ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{k\in Z}\frac{1}{(1+k^2)(2+k^2)(3+k^2)}=\frac{\pi}{2}\left(\frac{coth(\pi\sqrt{3})}{\sqrt{3}}-\sqrt{2}coth(\pi\sqrt{2})+coth(\pi)\right)$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Shobhit Jain-India

$$I=\sum_{k=-\infty}^{\infty}\frac{1}{(1+k^2)(2+k^2)(3+k^2)}=\sum_{k=-\infty}^{\infty}\left(\frac{\frac{1}{2}}{3+k^2}-\frac{1}{2+k^2}+\frac{\frac{1}{2}}{1+k^2}\right)=$$

$$=\frac{1}{2}f(\sqrt{3})-f(\sqrt{2})+\frac{1}{2}f(1)$$

$$here,\ \ f(x)=\sum_{k=-\infty}^{\infty}\frac{1}{x^2+k^2}.Now,\ \ \frac{sin(\pi x)}{\pi x}=\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

$$\Rightarrow\ ln(sin(\pi x))-ln(\pi x)=\sum_{k=1}^{\infty}ln\left(1-\frac{x^2}{k^2}\right)\ \ Now\ differentiate\ w.r.t\ x$$

$$\Rightarrow\pi\cot(\pi x)-\frac{1}{x}=\sum_{k=1}^{\infty}\frac{2x}{x^2-k^2}\Rightarrow\frac{\pi}{tan(\pi x)}=\sum_{k=-\infty}^{\infty}\frac{x}{x^2-k^2}$$

$$Now\ replace\ x\ by\ ix\ \Rightarrow\frac{\pi}{tan(i\pi x)}=\sum_{k=-\infty}^{\infty}\frac{ix}{-x^2-k^2}\qquad use,\ \ tan(i\pi x)=i\,tanh(\pi x)$$

$$\Rightarrow\frac{\pi}{i\,tanh(\pi x)}=\sum_{k=-\infty}^{\infty}\frac{ix}{-x^2-k^2}\Rightarrow\frac{\pi}{tanh(\pi x)}=\sum_{k=-\infty}^{\infty}\frac{x}{x^2+k^2}$$

$$\Rightarrow f(x)=\sum_{k=-\infty}^{\infty}\frac{1}{x^2+k^2}=\frac{\pi\,coth(\pi x)}{x}$$

$$\Rightarrow I=\frac{1}{2}f(\sqrt{3})-f(\sqrt{2})+\frac{1}{2}f(1)=\frac{1}{2}\left(\frac{\pi\,coth(\pi\sqrt{3})}{\sqrt{3}}\right)-\left(\frac{\pi\,coth(\pi\sqrt{2})}{\sqrt{2}}\right)+\frac{1}{2}(\pi\,coth(\pi))$$

$$\Rightarrow I=\frac{\pi}{2}\left(\frac{coth(\pi\sqrt{3})}{\sqrt{3}}-\sqrt{2}coth(\pi\sqrt{2})+coth(\pi)\right)$$

Solution 2 by Pratham Prasad-India

$$\sum_{q\epsilon Z}\frac{1}{(1+q^2)(2+q^2)(3+q^2)}=\frac{1}{2}\sum_{q\epsilon Z}\frac{1}{(1+q^2)}-\frac{2}{(2+q^2)}+\frac{1}{(3+q^2)}$$

$$=\frac{1}{2}\pi\left(\coth(\pi)-\sqrt{2}\coth(\pi\sqrt{2})+\frac{1}{\sqrt{3}}\coth(\pi\sqrt{3})\right)$$

$$=\frac{\pi}{2}\left(\frac{1}{\sqrt{3}}\coth(\pi\sqrt{3})-\sqrt{2}\coth(\pi\sqrt{2})+\coth(\pi)\right)$$

As by Fourier series:

$$\sum_{n\epsilon Z}\frac{1}{k^2+n^2}=\frac{\pi}{k}\coth(\pi k)$$