ROMANIAN MATHEMATICAL MAGAZINE

NAGEL'S CEVIANS REVISITED

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ABSTRACT: In this paper we will have a new approach involving elements of triangle and Nagel cevians.

We consider ABC triangle and we use :

$r_a = \frac{S}{p-a}$ (and analogs) ; 2S=$ah_a = bh_b = ch_c$=2pr.

$$\frac{1}{r_a} = \frac{p-a}{S} = \frac{1}{r} - \frac{2}{h_a} \rightarrow \frac{1}{r} = \frac{1}{r_a} + \frac{2}{h_a} \text{ (and analogs)(1)}$$

We use:

$$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} \text{ (and analogs)(2)}$$

From (1) and (2) →

$$\frac{n_a}{r_a} = \frac{n_a - \sqrt{4r^2+(b-c)^2}}{r} \text{ (and analogs)(3)}$$

From (3) →

$$\frac{r_a}{r} = \frac{n_a}{n_a - \sqrt{4r^2+(b-c)^2}} \text{ (and analogs)(4)}$$

From $\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$ (and analogs); bc= $2Rh_a$ (and analogs); $(p-b)(p-c)$= rr_a (and analogs); $\sin\frac{A}{2} = \sqrt{\frac{r_a - r}{4R}}$ (and analogs)→ $\frac{r_a-r}{4R} = \frac{r}{2R}\frac{r_a}{h_a} \rightarrow \frac{r_a}{h_a} = \frac{r_a - r}{2r}$ (and analogs);

From $\frac{r_a}{h_a} = \frac{r_a-r}{2r}$ (and analogs) and(4) :

$$2\frac{r_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{n_a - \sqrt{4r^2+(b-c)^2}} \text{ (and analogs)(5)}$$

From $\frac{r_a r_b r_c}{h_a h_b h_c} = \frac{R}{2r}$ →

$$\frac{4R}{r} = \prod \frac{\sqrt{4r^2+(b-c)^2}}{n_a - \sqrt{4r^2+(b-c)^2}} \text{ (6)}$$

From $2\frac{r_a}{h_a} = \frac{a}{p-a}$ (and analogs), obtain:

$$\frac{a}{p-a}=\frac{\sqrt{4r^2+(b-c)^2}}{n_a-\sqrt{4r^2+(b-c)^2}}\text{(and analogs)(7)}$$

From $2(r_a+r_b+r_c)=2(4R+r)$ and $2r_a=\frac{h_a\sqrt{4r^2+(b-c)^2}}{n_a-\sqrt{4r^2+(b-c)^2}}$:

$$2(4R+r)=\sum\frac{h_a\sqrt{4r^2+(b-c)^2}}{n_a-\sqrt{4r^2+(b-c)^2}}(8)$$

From $2\frac{r_a}{h_a}=\frac{a}{p-a}\to\frac{r_a}{h_a}=\frac{a}{b+c-a}$ (and analogs) and

$$\frac{b+c}{a}=1+\frac{h_a}{r_a}\text{ (and analogs) and use (5) :}$$

$$\frac{h_a}{r_a}=2\left[\frac{n_a}{\sqrt{4r^2+(b-c)^2}}-1\right]\text{(and analogs)(9)}$$

$$\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}-1\text{(and analogs)(10)}$$

From (4) and $r_a+r_b+r_c=4R+r$:

$$1+\frac{4R}{r}=\sum\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\ (11)$$

From (6) and (11):

$$1+\prod\frac{\sqrt{4r^2+(b-c)^2}}{n_a-\sqrt{4r^2+(b-c)^2}}=\sum\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\ (12)$$

From $\frac{r_a}{r}=\frac{p}{p-a}$ (and analogs) and (4):

$$\frac{p}{p-a}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\text{ (and analogs) (13)}$$

From (13) after simple manipulations:

$$\frac{a}{p}=\frac{\sqrt{4r^2+(b-c)^2}}{n_a}\text{ (and analogs) (14)}$$

From (14) after summation we obtain:

$$n_a+n_b+n_c=p\sum\frac{\sqrt{4r^2+(b-c)^2}}{a}(15)$$

From (13) and $p^2=n_a^2+2r_ah_a$(and analogs) and $r_br_c=p(p-a)$ (and analogs):

$n_a^2+2r_ah_a=\frac{n_ar_br_c}{n_a-\sqrt{4r^2+(b-c)^2}}$:

$$n_a+\frac{2r_ah_a}{n_a}=\frac{r_br_c}{n_a-\sqrt{4r^2+(b-c)^2}}\text{ (and analogs)(16)}$$

If $\sqrt{4r^2+(b-c)^2}$ = x (and analogs) ,we obtain:

$\frac{n_a}{r_a}=\frac{n_a-x}{r}$ (and analogs), $\frac{r_a}{r}=\frac{n_a}{n_a-x}$ (and analogs), $2\frac{r_a}{h_a}=\frac{x}{n_a-x}$(and analogs)

