sqrt[5]{\frac{r_a^5}{r_a^2}.r_b^{12}}\right)^5=$$

$$=\left(\sum\sqrt[5]{\frac{r_a^5}{r_a^2}.r_b^{10}.r_b^2}\right)^5=\left(\sum\sqrt[5]{\left(r_ar_b^2\right)^5}\right)^5=\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5$$

$$\frac{r_a^5}{r_b^2}+\frac{r_b^5}{r_c^2}+\frac{r_c^5}{r_a^2}\geq\frac{\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5}{\left(r_a^3+r_b^3+r_c^3\right)^4}$$

$$\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5\overset{AM-GM}{\geq}\left(3\sqrt[3]{r_a^3r_b^3r_c^3}\right)^5=(3s^2r)^5\overset{Mitrinovic}{\geq}$$

$$\geq\ (3\cdot 27\,r^3)^5=3^{20}r^{15}\ (1)$$

$$\sum r_a^3=\left(\sum r_a\right)^3-3\left(\sum r_a\right)\left(\sum r_ar_b\right)+3r_ar_br_c=$$

$$=(4R+r)^3-3(4R+r)s^2+3s^2r=(4R+r)^3-12s^2R\leq$$

$$\overset{Euler\,\&\,Mitrinovic}{\leq}\left(4R+\frac{R}{2}\right)^3-12.27r^2.2r=\frac{3^6R^3}{8}-3^4.8.r^3=\frac{3^4}{8}(9R^3-64r^3)\ (2)$$

$$\frac{\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5}{\left(r_a^3+r_b^3+r_c^3\right)^4}\overset{(1)\&(2)}{\geq}\frac{3^{20}r^{15}}{\left(\frac{3^4}{8}(9R^3-64r^3)\right)^4}=\frac{3^4.8^4.r^{15}}{(9R^3-64r^3)^4}$$

Equality holds for an equilateral triangle.

3094.

In any triangle ABC with area F, the following inequality holds :

$$((a+b)^2+2)((b+c)^2+2)((c+a)^2+2)\geq 144\sqrt{3}F$$

Proposed by D.M.Bătinețu Giurgiu, Claudia Nănuți

Solution by Soumava Chakraborty-Kolkata-India

Via Arkady Alt (2009), for all positive numbers x,y,z,t,

$$\text{the following inequality holds}: (x^2+t^2)(y^2+t^2)(z^2+t^2) \overset{(*)}{\geq} \frac{3t^4}{4}(x+y+z)^2$$

$$\text{and choosing } x=a+b, y=b+c, z=c+a \text{ and } t=\sqrt{2}, \text{we get}:$$

$$((a+b)^2+2)((b+c)^2+2)((c+a)^2+2) \geq \frac{3(\sqrt{2})^4}{4}\left(2\sum_{cyc} a\right)^2$$

$$=48s^2 \overset{\text{Mitrinovic}}{\geq} 48s.3\sqrt{3}r = 144.\sqrt{3}.F$$

$$\therefore ((a+b)^2+2)((b+c)^2+2)((c+a)^2+2) \geq 144.\sqrt{3}.F \ \forall \ \Delta \ ABC \text{ and}$$

$$\because \text{equality for } (*) \text{ holds when}: x=y=z=\frac{t}{\sqrt{2}} \therefore \text{equality holds when}:$$

$$a+b=b+c=c+a=1 \Rightarrow \text{equality holds when}: a=b=c=\frac{1}{2} \ (QED)$$

3095. If in $\Delta ABC, a \neq b \neq c \neq a$ the following relationship holds:

$$\frac{m_a^4+m_b^4+m_c^4-9F^2}{m_a^2+m_b^2+m_c^2-3\sqrt{3}F} > 2\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\sum m_a^4 - 9F^2 \overset{CBS}{\geq} \frac{(\sum m_a^2)^2}{3} - 9F^2 = \frac{1}{3}\left(\left(\sum m_a^2\right)^2 - 27F^2\right) =$$

$$=\frac{1}{3}\left(\sum m_a^2 + 3\sqrt{3}F\right)\left(\sum m_a^2 - 3\sqrt{3}F\right) = \frac{1}{3}\left(\frac{3}{4}\sum a^2 + 3\sqrt{3}F\right)\left(\sum m_a^2 - 3\sqrt{3}F\right) >$$

