ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\sum_{m=3}^{\infty}\sum_{n=2}^{m-1}\frac{H_{n-1}2^{-m}}{n(m-n)}, \qquad where\ H_k\ is\ the\ k\_th\ harmonic\ number$$

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Solution by Shobhit Jain-India

$$I=\sum_{1<n<}\sum_{m<\infty}\frac{H_{n-1}2^{-m}}{n(m-n)}=\sum_{n=2}^{\infty}\frac{H_{n-1}}{n}\sum_{m=n+1}^{\infty}\frac{2^{-m}}{(m-n)}\underbrace{=}_{m=k+n}\left(\sum_{n=2}^{\infty}\frac{H_{n-1}}{n2^n}\right)\left(\sum_{k=1}^{\infty}\frac{2^{-k}}{k}\right)$$

$$Use,\ \sum_{k=1}^{\infty}\frac{2^{-k}}{k}=-ln\left(1-2^{-1}\right)=ln2$$

$$Now,\qquad -ln(1-y)=\sum_{k=1}^{\infty}\frac{y^k}{k}\Rightarrow-\frac{ln(1-y)}{1-y}=\sum_{n=2}^{\infty}H_{n-1}y^{n-1}$$

$$Integrate\ w.r.t\ y\ from\ y=0\ to\ y=x$$

$$\Rightarrow\frac{ln^2(1-x)}{2}=\sum_{n=2}^{\infty}\frac{H_{n-1}}{n}x^n\underbrace{\Rightarrow}_{x=\frac{1}{2}}\frac{ln^2 2}{2}=\sum_{n=2}^{\infty}\frac{H_{n-1}}{n2^n}\Rightarrow I=\left(\frac{ln^2 2}{2}\right)(ln2)=\frac{1}{2}(ln2)^3$$