$\frac{p}{p-a}=\frac{n_a}{n_a-x}$ (and analogs). From (3), $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$,$(n_a+n_b+n_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)=\frac{n_a+n_b+n_c}{r}$

$(n_a+n_b+n_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\right)=\sum\frac{n_a}{r_a}+\sum\frac{n_b+n_c}{r_a}\rightarrow$

$$\sum\frac{\mathbf{n_b+n_c}}{\mathbf{r_a}}=\sum\frac{\mathbf{x}}{\mathbf{r}}\ \textbf{(17)}$$

From $p^2=n_a^2+2r_ah_a$(and analogs)→$(p-n_a)(p+n_a)=2r_ah_a$

$p-n_a=\frac{2r_ah_a}{p+n_a}\rightarrow\frac{p}{r_a}-\frac{n_a}{r_a}=\frac{2h_a}{p+n_a}$ (and analogs)

From $\frac{p}{r_a}-\frac{n_a}{r_a}=\frac{2h_a}{p+n_a}$ (and analogs), (3) and $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$ after simple manipulations:

$$\sum\frac{\mathbf{h_a}}{\mathbf{p+n_a}}=\frac{\mathbf{p-n_a-n_b-n_c}}{\mathbf{2r}}+\sum\frac{\sqrt{\mathbf{4r^2+(b-c)^2}}}{\mathbf{2r}}\ \textbf{(18)}$$

$$\sum\frac{\mathbf{h_a}}{\mathbf{p+n_a}}=\frac{\mathbf{p+x+y+z-n_a-n_b-n_c}}{\mathbf{2r}}\textbf{(19)}$$

From $(p-n_a)(p+n_a)=2r_ah_a$

$p+n_a=\frac{2r_ah_a}{p-n_a}\rightarrow\frac{p}{r_a}+\frac{n_a}{r_a}=\frac{2h_a}{p-n_a}$ (and analogs) and (3) after simple manipulations:

$$\sum\frac{\mathbf{h_a}}{\mathbf{p-n_a}}=\frac{\mathbf{p+n_a+n_b+n_c}}{\mathbf{2r}}-\sum\frac{\sqrt{\mathbf{4r^2+(b-c)^2}}}{\mathbf{2r}}\ \textbf{(20)}$$

$$\sum\frac{\mathbf{h_a}}{\mathbf{p-n_a}}=\frac{\mathbf{p+n_a+n_b+n_c-x-y-z}}{\mathbf{2r}}\ \textbf{(21)}$$

$\frac{r_br_c}{r^2}=\frac{n_bn_c}{(n_b-y)(n_c-z)}$,x=$\sqrt{4r^2+(b-c)^2}$ (and analogs), $r_br_c=p(p-a)$(and analogs);

$m_al_a\geq p(p-a)$(and analogs)(Panaitopol)

$$\frac{\mathbf{m_al_a}}{\mathbf{r^2}}\geq\frac{\mathbf{n_bn_c}}{\mathbf{(n_b-y)(n_c-z)}}\ \textbf{(and analogs)(22)}$$

From (22) after summation :

$$\frac{\mathbf{m_al_a+m_bl_b+m_cl_c}}{\mathbf{r^2}}\geq\sum\frac{\mathbf{n_bn_c}}{\mathbf{(n_b-y)(n_c-z)}}\textbf{(23)}$$

$$\frac{\mathbf{n_bn_c}}{\mathbf{r_br_c}}=\frac{\mathbf{(n_b-y)(n_c-z)}}{\mathbf{r^2}}\ \textbf{(and analogs) (24)}$$

From (24):

$$\frac{n_b}{r_c}+\frac{n_c}{r_b}\ge 2\frac{\sqrt{(n_b-y)(n_c-z)}}{r}\ \textbf{(and analogs)(25)}$$

$\sqrt{4r^2+(b-c)^2}$ = x (and analogs), $\frac{p}{a}=\frac{h_a}{2r}$ (and analogs) because $ah_a=bh_b=ch_c$=2pr→

$$\frac{p-a}{a}=\frac{h_a-2r}{2r}$$

From(13) and (7): $\frac{p}{a}\frac{a}{p-a}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$→$\frac{h_a}{2r}\frac{2r}{h_a-2r}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$

$$\frac{h_a}{h_a-2r}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\ \textbf{(and analogs)(26)}$$

From (26):

$$\frac{n_a}{h_a}=\frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}=\frac{n_a-x}{h_a-2r}\textbf{(and analogs)(27)}$$

From(27) and $\frac{R}{r}\ge 1+\frac{n_a}{h_a}$ (and analogs)→

$$\frac{R}{r}\ge 1+\frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}\textbf{(and analogs)(28)}$$

From (28) :

$$\left(\frac{R}{r}-1\right)^3\ge\frac{(n_a-x)(n_b-y)(n_c-z)}{(h_a-2r)(h_b-2r)(h_c-2r)}$$

