ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\prod_{p\,:\,prime\,\le n} p^{\sum_{k=1}^{n}\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor} = n!$$

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$$Consider,\ \ E_p(n!) = \sum_{r=1}\left\lfloor\frac{n}{p^r}\right\rfloor = exponent\ of\ prime\ p\ in\ n!$$

$$\Rightarrow \prod_{p\,:\,prime\,\le n} p^{E_p(n!)} = n!$$

$$Now,\ \ Consider\ the\ function\ f_p(n) = \sum_{k=1}^{\lfloor n\rfloor}\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor\quad n \ge p$$

$$\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor = 0,\ \ for\ \ 0 \le log_p\left(\frac{n}{k}\right) < 1 \Rightarrow \frac{n}{p} < k \le n \Rightarrow \left\lfloor\frac{n}{p}\right\rfloor < k \le \lfloor n\rfloor$$

$$\Rightarrow f_p(n) = \sum_{k=1}^{\lfloor n/p\rfloor}\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor \Rightarrow f_p(n) = 0\ \ for\ n < p$$

$$\Rightarrow f_p\left(\frac{n}{p}\right) = \sum_{k=1}^{\lfloor n/p^2\rfloor}\left\lfloor log_p\left(\frac{n/p}{k}\right)\right\rfloor = \sum_{k=1}^{\lfloor n/p^2\rfloor}\left\lfloor log_p\left(\frac{n/p}{k}\right)\right\rfloor + \sum_{k=\lfloor n/p^2\rfloor+1}^{\lfloor n/p\rfloor} 0 = \sum_{k=1}^{\lfloor n/p\rfloor}\left\lfloor log_p\left(\frac{n/p}{k}\right)\right\rfloor$$

$$= \sum_{k=1}^{\lfloor n/p\rfloor}\left\lfloor log_p\left(\frac{n}{k}\right) - 1\right\rfloor = \sum_{k=1}^{\lfloor n/p\rfloor}\left(\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor - 1\right) = \sum_{k=1}^{\lfloor n/p\rfloor}\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor - \left\lfloor\frac{n}{p}\right\rfloor = f_p(n) - \left\lfloor\frac{n}{p}\right\rfloor$$

$$\Rightarrow f_p\left(\frac{n}{p}\right) = f_p(n) - \left\lfloor\frac{n}{p}\right\rfloor$$

$$\Rightarrow f_p(n) = \left\lfloor\frac{n}{p}\right\rfloor + f_p\left(\frac{n}{p}\right) = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + f_p\left(\frac{n}{p^2}\right) = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \left\lfloor\frac{n}{p^3}\right\rfloor + f_p\left(\frac{n}{p^3}\right)$$

$$= \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \left\lfloor\frac{n}{p^3}\right\rfloor + \left\lfloor\frac{n}{p^4}\right\rfloor + \ \ldots\ldots + 0 = E_p(n!) \Rightarrow E_p(n!) = f_p(n) = \sum_{k=1}^{\lfloor n\rfloor}\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor\quad n \ge p$$

$$\Rightarrow \prod_{p\,:\,prime\,\le n} p^{\sum_{k=1}^{n}\left\lfloor log_p\left(\frac{n}{k}\right)\right\rfloor} = n!$$