$$\overset{Ionescu-Weitzenbock}{>} \frac{1}{3}\left(\frac{3}{4}\cdot 4\sqrt{3}F + 3\sqrt{3}F\right)\left(\sum m_a^2 - 3\sqrt{3}F\right) =$$

$$= 2\sqrt{3}F\left(\sum m_a^2 - 3\sqrt{3}F\right) \ (1)$$

$$\frac{m_a^4+m_b^4+m_c^4-9F^2}{m_a^2+m_b^2+m_c^2-3\sqrt{3}F} \overset{(1)}{>} \frac{2\sqrt{3}F(\sum m_a^2 - 3\sqrt{3}F)}{m_a^2+m_b^2+m_c^2-3\sqrt{3}F} = 2\sqrt{3}$$

3096. In any Δ ABC, the following relationship holds:

$$r_a^2+r_b^2+r_c^2+l_a^2+l_b^2+l_c^2 \geq 2(m_a^2+m_b^2+m_c^2)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{(b+c)^2}=\sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2}=2s.\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2}-\sum_{cyc}\frac{1}{b+c}$$

$$=\frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2-2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2}-\frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}$$

$$=\frac{\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}\Rightarrow$$

$$\sum_{cyc}\frac{a}{(b+c)^2}\overset{(\bullet)}{=}\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\text{Now, }\sum_{cyc}w_a^2=\sum_{cyc}\frac{4bcs(s-a)}{(b+c)^2}=\sum_{cyc}\frac{bc\left((b+c)^2-a^2\right)}{(b+c)^2}=\sum_{cyc}\left(bc-\frac{a^2bc}{(b+c)^2}\right)$$

$$\overset{\text{via }(\bullet)}{=}s^2+4Rr+r^2$$

$$+2Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(16R^2+8Rr+r^2)}{(s^2+2Rr+r^2)^2}$$

$$\therefore r_b^2+r_c^2+l_a^2+l_b^2+l_c^2\ge 2(m_a^2+m_b^2+m_c^2)\Leftrightarrow(4R+r)^2-2s^2$$

$$+\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(16R^2+8Rr+r^2)}{(s^2+2Rr+r^2)^2}$$

$$\ge 2.\frac{3}{2}.(s^2-4Rr-r^2)$$

$$\Leftrightarrow -4s^6+(16R^2-3r^2)s^4+r(64R^3+124R^2r+76Rr^2+6r^3)$$

$$+r^2(64R^4+144R^3r+128R^2r^2+44Rr^3+5r^4)\overset{(*)}{\ge}0$$

$$\text{Now, }P=-4s^4(s^2-4R^2-4Rr-3r^2)-(16Rr+15r^2)s^2(s^2-4R^2-4Rr-3r^2)$$

$$-r^2(32Rr+39r^2)(s^2-4R^2-4Rr-3r^2)\overset{\text{Gerretsen}}{\ge}0\ \therefore\ \text{in order to prove }(*),$$

$$\text{it suffices to prove : LHS of }(*)\ge P\Leftrightarrow 16t^4+4t^3-39t^2-52t-28\ge 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(16t^3+36t^2+33t+14)\ge 0\rightarrow\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)\text{ is true}$$

$$\therefore r_a^2+r_b^2+r_c^2+l_a^2+l_b^2+l_c^2\ge 2(m_a^2+m_b^2+m_c^2)\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$

3097. In ΔABC the following relationship holds:

$$\sum\frac{h_a+r}{h_a-r}\le\sum\frac{r_a+r}{r_a-r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{h_a+r}{h_a-r}=\frac{r\left(\frac{2s}{a}+1\right)}{r\left(\frac{2s}{a}-1\right)}=\frac{2s+a}{2s-a}=\frac{2a+(b+c)}{b+c}=\frac{2a}{b+c}+1$$

$$\sum\frac{h_a+r}{h_a-r}=\sum\left(\frac{2a}{b+c}+1\right)=3+2\sum\left(\frac{a}{b+c}\right)\overset{Nesbitt}{\ge}3+2\cdot\frac{3}{2}=6$$

$$\sum\frac{h_a+r}{h_a-r}=\sum\left(\frac{2a}{b+c}+1\right)=3+2\sum\left(\frac{a}{b+c}\right)\overset{AM-HM}{\le}$$