Because of (27) and $n_a\ge h_a$→$n_a-x\ge h_a-2r$ we obtain:

$$\frac{R}{r}\ge 1+\sqrt[3]{\frac{(n_a-x)(n_b-y)(n_c-z)}{(h_a-2r)(h_b-2r)(h_c-2r)}}\ge 2\ \textbf{(Euler Inequality Refinement) (29)}$$

$\sqrt{4r^2+(b-c)^2}$ = x (and analogs)

From $n_a\ge p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs)[3]→ $\frac{n_a}{h_a}\ge\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}$ and (27):

$$\frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}=\frac{n_a-x}{h_a-2r}\ge\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}\textbf{(and analogs)(30)}$$

From (4),(26) and $n_a\ge p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs):

$$\frac{h_a}{h_a-2r}\ge\frac{p_a}{n_a-\sqrt{4r^2+(b-c)^2}}\sqrt{\frac{l_a}{g_a}}\textbf{(and analogs)(31)}$$

$$\frac{r_a}{r}\ge\frac{p_a}{n_a-\sqrt{4r^2+(b-c)^2}}\sqrt{\frac{l_a}{g_a}}\textbf{(and analogs)(32)}$$

From(32) and $r_a + r_b + r_c = 4R + r$:

$$1+\frac{4R}{r} \geq \sum \frac{p_a}{n_a - \sqrt{4r^2+(b-c)^2}}\sqrt{\frac{l_a}{g_a}} \quad (33)$$

$1+\frac{4R}{r} \geq \sum \frac{p_a}{n_a - x}\sqrt{\frac{l_a}{g_a}}$, $\sqrt{4r^2 + (b-c)^2}$ = x (and analogs). From (6): $\frac{4R}{r} = \prod \frac{x}{n_a - x}$

From (5) and (26):

$$\frac{2r_a}{h_a - 2r} = \frac{n_a x}{(n_a - x)^2} \text{ (and analogs) } (34)$$

From (34) :

$$\prod \frac{2r_a}{h_a - 2r} = \prod \frac{n_a x}{(n_a - x)^2} \quad (35)$$

$$\frac{n_a n_b n_c}{r_a r_b r_c} = \frac{8(n_a - x)^2 (n_b - y)^2 (n_c - z)^2}{xyz(h_a - 2r)(h_b - 2r)(h_c - 2r)} \quad (36)$$

From $\frac{r_b r_c}{r^2} = \frac{n_b n_c}{(n_b - y)(n_c - z)}$ (and analogs), $r_b r_c = p(p-a)$ (and analogs):

$$\left(\frac{p}{r}\right)^2 = \sum \frac{n_b n_c}{(n_b - y)(n_c - z)}, x = \sqrt{4r^2 + (b-c)^2} \text{ (and analogs) } (37)$$

From $l_a = 2\frac{\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs) and $\frac{r_b r_c}{r^2} = \frac{n_b n_c}{(n_b - y)(n_c - z)}$ (and analogs)

$$\left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)\frac{l_a}{r} = 2\sqrt{\frac{n_b n_c}{(n_b - y)(n_c - z)}} \text{ (and analogs) } (38)$$

From (38):

$$\left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)\frac{l_a}{r} \leq \frac{n_b}{n_c - z} + \frac{n_c}{n_b - y} \text{ (and analogs) } (39)$$

From (38):

$$l_a l_b l_c \prod \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right) = (2r)^3 \frac{n_a n_b n_c}{(n_a - x)(n_b - y)(n_c - z)} \quad (40)$$

$x = \sqrt{4r^2 + (b-c)^2}$ (and analogs). From $bc = r_b r_c + r_a r$ (and analogs):

$$bc = \left[\frac{n_b n_c}{(n_b - y)(n_c - z)} + \frac{n_a}{n_a - x}\right] r^2 \text{ (and analogs) } (41)$$

From $bc = 2Rh_a$ (and analogs); $ah_a = 2pr = (a+b+c)r$

$\frac{h_a}{r} = 1 + \frac{b+c}{a}$ (and analogs) and (41) we obtain:

$$\frac{2R}{r}\left(1+\frac{b+c}{a}\right)=\frac{n_b n_c}{(n_b-y)(n_c-z)}+\frac{n_a}{n_a-x}$$ **(and analogs)(42)**

$\sqrt{4r^2+(b-c)^2}=x$ (and analogs)

References:

[1]. Bogdan Fuștei-100 OLD AND NEW INEQUALITIES AND IDENTITIES IN TRIANGLE-www.ssmrmh.ro

[2]. Bogdan Fuștei-THE AVALANCHE OF GEOMETRIC INEQUALITIES-www.ssmrmh.ro

[3]. Bogdan Fuștei-CONNECTIONS BETWEEN FAMOUS CEVIANS-www.ssmrmh.ro