ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{n=1}^{\infty}\frac{\sigma_5(5n)}{n^{12}}=\frac{244,218,749}{78,125}\zeta(7)\zeta(12)$$

Proposed by Fao Ler-Iraq

Solution by Shobhit Jain-India

$$We\ know,\quad \sigma_k(n)=\sum_{d|n}d^k=\sum_{d|n}I_k(d).¶\left(\frac{n}{d}\right)=(I_k*¶)(n)$$

$$here,\ \ I_k(n)=n^k\qquad and\qquad ¶(n)=1\qquad\qquad \forall\ n\in N$$

$$Note:\ \ \sigma_k\,,I_k\,,¶\ are\ multiplicative\ functions$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s}=\sum_{n=1}^{\infty}\frac{(I_k*¶)(n)}{n^s}=\left(\sum_{n=1}^{\infty}\frac{I_k(n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{¶(n)}{n^s}\right)=\left(\sum_{n=1}^{\infty}\frac{n^k}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right)$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s}=\zeta(s-k)\zeta(s)=\prod_{p\ is\ prime}\left(\sum_{\alpha=0}^{\infty}\frac{\sigma_k(p^\alpha)}{p^{\alpha s}}\right)$$

$$Now,\ \ \sigma_k(p^\alpha)=\sum_{r=0}^{\alpha}p^{rk}=\frac{p^{(\alpha+1)k}-1}{p^k-1}$$

$$Now,\ \ \sum_{n=1}^{\infty}\frac{\sigma_k(5n)}{n^s}=\left(\sum_{\alpha=0}^{\infty}\frac{\sigma_k(5^{\alpha+1})}{5^{\alpha s}}\right)\prod_{p(\neq5)\ is\ prime}\left(\sum_{\alpha=0}^{\infty}\frac{\sigma_k(p^\alpha)}{p^{\alpha s}}\right)$$

$$=\frac{\left(\sum_{\alpha=0}^{\infty}\frac{\sigma_k(5^{\alpha+1})}{5^{\alpha s}}\right)}{\left(\sum_{\alpha=0}^{\infty}\frac{\sigma_k(5^{\alpha})}{5^{\alpha s}}\right)}\prod_{p\ is\ prime}\left(\sum_{\alpha=0}^{\infty}\frac{\sigma_k(p^\alpha)}{p^{\alpha s}}\right)=\frac{\left(\sum_{\alpha=0}^{\infty}\frac{5^{(\alpha+2)k}-1}{5^{\alpha s}(5^k-1)}\right)}{\left(\sum_{\alpha=0}^{\infty}\frac{5^{(\alpha+1)k}-1}{5^{\alpha s}(5^k-1)}\right)}\zeta(s-k)\zeta(s)\,,s>k$$

$$=\frac{\sum_{\alpha=0}^{\infty}\left(5^{2k}5^{(k-s)\alpha}-5^{-\alpha s}\right)}{\sum_{\alpha=0}^{\infty}\left(5^{k}5^{(k-s)\alpha}-5^{-\alpha s}\right)}\zeta(s-k)\zeta(s)=\left(\frac{\frac{5^{2k}}{1-5^{k-s}}-\frac{1}{1-5^{-s}}}{\frac{5^{k}}{1-5^{k-s}}-\frac{1}{1-5^{-s}}}\right)\zeta(s-k)\zeta(s)$$

$$=\left(\frac{5^{2k}-5^{2k-s}+5^{k-s}-1}{5^k-5^{k-s}+5^{k-s}-1}\right)\zeta(s-k)\zeta(s)=\left(\frac{5^{2k}-1-5^{2k-s}+5^{k-s}}{5^k-1}\right)\zeta(s-k)\zeta(s)$$

$$=\left(\frac{5^{2k}-1-5^{k-s}\left(5^k-1\right)}{5^k-1}\right)\zeta(s-k)\zeta(s)=\left(\frac{5^{2k}-1}{5^k-1}-\frac{5^{k-s}\left(5^k-1\right)}{5^k-1}\right)\zeta(s-k)\zeta(s)$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{\sigma_k(5n)}{n^s}=\left(5^k+1-5^{k-s}\right)\zeta(s-k)\zeta(s)$$

$$\underbrace{\Rightarrow}_{\substack{k=5\\s=12}}\sum_{n=1}^{\infty}\frac{\sigma_5(5n)}{n^{12}}=\left(5^5+1-5^{-7}\right)\zeta(7)\zeta(12)=\left(\frac{5^{12}+5^7-1}{5^7}\right)\zeta(7)\zeta(12)$$

$$=\left(\frac{244,140,625+78,125-1}{78,125}\right)\zeta(7)\zeta(12)$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{\sigma_5(5n)}{n^{12}}=\frac{244,218,749}{78,125}\zeta(7)\zeta(12)$$