$$\le 3+\frac{2}{4}\sum\left(\frac{a}{b}+\frac{a}{c}\right)=3+\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)\ (1)$$

$$\sum\frac{r_a+r}{r_a-r}=\sum\frac{\frac{rs}{s-a}+r}{\frac{rs}{s-a}-r}=\sum\frac{2s-a}{a}=\sum\frac{b+c}{a}=\sum\left(\frac{b}{a}+\frac{c}{a}\right)=\sum\left(\frac{a}{b}+\frac{b}{a}\right)=$$

$$=\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)\overset{AM-GM}{\ge}\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}(2+2+2)=$$

$$=3+\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)\ (2)$$

$$\textit{From } (1)\&\ (2) \textit{we get}\sum\frac{h_a+r}{h_a-r}\le\sum\frac{r_a+r}{r_a-r}$$

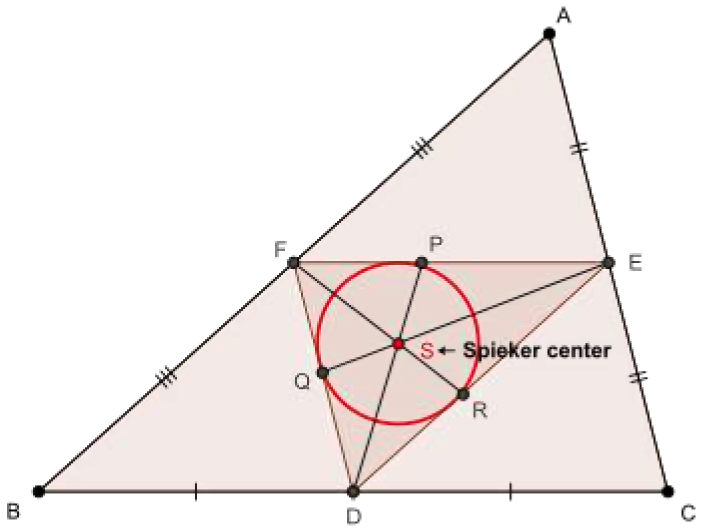
Equality holds for an equilateral triangle.

3098. In $any\ \Delta\ ABC$, the following relationship holds :

$$\frac{w_a m_a p_a}{h_a g_a n_a}+\frac{w_b m_b p_b}{h_b g_b n_b}+\frac{w_c m_c p_c}{h_c g_c n_c}\le\frac{1}{3}+\frac{4R}{3r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC } at \text{ X } and \text{ m}(\measuredangle BAX) = \alpha \ and \ m(\measuredangle CAX) = \beta \ (say)$$
$$and \text{ inradius of } \Delta \text{ DEF} = r' (say)$$

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right) - \sum_{cyc}\frac{a^4}{16} = \frac{1}{16}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)$$

$$= \frac{16r^2s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \ and \ m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$\text{Via } (1), (2) \ and \text{ using cosine law on } \Delta \text{ AFS } and \ \Delta \text{ AES, we arrive at}:$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}}$$
$$+ \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)$$

$$\text{Now}, b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}=$$
$$(2s+a).\frac{\begin{pmatrix}4(z+x)^2+4(x+y)^2-4(z+x)(x+y)\\+(y+z)((z+x)+(x+y)-2(y+z))\end{pmatrix}}{4}-\frac{a(b-c)^2}{4}$$
$$(a=y+z, b=z+x, c=x+y)$$
$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore(\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$\Rightarrow ap_a^2\overset{(\bullet\bullet\bullet)}{=}as(s-a)-\frac{a(b-c)^2}{4}+\frac{a(4s+a)^2}{(4s+2a)^2}.(b-c)^2$$
$$\text{Now},\frac{a(4s+a)^2}{(4s+2a)^2}=a.\frac{(4s+2a)^2-2a(4s+2a)+a^2}{(4s+2a)^2}$$
$$=a-\frac{(a+2s-2s)^2}{2s+a}+\frac{(a+2s-2s)^3}{(4s+2a)^2}$$
$$=a-(2s+a)+4s-\frac{4s^2}{2s+a}+\frac{1}{4}\left(\frac{(2s+a)^3-8s^3-3(2s+a)(2s)a}{(2s+a)^2}\right)$$
$$=2s-\frac{4s^2}{2s+a}+\frac{2s+a}{4}-\frac{2s^3}{(2s+a)^2}-\frac{3s(a+2s-2s)}{2(2s+a)}$$
$$=\frac{5s}{2}+\frac{a}{4}-\frac{4s^2}{2s+a}-\frac{2s^3}{(2s+a)^2}-\frac{3s}{2}+\frac{3s^2}{2s+a}$$

