ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{e^{(m+n)\pi}+e^{(m-n)\pi}}=\frac{1-G\sqrt{2}}{4(1-e^{\pi})}$$

where, G is Gauss's constant

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$$I=\sum_{m=1}^{\infty}\frac{1}{e^{m\pi}}\sum_{n=1}^{\infty}\frac{1}{e^{n\pi}+e^{-n\pi}}=M\times N$$

$$Now,\ \ M=\sum_{m=1}^{\infty}\frac{1}{e^{m\pi}}=\sum_{m=1}^{\infty}e^{-m\pi}=\frac{e^{-\pi}}{1-e^{-\pi}}=\frac{1}{e^{\pi}-1}$$

$$N=\sum_{n=1}^{\infty}\frac{1}{q^{-n}+q^{n}}=\sum_{n=1}^{\infty}\frac{q^{n}}{1+q^{2n}}=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{k}q^{(2k+1)n}=\sum_{k=0}^{\infty}(-1)^{k}\sum_{n=1}^{\infty}q^{(2k+1)n}=\sum_{k=0}^{\infty}\frac{(-1)^{k}q^{2k+1}}{1-q^{2k+1}}$$

$here,\ \ q=e^{-\pi}.\ Consider,\ \ \varphi(q)=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}$

$$here\ \ f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}},$$

$|ab|<1\quad (Ramanujan\ general\ theta\ function)$

$$Let\ \ g(x)={}_2F_1\left(\frac{1}{2},\frac{1}{2};1,x\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(1)_n n!}x^n=\sum_{n=0}^{\infty}\frac{\left({}^{2n}C_n\right)^2}{16}x^n\qquad |x|<1$$

$Consider\ complete\ elliptic\ integral\ of\ first\ kind,$

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{\pi}{2}g(k^2)\ \ and\ \ K(k')=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}=\frac{\pi}{2}g(k'^2)$$
$$=\frac{\pi}{2}g(1-k^2)$$

$k'=\sqrt{1-k^2}$

$$If\ \ k=\sqrt{1-\frac{\varphi^4(-q)}{\varphi^4(q)}}\qquad then,\ \ \varphi^2\left(e^{-\pi\frac{K(k')}{K(k)}}\right)=g(k^2)=\frac{2}{\pi}K(k)$$

$$Now,\ \ k=k'=\frac{1}{\sqrt{2}}\ \Rightarrow,\ \ \varphi^2(e^{-\pi})=g\left(\frac{1}{2}\right)=\frac{2}{\pi}K\left(\frac{1}{\sqrt{2}}\right)$$

$$K\left(\frac{1}{\sqrt{2}}\right)=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{2-sin^2\theta}}=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+cos^2\theta}}\underbrace{=}_{\theta=cos^{-1}x}\sqrt{2}\int_0^1\frac{dx}{\sqrt{1-x^4}}=\frac{\sqrt{2}}{4}B\left(\frac{1}{4},\frac{1}{2}\right)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}$$

$$\Rightarrow\varphi^2(e^{-\pi})=g\left(\frac{1}{2}\right)=\frac{2}{\pi}K\left(\frac{1}{\sqrt{2}}\right)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{\pi}}=G\sqrt{2}$$

$$\left(G=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2\pi}}=Gauss's\ Constant\right)$$

Now we can use the identity given in Ramanujan Notebook (Part – 3) page – 114, entry 8

$$\varphi^2(q)=1+4\sum_{k=0}^{\infty}\frac{(-1)^k q^{2k+1}}{1-q^{2k+1}}$$

$$Consider\sum_{n=-\infty}^{\infty}\frac{2}{q^{-n}+q^n}=1+4\sum_{n=1}^{\infty}\frac{1}{q^{-n}+q^n}=1+4\sum_{k=0}^{\infty}\frac{(-1)^k q^{2k+1}}{1-q^{2k+1}}=\varphi^2(q)$$

$$\underbrace{\Rightarrow}_{q=e^{-\pi}}\sum_{n=-\infty}^{\infty}\frac{2}{e^{n\pi}+e^{-n\pi}}=1+4\sum_{n=1}^{\infty}\frac{1}{e^{n\pi}+e^{-n\pi}}=1+4\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)\pi}}{1-e^{-(2k+1)\pi}}=\varphi^2(e^{-\pi})=G\sqrt{2}$$

$$\Rightarrow N=\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)\pi}}{1-e^{-(2k+1)\pi}}=\frac{G\sqrt{2}-1}{4}\Rightarrow I=M\times N=\left(\frac{1}{e^{\pi}-1}\right)\left(\frac{G\sqrt{2}-1}{4}\right)$$

$$\Rightarrow I=\frac{1-G\sqrt{2}}{4(1-e^{\pi})}$$