$$\therefore \frac{a(4s+a)^2}{(4s+2a)^2} \overset{(\bullet\bullet\bullet\bullet)}{=} s + \frac{a}{4} - \frac{s^2(4s+a)}{(2s+a)^2}$$

$$\therefore (\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet) \Rightarrow ap_a^2 = as(s-a) - \frac{a(b-c)^2}{4} + s(b-c)^2 + \frac{a(b-c)^2}{4}$$

$$-\frac{s^2(4s+a)}{(2s+a)^2}.(b-c)^2 \overset{a<s}{\leq} as(s-a) + s(b-c)^2 - \frac{s^2(4s+a)}{(2s+s)^2}.(b-c)^2$$

$$= as(s-a) + s(b-c)^2 - \frac{(4s+a)(b-c)^2}{9}$$

$$\Rightarrow ap_a^2 \leq as(s-a) + \frac{5s(b-c)^2}{9} - \frac{a(b-c)^2}{9} \text{ and analogs}$$

$$\therefore \sum_{cyc} ap_a^2 \leq s\left(2s^2 - 2(s^2 - 4Rr - r^2)\right) + \frac{10s}{9}\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)$$

$$-\frac{1}{9}\left(2s(s^2+4Rr+r^2) - 9abc\right)$$

$$= s(8Rr + 2r^2) + \frac{10s(s^2 - 12Rr - 3r^2)}{9} - \frac{2s(s^2 - 14Rr + r^2)}{9}$$

$$= \frac{2s(4s^2 - 10Rr - 7r^2)}{9} \Rightarrow \frac{1}{2rs}.\sum_{cyc} ap_a^2 \overset{\circledast}{\leq} \frac{4s^2 - 10Rr - 7r^2}{9r} \text{ and}$$

$$\frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \overset{CBS}{\leq} \sqrt{\sum_{cyc} \frac{p_a^2}{h_a}}.\sqrt{\sum_{cyc} \frac{1}{h_a}} = \sqrt{\frac{1}{2rs}.\sum_{cyc} ap_a^2}.\sqrt{\frac{1}{r}} \overset{\text{via }\circledast}{\leq} \sqrt{\frac{4s^2 - 10Rr - 7r^2}{9r^2}}$$

$$\overset{\text{Gerretsen}}{\leq} \sqrt{\frac{4(4R^2 + 4Rr + 3r^2) - 10Rr - 7r^2}{9r^2}} = \sqrt{\frac{16R^2 + 6Rr + 5r^2}{9r^2}} \overset{\text{Euler}}{\leq}$$

$$\sqrt{\frac{16R^2 + 6Rr + r^2 + 2Rr}{9r^2}} = \sqrt{\frac{(4R+r)^2}{9r^2}} \therefore \frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \leq \frac{1}{3} + \frac{4R}{3r} \rightarrow (m)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$ and $b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$ and via summation, we get :

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2) =$$

$$2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2) = 2(n_a^2 + g_a^2) +$$

$$a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b-c)^2$$

$$= 2(n_a^2 + g_a^2) \Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2 + g_a^2)$$

$$\Rightarrow n_a^2 + g_a^2 \overset{\circledcirc}{=} (b-c)^2 + 2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)}{bc}$$

$$= as^2 - as\left(\frac{a^2 - (b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - \frac{a^2 - (b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 \overset{\circledcirc\circledcirc}{=} s(s-a)+\frac{s}{a}(b-c)^2$$

Via $(\bullet)$ and $(\bullet\bullet)$, we get: $g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$

$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$

$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$

$$=(s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)\Rightarrow g_a^2 \overset{\circledcirc\circledcirc\circledcirc}{=}(s-a)\left(s-\frac{(b-c)^2}{a}\right)$$

$$\therefore (\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow n_a^2g_a^2=s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$

$$=s(s-a)\left(s(s-a)+s\frac{(b-c)^2}{a}-\frac{(b-c)^2}{a}(s-a)-\frac{(b-c)^4}{a^2}\right)$$

$$\Rightarrow n_a^2g_a^2 \overset{(\blacksquare)}{=} s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}\right)$$

$$\text{Again, } m_a^2w_a^2=\frac{(b-c)^2+4s(s-a)}{4}\cdot\frac{4bcs(s-a)}{(b+c)^2}$$

$$\Rightarrow m_a^2w_a^2 \overset{(\blacksquare\blacksquare)}{=} s(s-a)\frac{bc}{(b+c)^2}\left((b-c)^2+4s(s-a)\right)$$

$$\therefore (\blacksquare),(\blacksquare\blacksquare)\Rightarrow n_a^2g_a^2-m_a^2w_a^2$$

$$=s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}-\frac{bc}{(b+c)^2}\left((b-c)^2+4s(s-a)\right)\right)$$

$$=s(s-a)\begin{pmatrix} s(s-a)+(b-c)^2\left(\frac{a^2-(b-c)^2}{a^2}\right) \\ -\frac{bc}{(b+c)^2}\left((b-c)^2+(b+c)^2-a^2\right)\end{pmatrix}$$

$$=s(s-a)\left(s(s-a)-bc+\left(a^2-(b-c)^2\right)\left(\frac{(b-c)^2}{a^2}+\frac{bc}{(b+c)^2}\right)\right)$$

$$=\frac{s(s-a)}{4}\left(\left((b+c)^2-a^2-4bc\right)+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$=\frac{s(s-a)}{4}\left((b-c)^2-a^2+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$=\frac{s(s-a)}{4}\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}-1\right)$$

$$=\frac{s(s-a)}{4}.4(s-b)(s-c)\left(\frac{4(b-c)^2}{a^2}-\frac{(b-c)^2}{(b+c)^2}\right)=r^2s^2(b-c)^2\left(\frac{4}{a^2}-\frac{1}{(b+c)^2}\right)$$

$$=r^2s^2(b-c)^2\left(\frac{2}{a}+\frac{1}{b+c}\right)\left(\frac{2b+2c-a}{a(b+c)}\right)\geq 0$$

$$\Rightarrow n_a^2g_a^2\geq m_a^2w_a^2\Rightarrow n_ag_a\geq m_aw_a\to (n)$$

$$\therefore \frac{w_a m_a p_a}{h_a g_a n_a}+\frac{w_b m_b p_b}{h_b g_b n_b}+\frac{w_c m_c p_c}{h_c g_c n_c} \overset{\text{via (n)}}{\leq} \frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c} \overset{\text{via (m)}}{\leq} \frac{1}{3}+\frac{4R}{3r}$$

$$\therefore \frac{w_a m_a p_a}{h_a g_a n_a}+\frac{w_b m_b p_b}{h_b g_b n_b}+\frac{w_c m_c p_c}{h_c g_c n_c} \leq \frac{1}{3}+\frac{4R}{3r}\ \forall \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3099. In ΔABC the following relationship holds:

$$a^8+b^8+c^8+3 \geq 32F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Tapas Das-India

$$a^8+b^8+c^8+3=\sum(a^8+1) \overset{AM-GM}{\geq} \sum 2a^4 \overset{AM-GM}{\geq}$$

$$\geq 6(abc)^{\frac{4}{3}} \overset{Carlitz}{\geq} 6\left(\frac{4F}{\sqrt{3}}\right)^2=32F^2$$

Equality holds for an equilateral triangle with $a=b=c=1$.

3100. In ΔABC the following relationship holds:

$$\prod(a^m+b^m+c^2) \geq 4^{m+1}\left(\sqrt{3}\right)^{5-m}F^{m+1}, m \geq 0$$

Proposed by D.M.Bătinețu-Giurgiu,Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\prod(a^m+b^m+c^2) \overset{AM-GM}{\geq} \prod 3(a^m.b^m.c^2)^{\frac{1}{3}} \geq 27(a^{2m+2}.b^{2m+2}.c^{2m+2})^{\frac{1}{3}}=$$

$$=27((abc)^2)^{\frac{m+1}{3}} \overset{Carlitz}{\geq} \left(\sqrt{3}\right)^{6\left(\left(\frac{4F}{\sqrt{3}}\right)^3\right)^{\frac{m+1}{3}}} \geq 4^{m+1}\left(\sqrt{3}\right)^{5-m}F^{m+1}$$

Equality holds for an equilateral triangle with $a=b=c=1$ and $m=0$.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru